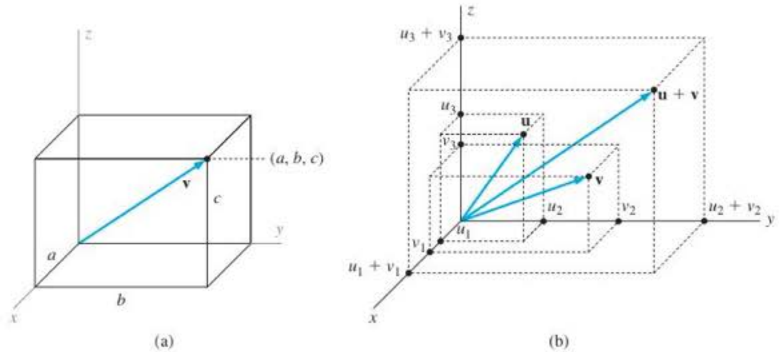

Figure 1.6 Vectors in $\mathcal{R}^3$

endpoint. (See Figure 1.6(a).) As is the case in $\mathcal{R}^2$, we can view two nonzero vectors in $\mathcal{R}^3$ as adjacent sides of a parallelogram, and we can represent their addition by using the parallelogram law. (See Figure 1.6(b).) In real life, motion takes place in 3-dimensional space, and we can depict quantities such as velocities and forces as vectors in $\mathcal{R}^3$.

EXERCISES

In Exercises 1–12, compute the indicated matrices, where

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 4 \end{bmatrix}.$$

1. $4A$
2. $-A$
3. $4A - 2B$
4. $3A + 2B$
5. $(2B)^T$
6. $A^T + 2B^T$
7. $A + B$
8. $(A + 2B)^T$
9. A^T
10. $A - B$
11. $-(B^T)$
12. $(-B)^T$

In Exercises 13–24, compute the indicated matrices, if possible, where

$$A = \begin{bmatrix} 3 & -1 & 2 & 4 \\ 1 & 5 & -6 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 0 \\ 2 & 5 \\ -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

13. $-A$
14. $3B$
15. $(-2)A$
16. $(2B)^T$
17. $A - B$
18. $A - B^T$
19. $A^T - B$
20. $3A + 2B^T$
21. $(A + B)^T$
22. $(4A)^T$
23. $B - A^T$
24. $(B^T - A)^T$

In Exercises 25–28, assume that $A = \begin{bmatrix} 3 & -2 \\ 0 & 1.6 \\ 2\pi & 5 \end{bmatrix}$.

25. Determine a_{12}.
26. Determine a_{21}.
27. Determine $\mathbf{a}_1$.
28. Determine $\mathbf{a}_2$.

In Exercises 29–32, assume that $C = \begin{bmatrix} 2 & -3 & 0.4 \\ 2e & 12 & 0 \end{bmatrix}$.

29. Determine $\mathbf{c}_1$.
30. Determine $\mathbf{c}_3$.
31. Determine the first row of C.
32. Determine the second row of C.

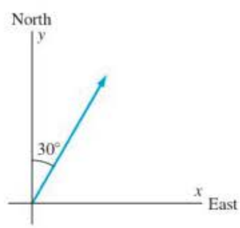

Figure 1.7 A view of the airplane from above

33. An airplane is flying with a ground speed of 300 mph at an angle of 30° east of due north. (See Figure 1.7.) In addition, the airplane is climbing at a rate of 10 mph. Determine the vector in $\mathcal{R}^3$ that represents the velocity (in mph) of the airplane.
34. A swimmer is swimming northeast at 2 mph in still water.
    (a) Give the velocity of the swimmer. Include a sketch.
    (b) A current in a northerly direction at 1 mph affects the velocity of the swimmer. Give the new velocity and speed of the swimmer. Include a sketch.
35. A pilot keeps her airplane pointed in a northeastward direction while maintaining an airspeed (speed relative to the surrounding air) of 300 mph. A wind from the west blows eastward at 50 mph.

(a) Find the velocity (in mph) of the airplane relative to the ground.

(b) What is the speed (in mph) of the airplane relative to the ground?

36. Suppose that in a medical study of 20 people, for each i, $1 \leq i \leq 20$, the 3×1 vector $\mathbf{u}_i$ is defined so that its components respectively represent the blood pressure, pulse rate, and cholesterol reading of the ith person. Provide an interpretation of the vector $\frac{1}{20}(\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_{20})$.

T&F *In Exercises 37–56, determine whether the statements are true or false.*

37. Matrices must be of the same size for their sum to be defined.
38. The transpose of a sum of two matrices is the sum of the transposed matrices.
39. Every vector is a matrix.
40. A scalar multiple of the zero matrix is the zero scalar.
41. The transpose of a matrix is a matrix of the same size.
42. A submatrix of a matrix may be a vector.
43. If B is a 3×4 matrix, then its rows are 4×1 vectors.
44. The $(3, 4)$-entry of a matrix lies in column 3 and row 4.
45. In a zero matrix, every entry is 0.
46. An $m \times n$ matrix has $m + n$ entries.
47. If $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v} = -3\mathbf{w}$, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.
48. If A and B are any $m \times n$ matrices, then
$$A - B = A + (-1)B.$$
49. The (i, j)-entry of A^T equals the (j, i)-entry of A.
50. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, then $A = B$.
51. In any matrix A, the sum of the entries of $3A$ equals three times the sum of the entries of A.
52. Matrix addition is commutative.
53. Matrix addition is associative.
54. For any $m \times n$ matrices A and B and any scalars c and d, $(cA + dB)^T = cA^T + dB^T$.
55. If A is a matrix, then cA is the same size as A for every scalar c.
56. If A is a matrix for which the sum $A + A^T$ is defined, then A is a square matrix.

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57. Let A and B be matrices of the same size.
    (a) Prove that the jth column of $A + B$ is $\mathbf{a}_j + \mathbf{b}_j$.
    (b) Prove that for any scalar c, the jth column of cA is $c\mathbf{a}_j$.
58. For any $m \times n$ matrix A, prove that $0A = O$, the $m \times n$ zero matrix.
59. For any $m \times n$ matrix A, prove that $1A = A$.
60. Prove Theorem 1.1(a).
61. Prove Theorem 1.1(c).
62. Prove Theorem 1.1(d).
63. Prove Theorem 1.1(e).
64. Prove Theorem 1.1(g).
65. Prove Theorem 1.2(b).
66. Prove Theorem 1.2(c).

A square matrix A is called a **diagonal matrix** *if $a_{ij} = 0$ whenever $i \neq j$. Exercises 67–70 are concerned with diagonal matrices.*

67. Prove that a square zero matrix is a diagonal matrix.
68. Prove that if B is a diagonal matrix, then cB is a diagonal matrix for any scalar c.
69. Prove that if B is a diagonal matrix, then B^T is a diagonal matrix.
70. Prove that if B and C are diagonal matrices of the same size, then $B + C$ is a diagonal matrix.

A (square) matrix A is said to be **symmetric** *if $A = A^T$. Exercises 71–78 are concerned with symmetric matrices.*

71. Give examples of 2×2 and 3×3 symmetric matrices.
72. Prove that the (i, j)-entry of a symmetric matrix equals the (j, i)-entry.
73. Prove that a square zero matrix is symmetric.
74. Prove that if B is a symmetric matrix, then so is cB for any scalar c.
75. Prove that if B is a square matrix, then $B + B^T$ is symmetric.
76. Prove that if B and C are $n \times n$ symmetric matrices, then so is $B + C$.
77. Is a square submatrix of a symmetric matrix necessarily a symmetric matrix? Justify your answer.
78. Prove that a diagonal matrix is symmetric.

A (square) matrix A is called **skew-symmetric** *if $A^T = -A$. Exercises 79–81 are concerned with skew-symmetric matrices.*

79. What must be true about the (i, i)-entries of a skew-symmetric matrix? Justify your answer.
80. Give an example of a nonzero 2×2 skew-symmetric matrix B. Now show that every 2×2 skew-symmetric matrix is a scalar multiple of B.
81. Show that every 3×3 matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
82.[4] The **trace** of an $n \times n$ matrix A, written trace(A), is defined to be the sum
$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$
Prove that, for any $n \times n$ matrices A and B and scalar c, the following statements are true:
    (a) $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$.
    (b) $\text{trace}(cA) = c \cdot \text{trace}(A)$.
    (c) $\text{trace}(A^T) = \text{trace}(A)$.
83. *Probability vectors* are vectors whose components are nonnegative and have a sum of 1. Show that if $\mathbf{p}$ and $\mathbf{q}$ are probability vectors and a and b are nonnegative scalars with $a + b = 1$, then $a\mathbf{p} + b\mathbf{q}$ is a probability vector.

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[4] This exercise is used in Sections 2.2, 7.1, and 7.5 (on pages 115, 495, and 533, respectively).

In the following exercise, use either a calculator with matrix capabilities or computer software such as MATLAB to solve the problem:

84. Consider the matrices

$$A = \begin{bmatrix} 1.3 & 2.1 & -3.3 & 6.0 \\ 5.2 & 2.3 & -1.1 & 3.4 \\ 3.2 & -2.6 & 1.1 & -4.0 \\ 0.8 & -1.3 & -12.1 & 5.7 \\ -1.4 & 3.2 & 0.7 & 4.4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2.6 & -1.3 & 0.7 & -4.4 \\ 2.2 & -2.6 & 1.3 & -3.2 \\ 7.1 & 1.5 & -8.3 & 4.6 \\ -0.9 & -1.2 & 2.4 & 5.9 \\ 3.3 & -0.9 & 1.4 & 6.2 \end{bmatrix}.$$

(a) Compute $A + 2B$.

(b) Compute $A - B$.

(c) Compute $A^T + B^T$.

SOLUTIONS TO THE PRACTICE PROBLEMS

1. (a) The $(1, 2)$-entry of A is 2.

   (b) The $(2, 2)$-entry of A is 3.

2. (a) $$A - B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \end{bmatrix}$$

   (b) $$2A = 2\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 6 & 0 & -4 \end{bmatrix}$$

   (c) $$A + 3B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$$
   $$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 0 \\ 6 & -3 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 1 \\ 9 & -3 & 10 \end{bmatrix}$$

3. (a) $$A^T = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

   (b) $$(3B)^T = \begin{bmatrix} 3 & 9 & 0 \\ 6 & -3 & 12 \end{bmatrix}^T = \begin{bmatrix} 3 & 6 \\ 9 & -3 \\ 0 & 12 \end{bmatrix}$$

   (c) $$(A + B)^T = \begin{bmatrix} 3 & 2 & 1 \\ 5 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$$

1.2 LINEAR COMBINATIONS, MATRIX–VECTOR PRODUCTS, AND SPECIAL MATRICES

In this section, we explore some applications involving matrix operations and introduce the product of a matrix and a vector.

Suppose that 20 students are enrolled in a linear algebra course, in which two tests, a quiz, and a final exam are given. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{20} \end{bmatrix}$, where u_i denotes the score of the ith student on the first test. Likewise, define vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ similarly for the second test, quiz, and final exam, respectively. Assume that the instructor computes a student's course average by counting each test score twice as much as a quiz score, and the final exam score three times as much as a test score. Thus the *weights* for the tests, quiz, and final exam score are, respectively, 2/11, 2/11, 1/11, 6/11 (the weights must sum to one). Now consider the vector

$$\mathbf{y} = \frac{2}{11}\mathbf{u} + \frac{2}{11}\mathbf{v} + \frac{1}{11}\mathbf{w} + \frac{6}{11}\mathbf{z}.$$

The first component y_1 represents the first student's course average, the second component y_2 represents the second student's course average, and so on. Notice that $\mathbf{y}$ is a sum of scalar multiples of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$. This form of vector sum is so important that it merits its own definition.

EXERCISES

In Exercises 1–8, two vectors **u** *and* **v** *are given. Compute the norms of the vectors and the distance d between them.*

1. $\mathbf{u} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

5. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

6. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

7. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

8. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

In Exercises 9–16, two vectors are given. Compute the dot product of the vectors, and determine whether the vectors are orthogonal.

9. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

10. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

11. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

13. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

14. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

15. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

16. $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

In Exercises 17–24, two orthogonal vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|^2$, $\|\mathbf{v}\|^2$, *and* $\|\mathbf{u}+\mathbf{v}\|^2$. *Use your results to illustrate the Pythagorean theorem.*

17. $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

18. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

19. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

20. $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$

21. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

23. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix}$

24. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$

In Exercises 25–32, two vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, *and* $\|\mathbf{u}+\mathbf{v}\|$. *Use your results to illustrate the triangle inequality.*

25. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$

26. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

27. $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

28. $\mathbf{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

29. $\mathbf{u} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

30. $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

31. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

32. $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$

In Exercises 33–40, two vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, *and* $\mathbf{u} \cdot \mathbf{v}$. *Use your results to illustrate the Cauchy–Schwarz inequality.*

33. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

34. $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

35. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

36. $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

37. $\mathbf{u} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

38. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

39. $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

40. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

In Exercises 41–48, a vector **u** *and a line* $\mathcal{L}$ *in* $\mathcal{R}^2$ *are given. Compute the orthogonal projection* **w** *of* **u** *on* $\mathcal{L}$, *and use it to compute the distance d from the endpoint of* **u** *to* $\mathcal{L}$.

41. $\mathbf{u} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $y = 0$

42. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $y = 2x$

43. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = -x$

44. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = -2x$

45. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $y = 3x$

46. $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $y = x$

47. $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $y = -3x$

48. $\mathbf{u} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $y = -4x$

For Exercises 49–54, suppose that **u**, **v**, *and* **w** *are vectors in* $\mathcal{R}^n$ *such that* $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 5$, $\mathbf{u} \cdot \mathbf{v} = -1$, $\mathbf{u} \cdot \mathbf{w} = 1$, *and* $\mathbf{v} \cdot \mathbf{w} = -4$.

49. Compute $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$.
50. Compute $\|4\mathbf{w}\|$.
51. Compute $\|\mathbf{u} + \mathbf{v}\|^2$.
52. Compute $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v}$.
53. Compute $\|\mathbf{v} - 4\mathbf{w}\|^2$.
54. Compute $\|2\mathbf{u} + 3\mathbf{v}\|^2$.

For Exercises 55–60, suppose that **u**, **v**, *and* **w** *are vectors in* $\mathcal{R}^n$ *such that* $\mathbf{u} \cdot \mathbf{u} = 14$, $\mathbf{u} \cdot \mathbf{v} = 7$, $\mathbf{u} \cdot \mathbf{w} = -20$, $\mathbf{v} \cdot \mathbf{v} = 21$, $\mathbf{v} \cdot \mathbf{w} = -5$, *and* $\mathbf{w} \cdot \mathbf{w} = 30$.

55. Compute $\|\mathbf{v}\|^2$.
56. Compute $\|3\mathbf{u}\|$.
57. Compute $\mathbf{v} \cdot \mathbf{u}$.
58. Compute $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v})$.
59. Compute $\|2\mathbf{u} - \mathbf{v}\|^2$.
60. Compute $\|\mathbf{v} + 3\mathbf{w}\|$.

T&F *In Exercises 61–80, determine whether the statements are true or false.*

61. Vectors must be of the same size for their dot product to be defined.
62. The dot product of two vectors in $\mathcal{R}^n$ is a vector in $\mathcal{R}^n$.
63. The norm of a vector equals the dot product of the vector with itself.
64. The norm of a multiple of a vector is the same multiple of the norm of the vector.
65. The norm of a sum of vectors is the sum of the norms of the vectors.
66. The squared norm of a sum of orthogonal vectors is the sum of the squared norms of the vectors.
67. The orthogonal projection of a vector on a line is a vector that lies along the line.
68. The norm of a vector is always a nonnegative real number.
69. If the norm of **v** equals 0, then $\mathbf{v} = \mathbf{0}$.
70. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
71. For all vectors **u** and **v** in $\mathcal{R}^n$, $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \cdot \|\mathbf{v}\|$.
72. For all vectors **u** and **v** in $\mathcal{R}^n$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
73. The distance between vectors **u** and **v** in $\mathcal{R}^n$ is $\|\mathbf{u} - \mathbf{v}\|$.
74. For all vectors **u** and **v** in $\mathcal{R}^n$ and every scalar c,
$$(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}).$$
75. For all vectors **u**, **v**, and **w** in $\mathcal{R}^n$,
$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$$
76. If A is an $n \times n$ matrix and **u** and **v** are vectors in $\mathcal{R}^n$, then $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A\mathbf{v}$.
77. For every vector **v** in $\mathcal{R}^n$, $\|\mathbf{v}\| = \|-\mathbf{v}\|$.
78. If **u** and **v** are orthogonal vectors in $\mathcal{R}^n$, then
$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|.$$
79. If **w** is the orthogonal projection of **u** on a line through the origin of $\mathcal{R}^2$, then $\mathbf{u} - \mathbf{w}$ is orthogonal to every vector on the line.
80. If **w** is the orthogonal projection of **u** on a line through the origin of $\mathcal{R}^2$, then **w** is the vector on the line closest to **u**.

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81. Prove (a) of Theorem 6.1.
82. Prove (b) of Theorem 6.1.
83. Prove (c) of Theorem 6.1.
84. Prove (e) of Theorem 6.1.
85. Prove (f) of Theorem 6.1.
86. Prove that if **u** is orthogonal to both **v** and **w**, then **u** is orthogonal to every linear combination of **v** and **w**.
87. Let $\{\mathbf{v}, \mathbf{w}\}$ be a basis for a subspace W of $\mathcal{R}^n$, and define
$$\mathbf{z} = \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$
Prove that $\{\mathbf{v}, \mathbf{z}\}$ is a basis for W consisting of orthogonal vectors.
88. Prove that the Cauchy–Schwarz inequality is an equality if and only if **u** is a multiple of **v** or **v** is a multiple of **u**.

89. Prove that the triangle inequality is an equality if and only if $\mathbf{u}$ is a nonnegative multiple of $\mathbf{v}$ or $\mathbf{v}$ is a nonnegative multiple of $\mathbf{u}$.

90. Use the triangle inequality to prove that $|\,\|\mathbf{v}\| - \|\mathbf{w}\|\,| \leq \|\mathbf{v} - \mathbf{w}\|$ for all vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{R}^n$.

91. Prove $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^n$.

92. Let $\mathbf{z}$ be a vector in $\mathcal{R}^n$. Let $W = \{\mathbf{u} \in \mathcal{R}^n : \mathbf{u} \cdot \mathbf{z} = 0\}$. Prove that W is a subspace of $\mathcal{R}^n$.

93. Let $\mathcal{S}$ be a subset of $\mathcal{R}^n$ and

$$W = \{\mathbf{u} \in \mathcal{R}^n : \mathbf{u} \cdot \mathbf{z} = 0 \text{ for all } \mathbf{z} \text{ in } \mathcal{S}\}.$$

Prove that W is a subspace of $\mathcal{R}^n$.

94. Let W denote the set of all vectors that lie along the line with equation $y = 2x$. Find a vector $\mathbf{z}$ in $\mathcal{R}^2$ such that $W = \{\mathbf{u} \in \mathcal{R}^2 : \mathbf{u} \cdot \mathbf{z} = 0\}$. Justify your answer.

95. Prove the *parallelogram law* for vectors in $\mathcal{R}^n$:

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

96. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal nonzero vectors in $\mathcal{R}^n$, then they are linearly independent.

97.[2] Let A be any $m \times n$ matrix.

(a) Prove that $A^T A$ and A have the same null space. *Hint:* Let $\mathbf{v}$ be a vector in $\mathcal{R}^n$ such that $A^T A\mathbf{v} = \mathbf{0}$. Observe that $A^T A\mathbf{v} \cdot \mathbf{v} = A\mathbf{v} \cdot A\mathbf{v} = 0$.

(b) Use (a) to prove that $\operatorname{rank} A^T A = \operatorname{rank} A$.

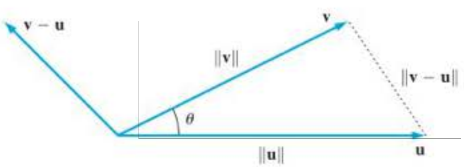

Figure 6.6

98.[3] Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathcal{R}^2$ or $\mathcal{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$ determine a triangle. (See Figure 6.6.) The relationship between the lengths of the sides of this triangle and θ is called the *law of cosines*. It states that

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta.$$

Use the law of cosines and Theorem 6.1 to derive the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta.$$

In Exercises 99–106, use the formula in Exercise 98 to determine the angle between the vectors $\mathbf{u}$ *and* $\mathbf{v}$.

99. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

100. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

101. $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

102. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

103. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

104. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

105. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

106. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathcal{R}^3$. *Define* $\mathbf{u} \times \mathbf{v}$ *to be the vector* $\begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$, *which is called the* **cross product** *of* $\mathbf{u}$ *and* $\mathbf{v}$.

For Exercises 107–120, use the preceding definition of the cross product.

107. For every vector $\mathbf{u}$ in $\mathcal{R}^3$, prove that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

108. Prove that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$.

109. For every vector $\mathbf{u}$ in $\mathcal{R}^3$, prove that $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

110. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

111. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$ and all scalars c, prove that

$$c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}.$$

112. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.$$

113. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

114. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

115. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

116. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

[2] This exercise is used in Section 6.7 (on page 439).

[3] This exercise is used in Section 6.9 (on page 471).

117. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}.$$

118. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

119. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. *Hint:* Use Exercises 98 and 118.

120. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove the *Jacobi identity*:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$$

Exercises 121–124 refer to the application regarding the two methods of computing average class size given in this section. In Exercises 121–123, data are given for students enrolled in a three-section seminar course. Compute the average $\overline{v}$ determined by the supervisor and the average v^* determined by the investigator.

121. Section 1 contains 8 students, section 2 contains 12 students, and section 3 contains 6 students.

122. Section 1 contains 15 students, and each of sections 2 and 3 contains 30 students.

123. Each of the three sections contains 22 students.

124. Use Exercise 88 to prove that the two averaging methods for determining class size are equal if and only if all of the class sizes are equal.

In Exercise 125, use either a calculator with matrix capabilities or computer software such as MATLAB to solve the problem.

125. In every triangle, the length of any side is less than the sum of the lengths of the other two sides. When this observation is generalized to $\mathcal{R}^n$, we obtain the *triangle inequality* (Theorem 6.4), which states

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -8 \\ -6 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2.01 \\ 4.01 \\ 6.01 \\ 8.01 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{v}_2 = \begin{bmatrix} 3.01 \\ 6.01 \\ 9.01 \\ 12.01 \end{bmatrix}.$$

(a) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}$.
(b) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}_1$.
(c) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}_2$.
(d) From what you have observed in (b) and (c), make a conjecture about when equality occurs in the triangle inequality.
(e) Interpret your conjecture in (d) geometrically in $\mathcal{R}^2$.

SOLUTIONS TO THE PRACTICE PROBLEMS

1. (a) We have $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 2^2 + 3^2} = 7$.

(b) We have $\|\mathbf{u} - \mathbf{v}\| = \left\| \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} \right\|$

$$= \sqrt{(-5)^2 + (-4)^2 + (-1)^2} = \sqrt{42}.$$

(c) We have

$$\left\| \frac{1}{\|\mathbf{u}\|}\mathbf{u} \right\| = \left\| \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$

and

$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left\| \frac{1}{7}\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{6}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix} \right\| = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1.$$

2. Taking dot products, we obtain

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (-2)(1) + (-5)(-1) + (3)(2) = 9 \\ \mathbf{u} \cdot \mathbf{w} &= (-2)(-3) + (-5)(1) + (3)(2) = 7 \\ \mathbf{v} \cdot \mathbf{w} &= (1)(-3) + (-1)(1) + (2)(2) = 0. \end{aligned}$$

So $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, but $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal, and $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal.

3. Let $\mathbf{w}$ be the required orthogonal projection. Then

$$\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{(-2)(1) + (-5)(-1) + (3)(2)}{1^2 + (-1)^2 + 2^2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

6.2 ORTHOGONAL VECTORS

It is easy to extend the property of orthogonality to any set of vectors. We say that a subset of $\mathcal{R}^n$ is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal. The subset is called an **orthonormal set** if it is an orthogonal set consisting entirely of unit vectors.