# Remarks on Chapter 5 in L. Trefethen: Spectral Methods in MATLAB

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## 1 Introduction

In this short note we give some comments and additional computations relating to the results in Chapter 5 of [3]. We assume that the reader is familiar with the basic results in Complex Analysis. We refer to the lecture notes [1] for results we need. The error estimate we give here for spectral differentiation based on the choice of Chebyshev points made in [3], is based on the paper [2].

## 2 Lagrange Interpolation

We state the Lagrange interpolation formula and derive an error estimate for it. We limit outselves to interpolation points in the interval $[-1,1]$. Let

$$x_0, x_1, \ldots, x_N \in [-1,1] \quad \text{be } N+1 \text{ distinct points.}$$

The Lagrange interpolation polynomials are defined by

$$L_j(x) = \prod_{\substack{k=0\\k\neq j}}^{N} \frac{x-x_k}{x_j-x_k}. \tag{2.1}$$

They have the property

$$L_j(x_k) = \begin{cases} 1, & \text{if } k=j,\\ 0, & \text{if } k\neq j. \end{cases} \tag{2.2}$$

Thus given a function $u$ on $[-1,1]$, let $u_j = u(x_j)$. Then the Lagrange interpolation formula is the following expression:

$$p_N(x) = \sum_{j=0}^{N} u_j L_j(x). \tag{2.3}$$

It is clear from (2.2) that $p_N$ is the interpolating polynomial, viz. $p_N(x_j) = u_j$, for all $j = 0, 1, \ldots, N$.

We will now derive an expression for the error term

$$R_N(x) = u(x) - p_N(x). \tag{2.4}$$

For this purpose we introduce the function

$$\omega_{N+1}(x) = \prod_{k=0}^{N} (x - x_k). \tag{2.5}$$

We have the following expression for the Lagrange polynomials:

$$L_j(x) = \frac{\omega_{N+1}(x)}{(x - x_j)\omega'_{N+1}(x_j)}. \tag{2.6}$$

Since the interpolating polynomial for a given set of nodes $\{(x_j, u_j)\}_{j=0,1,\ldots,N}$ is unique, it suffices to verify that the expression on the right hand side of (2.6) satisfies (2.2). We first assume that $k \neq j$. Since by definition $\omega_{N+1}(x_k) = 0$, the right hand side in (2.6) is zero for $x = x_k$. For $k = j$ we have to evaluate using a limit. We have, using $\omega_{N+1}(x_j) = 0$, i

$$\begin{aligned}
\lim_{x \to x_j} \frac{\omega_{N+1}(x)}{(x - x_j)\omega'_{N+1}(x_j)} &= \frac{1}{\omega'_{N+1}(x_j)} \lim_{x \to x_j} \frac{\omega_{N+1}(x) - \omega_{N+1}(x_j)}{x - x_j} \\
&= \frac{\omega'_{N+1}(x_j)}{\omega'_{N+1}(x_j)} = 1.
\end{aligned}$$

Thus equation (2.6) has been established.

In order to get an error estimate we now assume that the function $u$ extends to an analytic function in a domain $\Omega$, which contains the interval $[-1,1]$. The approximating polynomial (2.3) is defined for all $z \in \mathbf{C}$. Thus the error term (2.4) is defined for all $z \in \Omega$.

Now let $\Gamma$ be a simple, positively oriented, closed contour, containing $[-1,1]$ in its interior and contained in $\Omega$. Then we have the following formula for the error term, valid for $z$ inside the contour $\Gamma$.

$$\begin{aligned} R_N(z) &= u(z) - p_N(z) \\ &= u(z) - \sum_{j=0}^{N} \frac{\omega_{N+1}(z)u(x_j)}{(z-x_j)\omega'_{N+1}(x_j)} \\ &= \frac{\omega_{N+1}(z)}{2\pi i}\int_\Gamma \frac{u(\zeta)}{(\zeta - z)\omega_{N+1}(\zeta)} d\zeta. \end{aligned} \tag{2.7}$$

The last equality in (2.7) is a consequence of the calculus of residues. See Theorem 7.5 in [1]. More precisely, the integrand in (2.7) has simple poles at the point $z$ and at the points $x_0, x_1, \dots, x_N$. The residue at the point $z$ is

$$\frac{u(z)}{\omega_{N+1}(z)},$$

and the residue at a point $x_j$ is

$$\frac{u(x_j)}{\omega'_{N+1}(x_j)(x_j - z)}.$$

Thus the residue theorem implies that (2.7) holds (note that the denominator has the factor $(x_j - z)$, accounting for the sign in front of the sum in (2.7)).

# 3 Error Estimates in Polynomial Interpolation

The formula (2.7) allows one to obtain error estimates in the polynomial interpolation, if the function $u$ has an analytic continuation to a neighborhood of the interval $[-1, 1]$. Assume this is the case. Then we get from (2.7), taking $z = x \in [-1, 1]$,

$$|R_N(x)| \le \left[\frac{1}{2\pi}\frac{\max_{z\in\Gamma}|u(z)|}{\min_{z\in\Gamma}|\omega_{N+1}(z)|}\int_\Gamma \frac{1}{|\zeta - x|}|d\zeta|\right]|\omega_{N+1}(x)|. \tag{3.1}$$

The integral above is with respect to arc length. It follows from the above formula that the approximation error for large $N$ is governed by the behavior of $\omega_{N+1}(z)$ for large $N$, both on the curve $\Gamma$, and on the interval $[-1, 1]$. This explains why the author in [3] concentrates on the study of this function, in the book denoted by $p(z)$, see page 43 in [3].

Now let us try to explain Theorem 5 in [3]. We introduce the function

$$\phi_{N+1}(z) = \frac{1}{N+1} \sum_{k=0}^{N} \log|z - x_k|. \tag{3.2}$$

We also as in [3] assume that the distribution of the points $\{x_k\}$ for $N$ large is given by a density function $\rho(x)$. We define the associated potential function

$$\phi(z) = \int_{-1}^{1} \rho(x) \log|z - x| dx. \tag{3.3}$$

Let

$$\phi_{[-1,1]} = \max_{x \in [-1,1]} |\phi(x)|.$$

Assume that there exists a constant $\phi_u > \phi_{[-1,1]}$, such that $u$ is analytic in a domain slightly larger than the region

$$\{x \mid \phi(z) \leq \phi_u\}.$$

Now to explain the estimate in Theorem 5, we first note that for $N$ large we have $|\omega_{N+1}(z)| \approx e^{(N+1)\phi(z)}$, as explained in [3]. Suppose we can take a curve $\Gamma$, such that $\phi(z) \approx \phi_u$ for $z \in \Gamma$. Then we can conclude that

$$\min_{z \in \Gamma} |\omega_{N+1}(z)| \geq ce^{(N+1)\phi(z)} \geq ce^{(N+1)\phi_u},$$

for large $N$. Similarly, for $x \in [-1, 1]$, we have

$$|\omega_{N+1}(x)| \leq ce^{(N+1)\phi(x)} \leq ce^{(N+1)\phi_{[-1,1]}}.$$

Using these two estimates in (3.1), we get

$$|R_N(x)| \leq Ce^{-(N+1)(\phi_u - \phi_{[-1,1]})},$$

which is the estimate in Theorem 5.

We should note that the detailed justification of the result in Theorem 5 uses a number of properties of the functions $\phi_{N+1}$ and $\phi$. Both functions are harmonic, which means (identifying points in the complex plane $z = x + iy$ with points in $\mathbf{R}^2$) that

$$\frac{\partial^2 \phi}{\partial x^2}(x, y) + \frac{\partial^2 \phi}{\partial y^2}(x, y) = 0$$

for all $z = z + iy$ not in the interval $[-1, 1]$, and analogously for $\phi_{N+1}$. Harmonic functions have many nice properties. For example, they are infinitely often differentiable, and they obey the maximum principle, meaning that a harmonic function cannot have a local maximum or a local minimum in the interior of a bounded domain. It is these facts that allow us to choose an integration contour above.

## 4 An Error Estimate for Spectral Differentiation

We will briefly show how the error formula derived above can be used to get error estimates for spectral differentiation based on the Chebyshev points. We start with a general formula, valid for any set of nodes $\{x_j\}$. Take the formula (2.7) for $z = x \in [-1, 1]$ and differentiate with respect to $x$ to get the formula

$$\begin{aligned} R'_N(x) &= u'(x) - p'_N(x) \\ &= \frac{1}{2\pi i} \int_\Gamma \Big( \frac{\omega'_{N+1}(x)}{\zeta - x} + \frac{\omega_{N+1}(x)}{(\zeta - x)^2} \Big) \frac{u(\zeta)}{\omega_{N+1}(\zeta)} d\zeta. \end{aligned} \tag{4.1}$$

Now if we evaluate at the nodes and use that $\omega_{N+1}(x_j) = 0$ for all $j = 0, 1, \dots, N$, we get the general formula

$$u'(x_j) - p'_N(x_j) = \frac{\omega'_{N+1}(x_j)}{2\pi i} \int_\Gamma \frac{u(\zeta)}{\omega_{N+1}(\zeta)(\zeta - x_j)} d\zeta. \tag{4.2}$$

From now on we assume that the points $x_j$ are the Chebyshev points chosen in [3], i.e.

$$x_j = \cos(\pi j/N), \quad j = 0, 1, 2, \dots, N.$$

We let $T_N(x)$ denote the Chebyshev polynomial of degree $N$, see [3]. The Chebyshev points give the locations of the local extrema of $T_N(x)$ on the interval $[-1, 1]$. By using properties of the Chebyshev polynomials one can show that there is a constant $c$ such that

$$\omega_{N+1}(x) = c(T_{N+1}(x) - T_{N-1}(x)).$$

Since in (4.2) only a ratio of two $\omega_{N+1}$ enters, we can change the definition of $\omega_{N+1}$ to eliminate this constant. Thus from now on $\omega_{N+1}$ denotes this 'renormalized' function.

Now we need to chose an appropriate curve $\Gamma$ in order to use (4.2) to estimate the accuracy of spectral differentiation. As can be seen from the computations in [3] an ellipse with foci in $-1$ and 1 seems to be a good choice. Thus we take a parameter $\delta > 1$ and let $\Gamma_\delta$ denote the ellipse given by

$$z(\theta) = \tfrac{1}{2}(\delta e^{i\theta} + \delta^{-1} e^{-i\theta}), \quad 0 \le \theta \le 2\pi. \tag{4.3}$$

It will follow from the estimates below that $|\omega_{N+1}(z)|$ is nearly constant for large $N$ on the curve $\Gamma_\delta$.

Now we need one of the properties of the Chebyshev polynomials. We have

$$T_N(z) = \tfrac{1}{2}\big((z+\sqrt{z^2-1})^N + (z-\sqrt{z^2-1})^N\big). \tag{4.4}$$

If $z \in \Gamma_\delta$, a simple calculation using (4.3) shows that

$$z+\sqrt{z^2-1} = \delta e^{-\theta}, \tag{4.5}$$

$$z-\sqrt{z^2-1} = \delta^{-1} e^{-i\theta}. \tag{4.6}$$

Inserted into (4.4) this leads to

$$T_N(z) = \tfrac{1}{2}\big((\delta e^{i\theta})^N + (\delta^{-1}e^{-i\theta})^N\big) \tag{4.7}$$

Then we get

$$\begin{aligned}\omega_{N+1}(z) &= T_{N+1}(z) - T_{N-1}(z)\\ &= \tfrac{1}{2}\big((\delta e^{i\theta})^{N+1} + (\delta^{-1}e^{-i\theta})^{N+1}\big) - \tfrac{1}{2}\big((\delta e^{i\theta})^{N-1} + (\delta^{-1}e^{-i\theta})^{N-1}\big)\\ &= \tfrac{1}{2}\big(\delta e^{i\theta} - \delta^{-1}e^{-i\theta}\big)\big(\delta^N e^{iN\theta} - \delta^{-N}e^{-iN\theta}\big).\end{aligned}$$

Now

$$\begin{aligned}|\delta^N e^{iN\theta} - \delta^{-N}e^{-iN\theta}| &= |(\delta^N - \delta^{-N})\cos(N\theta) + i(\delta^N + \delta^{-N})\sin(N\theta)|\\ &= \sqrt{\delta^{2N} + \delta^{-2N} - 2\cos(2N\theta)},\end{aligned}$$

where in the last step we used the double angle formula for the cosine. This computation, also used for $N=1$, gives the result

$$|\omega_{N+1}(z)| = \tfrac{1}{2}\sqrt{\delta^2+\delta^{-2}-2\cos(2\theta)}\sqrt{\delta^{2N}+\delta^{-2N}-2\cos(2N\theta)}. \tag{4.8}$$

We use this result to get upper and lower bounds on $|\omega_{N+1}(z)|$. The minimal value occurs for $\theta = 0$, giving the lower bound

$$\tfrac{1}{2}(\delta - \delta^{-1})(\delta^N - \delta^{-N}).$$

To get the upper bound we maximize the two terms individually. The maximal value of each term occurs for values of $\theta$ such that the cosine term equals 1. This leads to an upper bound

$$\tfrac{1}{2}(\delta + \delta^{-1})(\delta^N + \delta^{-N}).$$

To use these bounds, let $\eta = \log\delta$, such that $\delta = e^\eta$. Note that $\eta > 0$, since we assume $\delta > 1$. Then we have for example

$$\tfrac{1}{2}(\delta + \delta^{-1}) = \tfrac{1}{2}(e^\eta + e^{-\eta}) = \cosh(\eta).$$

This identity and similar ones then allow us to summarize the lower and upper bounds as

$$2\sinh(\eta)\sinh(N\eta) \le |\omega_{N+1}(z)| \le 2\cosh(\eta)\cosh(N\eta). \tag{4.9}$$

We need a bound on the derivative $\omega'_{N+1}(x_j)$. This bound can be obtained from standard bounds on the derivative of Chebyshev polynomials. The result is

$$\max_{0\le j\le N}|\omega'_{N+1}(x_j)| = 4N.$$

Furthermore, we need an estimate for the arc length for the ellipse $\Gamma_\delta$. We denote this arc length by $\ell_\delta$. From calculus we have that

$$\ell_\delta = \int_0^{2\pi} |z'(\theta)|d\theta.$$

Since

$$|z'(\theta)| = |\tfrac{i}{2}(\delta e^{i\theta} - \delta^{-1}e^{-i\theta})| \le \tfrac{1}{2}(\delta+\delta^{-1}),$$

we have the simple estimate

$$\ell_\delta \le \pi(\delta+\delta^{-1}). \tag{4.10}$$

We also need an estimate for the minimal distance from points on the ellipse $\Gamma_\delta$ to points in the interval $[-1,1]$. This distance can be computed exactly. It is a very easy exercise in calculus (or geometry), so I just state the result. The minimal distance is

$$d_\delta = \tfrac{1}{2}(\delta+\delta^{-1}) - 1. \tag{4.11}$$

Now we can state the estimate for the approximation error. We assume that $u$ is analytic in a region larger than the closed ellipse bounded by $\Gamma_\delta$. We define

$$C_\delta = \max_{z\in\Gamma_\delta}|u(z)|.$$

Then using (4.2) and the above estimates we can estimate the approximation error as follows.

$$|u'(x_j) - p'_N(x_j)| \le \frac{|\omega'_{N+1}(x_j)|C_\delta}{2\pi \min_{z\in\Gamma_\delta}|\omega_{N+1}(z)|}\int_{\Gamma_\delta}\frac{1}{|\zeta-x_j|}|d\zeta| \tag{4.12}$$

$$\le \frac{4NC_\delta}{2\pi}\frac{1}{\sinh(\eta)\sinh(N\eta)}\frac{\ell_\delta}{d_\delta}. \tag{4.13}$$

Now we need to look at some of the terms individually. First, we have

$$\frac{\ell_\delta}{d_\delta} = \frac{\pi(\delta + \delta^{-1})}{\frac{1}{2}(\delta + \delta^{-1}) - 1} = \frac{2\pi\cosh(\eta)}{\cosh(\eta) - 1}.$$

This quantity tends to infinity as $\eta \to 0$. It is easy to estimate its size numerically. For example, it is less than 100 for $\eta \geq 0.365$, corresponding to $\delta > 1.441$.

The exponential decay for $N \to \infty$ comes from the $\sinh(N\eta)$ term. A simple estimate is

$$\frac{1}{\sinh(t)} = \frac{\cosh(t)}{\sinh(t)}\frac{1}{\cosh(t)} = \coth(t)\frac{2}{e^t + e^{-t}} \leq \coth(t)2e^{-t},$$

valid for any $t > 0$. Thus if we assume as above $\eta \geq 0.365$, we have $\coth(N\eta) \leq 2.9$, for all $N \geq 1$. Thus the decay rate is

$$e^{-N\eta} = (\delta^{-1})^N.$$

Now finally to compare with Theorem 6 in [3], we note that $\delta$ is the sum of the major and minor semi-axes in the ellipse $\Gamma_\delta$, which is called $K$ in Theorem 6.

# References


[1] Arne Jensen, *A Short Introduction to Complex Analysis*, Lecture Notes, Department of Mathematical Sciences, Aalborg University, 2005.

[2] S. C. Reddy and J. A. C. Weideman, The accuracy of the Chebyshev differencing method for analytic functions, *SIAM J. Numer. Anal.* **42** (2005), 2176–2187.

[3] Lloyd N. Trefethen, *Spectral Methods in* MATLAB, SIAM 2000.