Introduction to Pseudospectra and their Applications

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Motivation

Consider the problem

$$\frac{du}{dt} = Au, \qquad u(0) = u_0.$$

Solution is formally

$$u(t) = e^{tA}u_0.$$

Interpretation depends on the context.

- A operator on a finite dimensional space $\mathcal{H}$.

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I will limit the discussion to the first two cases.

Motivation

A result in the finite dimensional case. A an $N \times N$ matrix with

$$\operatorname{Re}\lambda < 0 \quad \text{for all } \lambda \in \sigma(A).$$

Then

$$u(t) = e^{tA}u_0 \to 0 \quad \text{as} \quad t \to \infty$$

for all initial conditions u_0.

Some questions.

- How fast will the solution approach zero.

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- How fast will the solution approach zero.
- How large will the solution get before approaching zero.

It is this last quantative question we will try to answer.

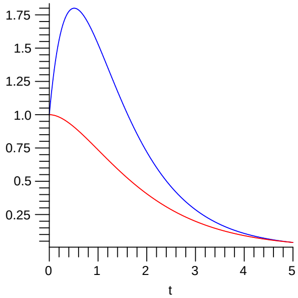
Motivation

An example:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}.$$

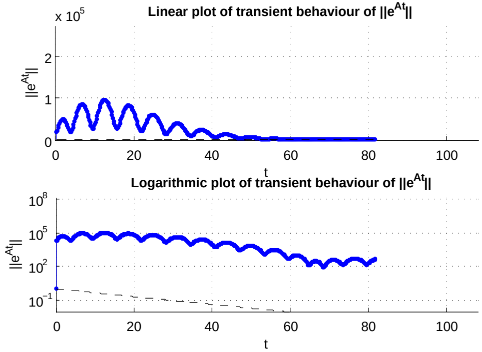
We plot $\|e^{tA}\|$ and $\|e^{tB}\|$ as functions of t.

Which is which?



Motivation

Here is an example from real life: Boing aircraft wing flutter. Matrix is 55×55.



Motivation—Resolvent and Spectrum

Operator equation $Ax - \lambda x = b, \quad \lambda \notin \sigma(A)$. Solution

$$x = (A - \lambda I)^{-1} b.$$

Perturb right hand side, $\|u\| \leq \varepsilon$, i.e. consider $Ay - \lambda y = b + u$. We have

$$\|x - y\| \leq \varepsilon \|(A - \lambda I)^{-1}\|.$$

Thus the effect depends on the norm of the resolvent. It may be large far from the spectrum. Recall general result:

$$\|(A - zI)^{-1}\| \geq \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

Definition of Pseudospectrum

Definition

Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. The ε-pseudospectrum of A is given by

$$\sigma_\varepsilon(A) = \sigma(A) \cup \{z \in \mathbf{C} \setminus \sigma(A) \mid \|(A - zI)^{-1}\| > \varepsilon^{-1}\}.$$

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Theorem

Let $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then the following three statements are equivalent.

(i) $z \in \sigma_\varepsilon(A)$.

(ii) *There exists $B \in \mathcal{B}(\mathcal{H})$ with $\|B\| < \varepsilon$ such that $z \in \sigma(A + B)$.*

(iii) *$z \in \sigma(A)$ or there exists $v \in \mathcal{H}$ with $\|v\| = 1$ such that $\|(A - zI)v\| < \varepsilon$.*

Definition of Pseudospectrum, Finite Dimension

Let T be an $N \times N$ matrix. The eigenvalues of $(T^*T)^{1/2}$ are called the singular values of T. The smallest singular value is denoted $s_{\min}(T)$.

Theorem

Assume that $\mathcal{H}$ is finite dimensional and $T \in \mathcal{B}(\mathcal{H})$. Let $\varepsilon > 0$. Then $z \in \sigma_\varepsilon(T)$ if and only if $s_{\min}(T - zI) < \varepsilon$.

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Since the singular values of a matrix can be computed numerically, this result provides a method for plotting the pseudospectra of a given matrix. One chooses a finite grid of points in the complex plane, and evaluates $s_{\min}(T - zI)$ at each point. Plotting level curves for these points provides a picture of the pseudospectra of T.

Properties of $\sigma_\varepsilon(A)$

Define $D_\delta = \{z \in \mathbf{C} \mid |z| < \delta\}$.

Proposition

Let $A \in \mathcal{B}(\mathcal{H})$. Each $\sigma_\varepsilon(A)$ is a bounded open subset of $\mathbf{C}$. We have $\sigma_{\varepsilon_1}(A) \subset \sigma_{\varepsilon_2}(A)$ for $0 < \varepsilon_1 < \varepsilon_2$. Furthermore, $\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A)$. For $\delta > 0$ we have $D_\delta + \sigma_\varepsilon(A) \subseteq \sigma_{\varepsilon+\delta}(A)$. We have $\sigma_\varepsilon(A^*) = \overline{\sigma_\varepsilon(A)}$.

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Proposition

Let $A \in \mathcal{B}(\mathcal{H})$ and assume that $V \in \mathcal{B}(\mathcal{H})$ is invertible. Let $\kappa = \mathrm{cond}(V)(= \|V\| \cdot \|V^{-1}\|)$. Let $B = VAV^{-1}$. Then

$$\sigma(B) = \sigma(A),$$

and for $\varepsilon > 0$ we have

$$\sigma_{\varepsilon/\kappa}(A) \subseteq \sigma_\varepsilon(B) \subseteq \sigma_{\kappa\varepsilon}(A).$$

Properties of $\sigma_\varepsilon(A)$

Proposition

Let $A \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$. *Then*

$$\{z \mid \operatorname{dist}(z, \sigma(A)) < \varepsilon\} \subseteq \sigma_\varepsilon(A).$$

If A *is normal, then*

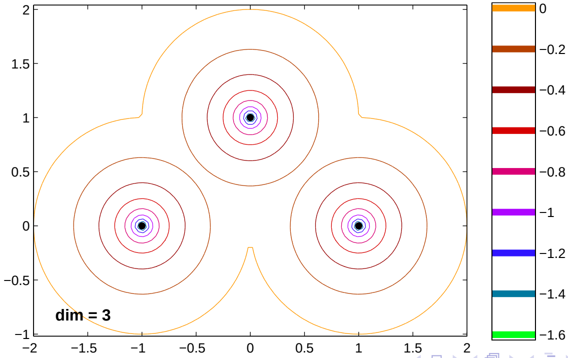
$$\sigma_\varepsilon(A) = \{z \mid \operatorname{dist}(z, \sigma(A)) < \varepsilon\}.$$

Example 1

Take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{bmatrix}$$

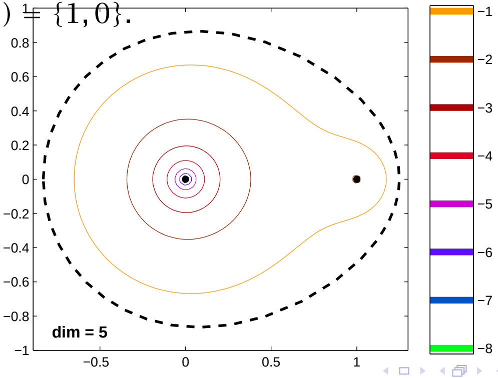
with $\sigma(A) = \{1, -1, i\}$.



Example 2

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $\sigma(B) = \{1, 0\}$.



Example 2

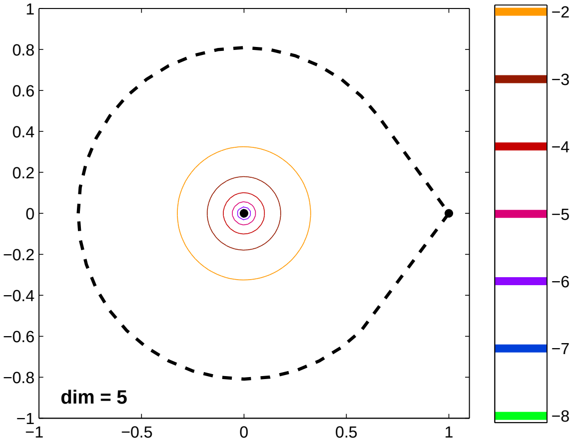
Let us look at the Jordan canonical form of this matrix. We have $J = Q^{-1}BQ$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We have

$$\operatorname{cond}(Q) = 3 + 2\sqrt{2} \approx 5.828427125.$$

Example 2



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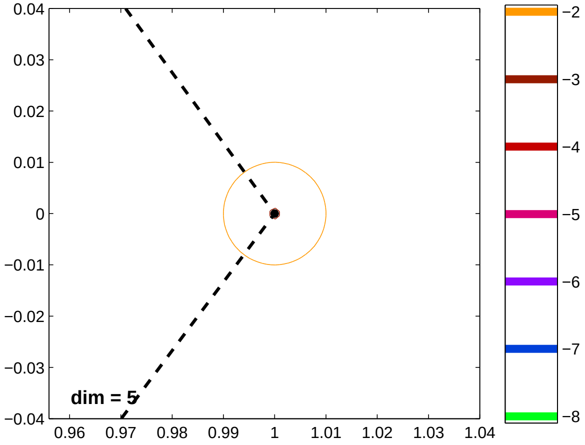

0.04
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
−0.04
0.96
0.97
0.98
0.99
1
1.01
1.02
1.03
1.04
dim = 5
−2
−3
−4
−5
−6
−7
−8

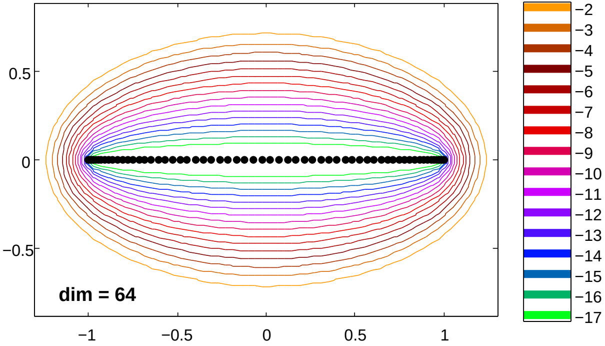
Example 3

A Toeplitz matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1/4 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1/4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1/4 & 0 \end{bmatrix}$$

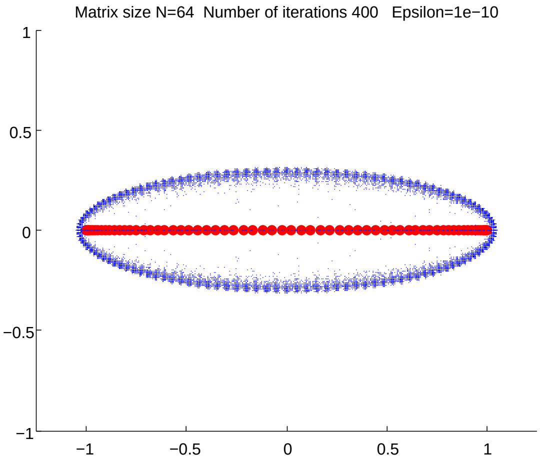
We have $A = SDS^{-1}$ with D diagonal.

Example 3



Example 3

$A + E$, E random matrix with $\|E\| < 10^{-10}$. Plot of spectra: blue. Spectrum of A: red.



Some definitions

Definition

Let $A \in \mathcal{B}(\mathcal{H})$. The numerical range of A is the set

$$W(A) = \{\langle u, Au\rangle \mid \|u\| = 1\}.$$

Theorem (Toeplitz-Hausdorff)

The numerical range $W(A)$ is always a convex set. If $\mathcal{H}$ is finite dimensional, then $W(A)$ is a compact set.

Proposition

Let $A \in \mathcal{B}(\mathcal{H})$. Then $\sigma(A) \subseteq \mathrm{cl}(W(A))$.

Some definitions

Definition

We define the following quantities.

$$\begin{aligned} \alpha(A) &= \sup\{\operatorname{Re} z \mid z \in \sigma(A)\}, \\ \alpha_\varepsilon(A) &= \sup\{\operatorname{Re} z \mid z \in \sigma_\varepsilon(A)\}, \\ \omega(A) &= \sup\{\operatorname{Re} z \mid z \in W(A)\}. \end{aligned}$$

We recall the relations

$$(A - zI)^{-1} = \int_0^\infty e^{-tz} e^{tA} dt \quad \text{for } \operatorname{Re} z \text{ sufficiently large.}$$

$$e^{tA} = \frac{-1}{2\pi i} \int_\gamma e^{tz} (A - zI)^{-1} dz$$

where γ is a simple closed contour enclosing $\sigma(A)$.

Estimates on $\|e^{tA}\|$

We have the following results from semigroup theory:

Theorem

$$\alpha(A) = \lim_{t\to\infty} \frac{1}{t} \log\|e^{tA}\|$$

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Theorem

$$\omega(A) = \frac{d}{dt}\,\|e^{tA}\|\Big|_{t=0} = \lim_{t\downarrow 0} \frac{1}{t} \log\|e^{tA}\|$$

Estimates on $\|e^{tA}\|$

Some simple estimates:

$$\|e^{tA}\| \geq e^{t\alpha(A)} \quad \text{for all } t \geq 0$$

$$\|e^{tA}\| \leq e^{t\omega(A)} \quad \text{for all } t \geq 0$$

An estimate based on pseudospectra:

Theorem

For all $\varepsilon > 0$ we have

$$\sup_{t \geq 0} \|e^{tA}\| \geq \frac{\alpha_\varepsilon(A)}{\varepsilon}$$

Estimates on $\|e^{tA}\|$

Define the Kreiss constant

$$\mathcal{K}(A) = \sup_{\varepsilon>0} \frac{\alpha_\varepsilon(A)}{\varepsilon}$$

Theorem (Kreiss Matrix Theorem)

If A is an $N \times N$ matrix, then we have

$$\|e^{tA}\| \leq eN\mathcal{K}(A).$$

Estimates on $\|e^{tA}\|$

Some further results

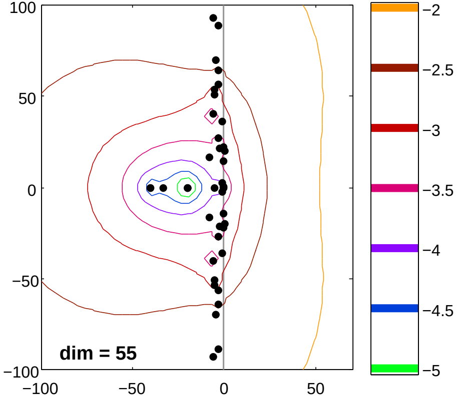
Theorem

Let $a = \operatorname{Re} z$. *Let* $K = \operatorname{Re} z \|(A - zI)^{-1}\|$. *Then for* $\tau > 0$ *we have*

$$\sup_{0 \le t \le \tau} \|e^{tA}\| \ge e^{a t} \left(1 + \frac{e^{at} - 1}{K}\right)^{-1}$$

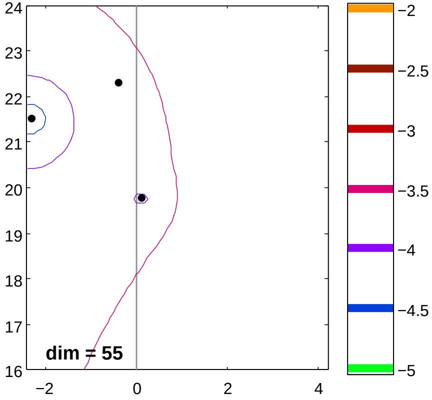
Some examples

Unstable Boing wing flutter matrix.



Some examples

Unstable Boing wing flutter matrix. Close-up near eigenvalue with Re $z > 0$



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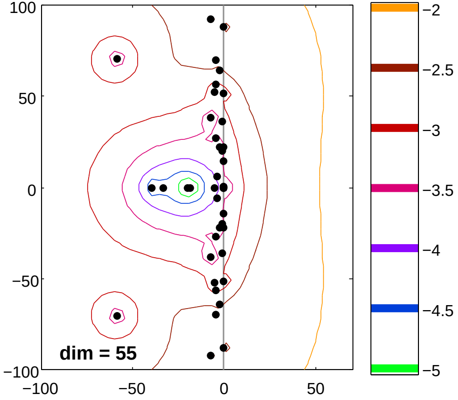

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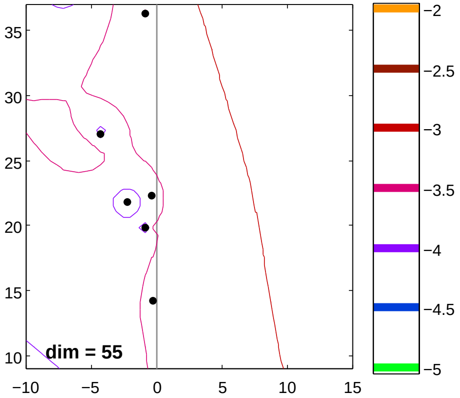
Stable Boing wing flutter matrix. Close-up near eigenvalue with $\mathrm{Re}\, z \sim 0$



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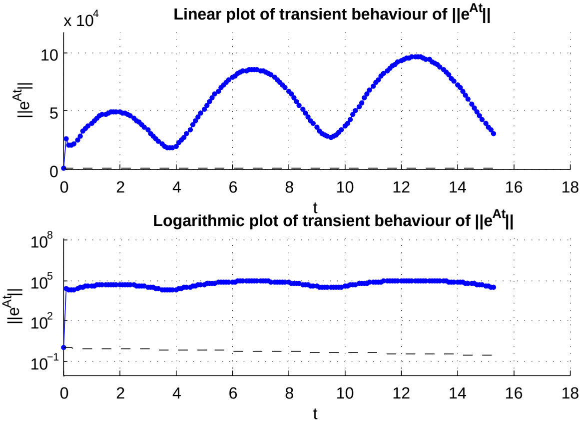
Stable Boing wing flutter matrix. Initial behavior.



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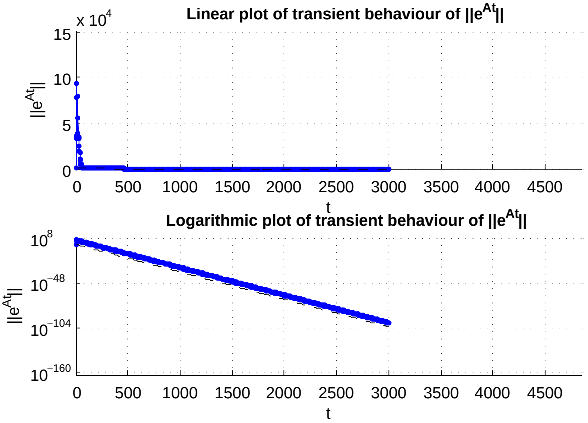
Stable Boing wing flutter matrix. Long time behavior.



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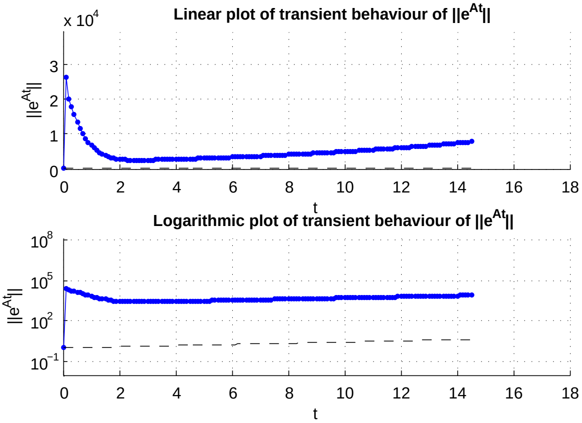

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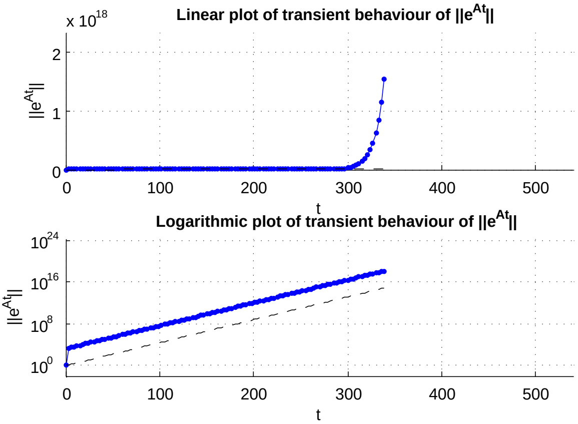
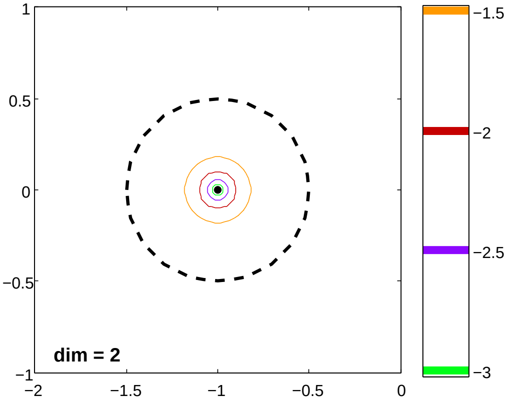


The first examples

Matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Boundary of numerical range included.



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