CHAPTER 1

MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

The most common use of linear algebra is to solve systems of linear equations, which arise in applications to such diverse disciplines as physics, biology, economics, engineering, and sociology. In this chapter, we describe the most efficient algorithm for solving systems of linear equations, *Gaussian elimination.* This algorithm, or some variation of it, is used by most mathematics software (such as MATLAB).

We can write systems of linear equations compactly, using arrays called *matrices* and *vectors.* More importantly, the arithmetic properties of these arrays enable us to compute solutions of such systems or to determine if no solutions exist. This chapter begins by developing the basic properties of matrices and vectors. In Sections 1.3 and 1.4, we begin our study of systems of linear equations. In Sections 1.6 and 1.7, we introduce two other important concepts of vectors, namely, generating sets and linear independence, which provide information about the existence and uniqueness of solutions of a system of linear equations.

1.1 MATRICES AND VECTORS

Many types of numerical data are best displayed in two-dimensional arrays, such as tables.

For example, suppose that a company owns two bookstores, each of which sells newspapers, magazines, and books. Assume that the sales (in hundreds of dollars) of the two bookstores for the months of July and August are represented by the following tables:

	July	
Store	1	2
Newspapers	6	8
Magazines	15	20
Books	45	64

and

	August	
Store	1	2
Newspapers	7	9
Magazines	18	31
Books	52	68

The first column of the July table shows that store 1 sold \$1500 worth of magazines and \$4500 worth of books during July. We can represent the information on July sales more simply as

$$\begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix}.$$

Such a rectangular array of real numbers is called a *matrix*.[1] It is customary to refer to real numbers as **scalars** (originally from the word *scale*) when working with a matrix. We denote the set of real numbers by $\mathcal{R}$.

Definitions A **matrix** (*plural, matrices*) is a rectangular array of scalars. If the matrix has m rows and n columns, we say that the **size** of the matrix is m **by** n, written $m \times n$. The matrix is **square** if $m = n$. The scalar in the ith row and jth column is called the (i, j)**-entry** of the matrix.

If A is a matrix, we denote its (i,j)-entry by a_{ij}. We say that two matrices A and B are **equal** if they have the same size and have equal corresponding entries; that is, $a_{ij} = b_{ij}$ for all i and j. Symbolically, we write $A = B$.

In our bookstore example, the July and August sales are contained in the matrices

$$B = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 7 & 9 \\ 18 & 31 \\ 52 & 68 \end{bmatrix}.$$

Note that $b_{12} = 8$ and $c_{12} = 9$, so $B \neq C$. Both B and C are 3×2 matrices. Because of the context in which these matrices arise, they are called *inventory matrices*.

Other examples of matrices are

$$\begin{bmatrix} \frac{2}{3} & -4 & 0 \\ \pi & 1 & 6 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -2 & 0 & 1 & 1 \end{bmatrix}.$$

The first matrix has size 2×3, the second has size 3×1, and the third has size 1×4.

Practice Problem 1 ▶ Let $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.

(a) What is the $(1, 2)$-entry of A?

(b) What is a_{22}? ◀

Sometimes we are interested in only a part of the information contained in a matrix. For example, suppose that we are interested in only magazine and book sales in July. Then the relevant information is contained in the last two rows of B; that is, in the matrix E defined by

$$E = \begin{bmatrix} 15 & 20 \\ 45 & 64 \end{bmatrix}.$$

E is called a *submatrix* of B. In general, a **submatrix** of a matrix M is obtained by deleting from M entire rows, entire columns, or both. It is permissible, when forming a submatrix of M, to delete none of the rows or none of the columns of M. As another example, if we delete the first row and the second column of B, we obtain the submatrix

$$\begin{bmatrix} 15 \\ 45 \end{bmatrix}.$$

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[1] James Joseph Sylvester (1814–1897) coined the term *matrix* in the 1850s.

MATRIX SUMS AND SCALAR MULTIPLICATION

Matrices are more than convenient devices for storing information. Their usefulness lies in their *arithmetic*. As an example, suppose that we want to know the total numbers of newspapers, magazines, and books sold by both stores during July and August. It is natural to form one matrix whose entries are the sum of the corresponding entries of the matrices B and C, namely,

$$\begin{array}{r c} \text{Store} & \begin{array}{cc} 1 & \;\;2 \end{array} \\ \begin{array}{r} \text{Newspapers} \\ \text{Magazines} \\ \text{Books} \end{array} & \begin{bmatrix} 13 & 17 \\ 33 & 51 \\ 97 & 132 \end{bmatrix} \end{array}.$$

If A and B are $m \times n$ matrices, the **sum** of A and B, denoted by $A + B$, is the $m \times n$ matrix obtained by adding the corresponding entries of A and B; that is, $A + B$ is the $m \times n$ matrix whose (i,j)-entry is $a_{ij} + b_{ij}$. Notice that the matrices A and B must have the same size for their sum to be defined.

Suppose that in our bookstore example, July sales were to double in all categories. Then the new matrix of July sales would be

$$\begin{bmatrix} 12 & 16 \\ 30 & 40 \\ 90 & 128 \end{bmatrix}.$$

We denote this matrix by $2B$.

Let A be an $m \times n$ matrix and c be a scalar. The **scalar multiple** cA is the $m \times n$ matrix whose entries are c times the corresponding entries of A; that is, cA is the $m \times n$ matrix whose (i,j)-entry is ca_{ij}. Note that $1A = A$. We denote the matrix $(-1)A$ by $-A$ and the matrix $0A$ by O. We call the $m \times n$ matrix O in which each entry is 0 the $m \times n$ **zero matrix**.

Example 1 Compute the matrices $A + B$, $3A$, $-A$, and $3A + 4B$, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix}.$$

Solution We have

$$A + B = \begin{bmatrix} -1 & 5 & 2 \\ 7 & -9 & 1 \end{bmatrix}, \quad 3A = \begin{bmatrix} 9 & 12 & 6 \\ 6 & -9 & 0 \end{bmatrix}, \quad -A = \begin{bmatrix} -3 & -4 & -2 \\ -2 & 3 & 0 \end{bmatrix},$$

and

$$3A + 4B = \begin{bmatrix} 9 & 12 & 6 \\ 6 & -9 & 0 \end{bmatrix} + \begin{bmatrix} -16 & 4 & 0 \\ 20 & -24 & 4 \end{bmatrix} = \begin{bmatrix} -7 & 16 & 6 \\ 26 & -33 & 4 \end{bmatrix}.$$

Just as we have defined addition of matrices, we can also define **subtraction**. For any matrices A and B of the same size, we define $A - B$ to be the matrix obtained by subtracting each entry of B from the corresponding entry of A. Thus the (i,j)-entry of $A - B$ is $a_{ij} - b_{ij}$. Notice that $A - A = O$ for all matrices A.

If, as in Example 1, we have

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix}, \quad \text{and} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$-B = \begin{bmatrix} 4 & -1 & 0 \\ -5 & 6 & -1 \end{bmatrix}, \quad A - B = \begin{bmatrix} 7 & 3 & 2 \\ -3 & 3 & -1 \end{bmatrix}, \quad \text{and} \quad A - O = \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix}.$$

Practice Problem 2 ▶ Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$. Compute the following matrices:

(a) $A - B$
(b) $2A$
(c) $A + 3B$ ◀

We have now defined the operations of matrix addition and scalar multiplication. The power of linear algebra lies in the natural relations between these operations, which are described in our first theorem.

THEOREM 1.1

(Properties of Matrix Addition and Scalar Multiplication) Let A, B, and C be $m \times n$ matrices, and let s and t be any scalars. Then

(a) $A + B = B + A$. (commutative law of matrix addition)
(b) $(A + B) + C = A + (B + C)$. (associative law of matrix addition)
(c) $A + O = A$.
(d) $A + (-A) = O$.
(e) $(st)A = s(tA)$.
(f) $s(A + B) = sA + sB$.
(g) $(s + t)A = sA + tA$.

PROOF We prove parts (b) and (f). The rest are left as exercises.

(b) The matrices on each side of the equation are $m \times n$ matrices. We must show that each entry of $(A + B) + C$ is the same as the corresponding entry of $A + (B + C)$. Consider the (i,j)-entries. Because of the definition of matrix addition, the (i,j)-entry of $(A + B) + C$ is the sum of the (i,j)-entry of $A + B$, which is $a_{ij} + b_{ij}$, and the (i,j)-entry of C, which is c_{ij}. Therefore this sum equals $(a_{ij} + b_{ij}) + c_{ij}$. Similarly, the (i,j)-entry of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$. Because the associative law holds for addition of scalars, $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$. Therefore the (i,j)-entry of $(A + B) + C$ equals the (i,j)-entry of $A + (B + C)$, proving (b).

(f) The matrices on each side of the equation are $m \times n$ matrices. As in the proof of (b), we consider the (i,j)-entries of each matrix. The (i,j)-entry of $s(A + B)$ is defined to be the product of s and the (i,j)-entry of $A + B$, which is $a_{ij} + b_{ij}$. This product equals $s(a_{ij} + b_{ij})$. The (i,j)-entry of $sA + sB$ is the sum of the (i,j)-entry of sA, which is sa_{ij}, and the (i,j)-entry of sB, which is sb_{ij}. This sum is $sa_{ij} + sb_{ij}$. Since $s(a_{ij} + b_{ij}) = sa_{ij} + sb_{ij}$, (f) is proved. ■

Because of the associative law of matrix addition, sums of three or more matrices can be written unambiguously without parentheses. Thus we may write $A + B + C$ instead of either $(A + B) + C$ or $A + (B + C)$.

MATRIX TRANSPOSES

In the bookstore example, we could have recorded the information about July sales in the following form:

Store	Newspapers	Magazines	Books
1	6	15	45
2	8	20	64

This representation produces the matrix

$$\begin{bmatrix} 6 & 15 & 45 \\ 8 & 20 & 64 \end{bmatrix}.$$

Compare this with

$$B = \begin{bmatrix} 6 & 8 \\ 15 & 20 \\ 45 & 64 \end{bmatrix}.$$

The rows of the first matrix are the columns of B, and the columns of the first matrix are the rows of B. This new matrix is called the *transpose* of B. In general, the **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix denoted by A^T whose (i,j)-entry is the (j,i)-entry of A.

The matrix C in our bookstore example and its transpose are

$$C = \begin{bmatrix} 7 & 9 \\ 18 & 31 \\ 52 & 68 \end{bmatrix} \quad \text{and} \quad C^T = \begin{bmatrix} 7 & 18 & 52 \\ 9 & 31 & 68 \end{bmatrix}.$$

Practice Problem 3 ▶ Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$. Compute the following matrices:

(a) A^T

(b) $(3B)^T$

(c) $(A+B)^T$ ◀

The following theorem shows that the transpose preserves the operations of matrix addition and scalar multiplication:

THEOREM 1.2

(Properties of the Transpose) Let A and B be $m \times n$ matrices, and let s be any scalar. Then

(a) $(A+B)^T = A^T + B^T$.

(b) $(sA)^T = sA^T$.

(c) $(A^T)^T = A$.

PROOF We prove part (a). The rest are left as exercises.

(a) The matrices on each side of the equation are $n \times m$ matrices. So we show that the (i,j)-entry of $(A+B)^T$ equals the (i,j)-entry of $A^T + B^T$. By the definition of transpose, the (i,j)-entry of $(A+B)^T$ equals the (j,i)-entry of $A+B$, which is $a_{ji} + b_{ji}$. On the other hand, the (i,j)-entry of $A^T + B^T$ equals the sum of the (i,j)-entry of A^T and the (i,j)-entry of B^T, that is, $a_{ji} + b_{ji}$. Because the (i,j)-entries of $(A+B)^T$ and $A^T + B^T$ are equal, (a) is proved. ■

VECTORS

A matrix that has exactly one row is called a **row vector**, and a matrix that has exactly one column is called a **column vector**. The term *vector* is used to refer to either a row vector or a column vector. The entries of a vector are called **components**. In this book, we normally work with column vectors, and we denote the set of all column vectors with n components by $\mathcal{R}^n$.

We write vectors as boldface lower case letters such as $\mathbf{u}$ and $\mathbf{v}$, and denote the ith component of the vector $\mathbf{u}$ by u_i. For example, if $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$, then $u_2 = -4$. Occasionally, we identify a vector $\mathbf{u}$ in $\mathcal{R}^n$ with an n-tuple, $(u_1, u_2, \ldots, u_n)$.

Because vectors are special types of matrices, we can add them and multiply them by scalars. In this context, we call the two arithmetic operations on vectors **vector addition** and **scalar multiplication**. These operations satisfy the properties listed in Theorem 1.1. In particular, the vector in $\mathcal{R}^n$ with all zero components is denoted by $\mathbf{0}$ and is called the **zero vector**. It satisfies $\mathbf{u} + \mathbf{0} = \mathbf{u}$ and $0\mathbf{u} = \mathbf{0}$ for every $\mathbf{u}$ in $\mathcal{R}^n$.

Example 2

Let $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 7 \\ -1 \\ 7 \end{bmatrix}, \quad \mathbf{u} - \mathbf{v} = \begin{bmatrix} -3 \\ -7 \\ 7 \end{bmatrix}, \quad \text{and} \quad 5\mathbf{v} = \begin{bmatrix} 25 \\ 15 \\ 0 \end{bmatrix}.$$

For a given matrix, it is often advantageous to consider its rows and columns as vectors. For example, for the matrix $\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & -2 \end{bmatrix}$, the **rows** are $\begin{bmatrix} 2 & 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}$, and the **columns** are $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Because the columns of a matrix play a more important role than the rows, we introduce a special notation. When a capital letter denotes a matrix, we use the corresponding lower case letter in boldface with a subscript j to represent the jth column of that matrix. So if A is an $m \times n$ matrix, its jth column is

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

GEOMETRY OF VECTORS

For many applications,[2] it is useful to represent vectors geometrically as directed line segments, or arrows. For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a vector in $\mathcal{R}^2$, we can represent $\mathbf{v}$ as an arrow from the origin to the point (a, b) in the xy-plane, as shown in Figure 1.1.

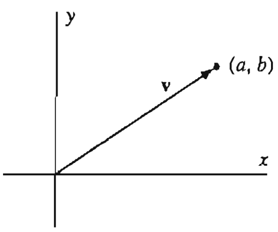

Figure 1.1 A vector in $\mathcal{R}^2$

[2] The importance of vectors in physics was recognized late in the nineteenth century. The algebra of vectors, developed by Oliver Heaviside (1850–1925) and Josiah Willard Gibbs (1839–1903), won out over the algebra of quaternions to become the language of physicists.

Example 3

Velocity Vectors A boat cruises in still water toward the northeast at 20 miles per hour. The velocity $\mathbf{u}$ of the boat is a vector that points in the direction of the boat's motion, and whose length is 20, the boat's speed. If the positive y-axis represents north and the positive x-axis represents east, the boat's direction makes an angle of $45°$ with the x-axis. (See Figure 1.2.) We can compute the components of $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ by using trigonometry:

$$u_1 = 20\cos 45° = 10\sqrt{2} \qquad \text{and} \qquad u_2 = 20\sin 45° = 10\sqrt{2}.$$

Therefore, $\mathbf{u} = \begin{bmatrix} 10\sqrt{2} \\ 10\sqrt{2} \end{bmatrix}$, where the units are in miles per hour.

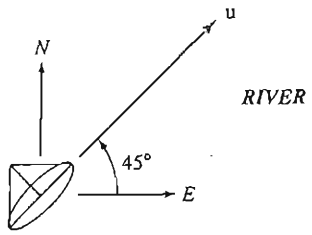

Figure 1.2

VECTOR ADDITION AND THE PARALLELOGRAM LAW

We can represent vector addition graphically, using arrows, by a result called the *parallelogram law*.[3] To add nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, first form a parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$. Then the sum $\mathbf{u}+\mathbf{v}$ is the arrow along the diagonal of the parallelogram as shown in Figure 1.3.

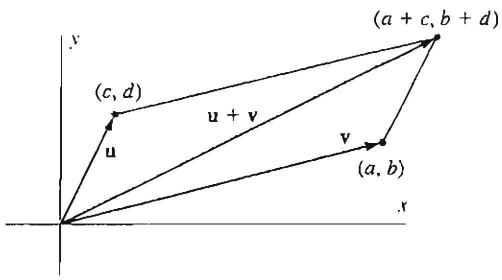

Figure 1.3 The parallelogram law of vector addition

Velocities can be combined by adding vectors that represent them.

Example 4

Imagine that the boat from the previous example is now cruising on a river, which flows to the east at 7 miles per hour. As before, the bow of the boat points toward the northeast, and its speed relative to the water is 20 miles per hour. In this case, the vector $\mathbf{u} = \begin{bmatrix} 10\sqrt{2} \\ 10\sqrt{2} \end{bmatrix}$, which we calculated in the previous example, represents the boat's velocity (in miles per hour) relative to the river. To find the velocity of the boat relative to the shore, we must add a vector $\mathbf{v}$, representing the velocity of the river, to the vector $\mathbf{u}$. Since the river flows toward the east at 7 miles per hour, its velocity vector is $\mathbf{v} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$. We can represent the sum of the vectors $\mathbf{u}$ and $\mathbf{v}$ by using the parallelogram law, as shown in Figure 1.4. The velocity of the boat relative to the shore (in miles per hour) is the vector

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 10\sqrt{2}+7 \\ 10\sqrt{2} \end{bmatrix}.$$

[3] A justification of the parallelogram law by Heron of Alexandria (first century C.E.) appears in his *Mechanics*.

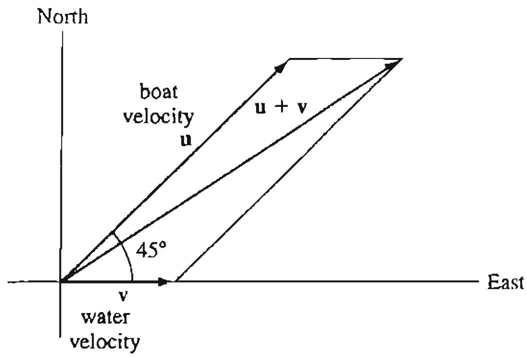

Figure 1.4

To find the speed of the boat, we use the Pythagorean theorem, which tells us that the length of a vector with endpoint (p, q) is $\sqrt{p^2 + q^2}$. Using the fact that the components of $\mathbf{u} + \mathbf{v}$ are $p = 10\sqrt{2} + 7$ and $q = 10\sqrt{2}$, respectively, it follows that the speed of the boat is

$$\sqrt{p^2 + q^2} \approx 25.44 \text{ mph}.$$

SCALAR MULTIPLICATION

We can also represent scalar multiplication graphically, using arrows. If $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a vector and c is a positive scalar, the scalar multiple $c\mathbf{v}$ is a vector that points in the same direction as $\mathbf{v}$, and whose length is c times the length of $\mathbf{v}$. This is shown in Figure 1.5(a). If c is negative, $c\mathbf{v}$ points in the opposite direction from $\mathbf{v}$, and has length $|c|$ times the length of $\mathbf{v}$. This is shown in Figure 1.5(b). We call two vectors **parallel** if one of them is a scalar multiple of the other.

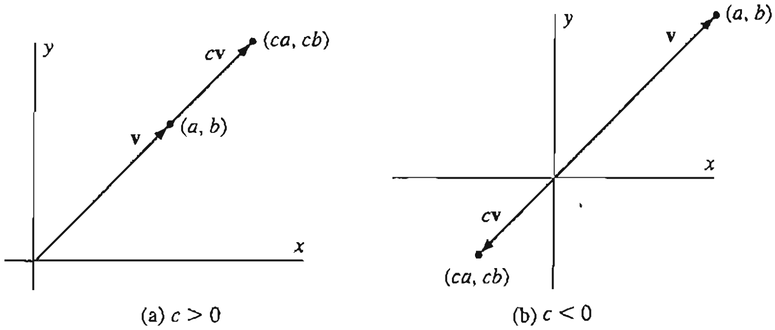

Figure 1.5 Scalar multiplication of vectors

VECTORS IN $\mathcal{R}^3$

If we identify $\mathcal{R}^3$ as the set of all ordered triples, then the same geometric ideas that hold in $\mathcal{R}^2$ are also true in $\mathcal{R}^3$. We may depict a vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathcal{R}^3$ as an arrow emanating from the origin of the xyz-coordinate system, with the point (a, b, c) as its

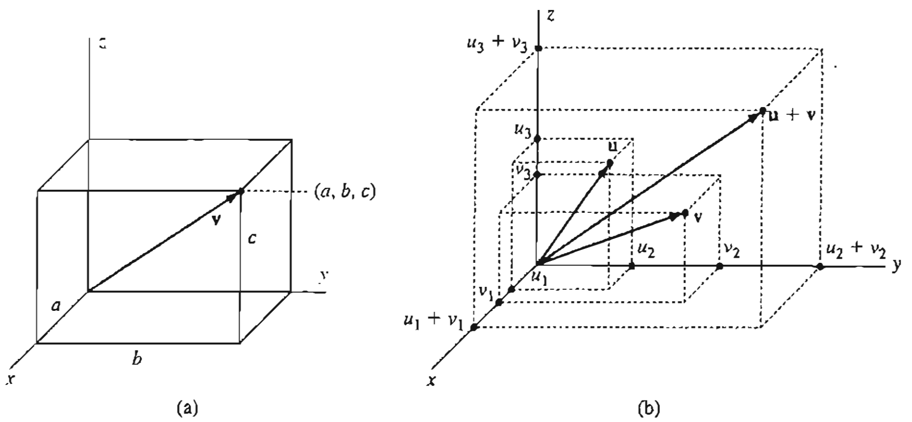

Figure 1.6 Vectors in $\mathcal{R}^3$

endpoint. (See Figure 1.6(a).) As is the case in $\mathcal{R}^2$, we can view two nonzero vectors in $\mathcal{R}^3$ as adjacent sides of a parallelogram, and we can represent their addition by using the parallelogram law. (See Figure 1.6(b).) In real life, motion takes place in 3-dimensional space, and we can depict quantities such as velocities and forces as vectors in $\mathcal{R}^3$.

EXERCISES

In Exercises 1–12, compute the indicated matrices, where

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 4 \end{bmatrix}.$$

1. $4A$
2. $-A$
3. $4A - 2B$
4. $3A + 2B$
5. $(2B)^T$
6. $A^T + 2B^T$
7. $A + B$
8. $(A + 2B)^T$
9. A^T
10. $A - B$
11. $-(B^T)$
12. $(-B)^T$

In Exercises 13–24, compute the indicated matrices, if possible, where

$$A = \begin{bmatrix} 3 & -1 & 2 & 4 \\ 1 & 5 & -6 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 0 \\ 2 & 5 \\ -1 & -3 \\ 0 & 2 \end{bmatrix}.$$

13. $-A$
14. $3B$
15. $(-2)A$
16. $(2B)^T$
17. $A - B$
18. $A - B^T$
19. $A^T - B$
20. $3A + 2B^T$
21. $(A + B)^T$
22. $(4A)^T$
23. $B - A^T$
24. $(B^T - A)^T$

In Exercises 25–28, assume that $A = \begin{bmatrix} 3 & -2 \\ 0 & 1.6 \\ 2\pi & 5 \end{bmatrix}$.

25. Determine a_{12}.
26. Determine a_{21}.
27. Determine $\mathbf{a}_1$.
28. Determine $\mathbf{a}_2$.

In Exercises 29–32, assume that $C = \begin{bmatrix} 2 & -3 & 0.4 \\ 2e & 12 & 0 \end{bmatrix}$.

29. Determine $\mathbf{c}_1$.
30. Determine $\mathbf{c}_3$.
31. Determine the first row of C.
32. Determine the second row of C.

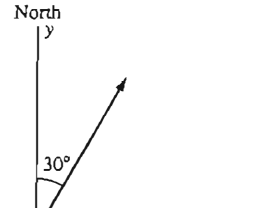

Figure 1.7 A view of the airplane from above

33. An airplane is flying with a ground speed of 300 mph at an angle of 30° east of due north. (See Figure 1.7.) In addition, the airplane is climbing at a rate of 10 mph. Determine the vector in $\mathcal{R}^3$ that represents the velocity (in mph) of the airplane.
34. A swimmer is swimming northeast at 2 mph in still water.
    (a) Give the velocity of the swimmer. Include a sketch.
    (b) A current in a northerly direction at 1 mph affects the velocity of the swimmer. Give the new velocity and speed of the swimmer. Include a sketch.
35. A pilot keeps her airplane pointed in a northeastward direction while maintaining an airspeed (speed relative to the surrounding air) of 300 mph. A wind from the west blows eastward at 50 mph.

(a) Find the velocity (in mph) of the airplane relative to the ground.

(b) What is the speed (in mph) of the airplane relative to the ground?

36. Suppose that in a medical study of 20 people, for each i, $1 \leq i \leq 20$, the 3×1 vector $\mathbf{u}_i$ is defined so that its components respectively represent the blood pressure, pulse rate, and cholesterol reading of the ith person. Provide an interpretation of the vector $\frac{1}{20}(\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_{20})$.

T&F *In Exercises 37–56, determine whether the statements are true or false.*

37. Matrices must be of the same size for their sum to be defined.
38. The transpose of a sum of two matrices is the sum of the transposed matrices.
39. Every vector is a matrix.
40. A scalar multiple of the zero matrix is the zero scalar.
41. The transpose of a matrix is a matrix of the same size.
42. A submatrix of a matrix may be a vector.
43. If B is a 3×4 matrix, then its rows are 4×1 vectors.
44. The $(3,4)$-entry of a matrix lies in column 3 and row 4.
45. In a zero matrix, every entry is 0.
46. An $m \times n$ matrix has $m + n$ entries.
47. If $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v} = -3\mathbf{w}$, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.
48. If A and B are any $m \times n$ matrices, then
$$A - B = A + (-1)B.$$
49. The (i,j)-entry of A^T equals the (j,i)-entry of A.
50. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, then $A = B$.
51. In any matrix A, the sum of the entries of $3A$ equals three times the sum of the entries of A.
52. Matrix addition is commutative.
53. Matrix addition is associative.
54. For any $m \times n$ matrices A and B and any scalars c and d, $(cA + dB)^T = cA^T + dB^T$.
55. If A is a matrix, then cA is the same size as A for every scalar c.
56. If A is a matrix for which the sum $A + A^T$ is defined, then A is a square matrix.

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57. Let A and B be matrices of the same size.
    (a) Prove that the jth column of $A + B$ is $\mathbf{a}_j + \mathbf{b}_j$.
    (b) Prove that for any scalar c, the jth column of cA is $c\mathbf{a}_j$.
58. For any $m \times n$ matrix A, prove that $0A = O$, the $m \times n$ zero matrix.
59. For any $m \times n$ matrix A, prove that $1A = A$.
60. Prove Theorem 1.1(a).
61. Prove Theorem 1.1(c).
62. Prove Theorem 1.1(d).
63. Prove Theorem 1.1(e).
64. Prove Theorem 1.1(g).
65. Prove Theorem 1.2(b).
66. Prove Theorem 1.2(c).

A square matrix A is called a **diagonal matrix** *if $a_{ij} = 0$ whenever $i \neq j$. Exercises 67–70 are concerned with diagonal matrices.*

67. Prove that a square zero matrix is a diagonal matrix.
68. Prove that if B is a diagonal matrix, then cB is a diagonal matrix for any scalar c.
69. Prove that if B is a diagonal matrix, then B^T is a diagonal matrix.
70. Prove that if B and C are diagonal matrices of the same size, then $B + C$ is a diagonal matrix.

A (square) matrix A is said to be **symmetric** *if $A = A^T$. Exercises 71–78 are concerned with symmetric matrices.*

71. Give examples of 2×2 and 3×3 symmetric matrices.
72. Prove that the (i,j)-entry of a symmetric matrix equals the (j,i)-entry.
73. Prove that a square zero matrix is symmetric.
74. Prove that if B is a symmetric matrix, then so is cB for any scalar c.
75. Prove that if B is a square matrix, then $B + B^T$ is symmetric.
76. Prove that if B and C are $n \times n$ symmetric matrices, then so is $B + C$.
77. Is a square submatrix of a symmetric matrix necessarily a symmetric matrix? Justify your answer.
78. Prove that a diagonal matrix is symmetric.

A (square) matrix A is called **skew-symmetric** *if $A^T = -A$. Exercises 79–81 are concerned with skew-symmetric matrices.*

79. What must be true about the (i,i)-entries of a skew-symmetric matrix? Justify your answer.
80. Give an example of a nonzero 2×2 skew-symmetric matrix B. Now show that every 2×2 skew-symmetric matrix is a scalar multiple of B.
81. Show that every 3×3 matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.
82.[4] The **trace** of an $n \times n$ matrix A, written trace(A), is defined to be the sum
$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$
Prove that, for any $n \times n$ matrices A and B and scalar c, the following statements are true:
    (a) $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$.
    (b) $\text{trace}(cA) = c \cdot \text{trace}(A)$.
    (c) $\text{trace}(A^T) = \text{trace}(A)$.
83. *Probability vectors* are vectors whose components are nonnegative and have a sum of 1. Show that if $\mathbf{p}$ and $\mathbf{q}$ are probability vectors and a and b are nonnegative scalars with $a + b = 1$, then $a\mathbf{p} + b\mathbf{q}$ is a probability vector.

---

[4] This exercise is used in Sections 2.2, 7.1, and 7.5 (on pages 115, 495, and 533, respectively).

In the following exercise, use either a calculator with matrix capabilities or computer software such as MATLAB to solve the problem:

84. Consider the matrices

$$A = \begin{bmatrix} 1.3 & 2.1 & -3.3 & 6.0 \\ 5.2 & 2.3 & -1.1 & 3.4 \\ 3.2 & -2.6 & 1.1 & -4.0 \\ 0.8 & -1.3 & -12.1 & 5.7 \\ -1.4 & 3.2 & 0.7 & 4.4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2.6 & -1.3 & 0.7 & -4.4 \\ 2.2 & -2.6 & 1.3 & -3.2 \\ 7.1 & 1.5 & -8.3 & 4.6 \\ -0.9 & -1.2 & 2.4 & 5.9 \\ 3.3 & -0.9 & 1.4 & 6.2 \end{bmatrix}.$$

(a) Compute $A + 2B$.

(b) Compute $A - B$.

(c) Compute $A^T + B^T$.

SOLUTIONS TO THE PRACTICE PROBLEMS

1. (a) The $(1, 2)$-entry of A is 2.

(b) The $(2, 2)$-entry of A is 3.

2. (a) $A - B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \end{bmatrix}$

(b) $2A = 2\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 2 \\ 6 & 0 & -4 \end{bmatrix}$

(c) $A + 3B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 0 \\ 6 & -3 & 12 \end{bmatrix}$

$= \begin{bmatrix} 5 & 8 & 1 \\ 9 & -3 & 10 \end{bmatrix}$

3. (a) $A^T = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$

(b) $(3B)^T = \begin{bmatrix} 3 & 9 & 0 \\ 6 & -3 & 12 \end{bmatrix}^T = \begin{bmatrix} 3 & 6 \\ 9 & -3 \\ 0 & 12 \end{bmatrix}$

(c) $(A + B)^T = \begin{bmatrix} 3 & 2 & 1 \\ 5 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$

1.2 LINEAR COMBINATIONS, MATRIX-VECTOR PRODUCTS, AND SPECIAL MATRICES

In this section, we explore some applications involving matrix operations and introduce the product of a matrix and a vector.

Suppose that 20 students are enrolled in a linear algebra course, in which two tests, a quiz, and a final exam are given. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{20} \end{bmatrix}$, where u_i denotes the score of the ith student on the first test. Likewise, define vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ similarly for the second test, quiz, and final exam, respectively. Assume that the instructor computes a student's course average by counting each test score twice as much as a quiz score, and the final exam score three times as much as a test score. Thus the *weights* for the tests, quiz, and final exam score are, respectively, 2/11, 2/11, 1/11, 6/11 (the weights must sum to one). Now consider the vector

$$\mathbf{y} = \frac{2}{11}\mathbf{u} + \frac{2}{11}\mathbf{v} + \frac{1}{11}\mathbf{w} + \frac{6}{11}\mathbf{z}.$$

The first component y_1 represents the first student's course average, the second component y_2 represents the second student's course average, and so on. Notice that $\mathbf{y}$ is a sum of scalar multiples of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$. This form of vector sum is so important that it merits its own definition.

Definitions A **linear combination** of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is a vector of the form

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars. These scalars are called the **coefficients** of the linear combination.

Note that a linear combination of one vector is simply a scalar multiple of that vector.

In the previous example, the vector $\mathbf{y}$ of the students' course averages is a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$. The coefficients are the weights. Indeed, any weighted average produces a linear combination of the scores.

Notice that

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, with coefficients -3, 4, and 1. We can also write

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This equation also expresses $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, but now the coefficients are 1, 2, and -1. So the set of coefficients that express one vector as a linear combination of the others need not be unique.

Example 1

(a) Determine whether $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) Determine whether $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) Determine whether $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Solution (a) We seek scalars x_1 and x_2 such that

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 + x_2 \end{bmatrix}.$$

That is, we seek a solution of the system of equations

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 3x_1 + x_2 &= -1. \end{aligned}$$

Because these equations represent nonparallel lines in the plane, there is exactly one solution, namely, $x_1 = -1$ and $x_2 = 2$. Therefore $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a (unique) linear combination of the vectors $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$, namely,

$$\begin{bmatrix}4\\-1\end{bmatrix} = (-1)\begin{bmatrix}2\\3\end{bmatrix} + 2\begin{bmatrix}3\\1\end{bmatrix}.$$

(See Figure 1.8.)

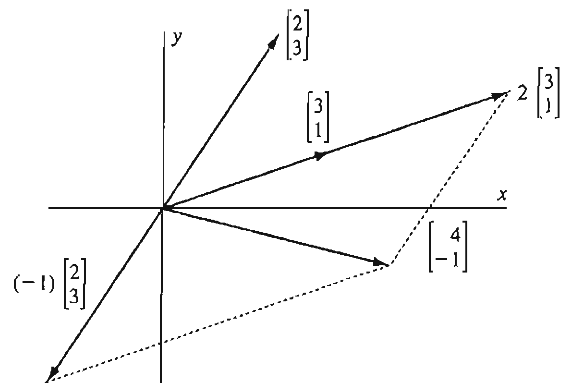


Figure 1.8 The vector $\begin{bmatrix}4\\-1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$.

(b) To determine whether $\begin{bmatrix}-4\\-2\end{bmatrix}$ is a linear combination of $\begin{bmatrix}6\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$, we perform a similar computation and produce the set of equations

$$\begin{aligned} 6x_1 + 2x_2 &= -4 \\ 3x_1 + x_2 &= -2. \end{aligned}$$

Since the first equation is twice the second, we need only solve $3x_1 + x_2 = -2$. This equation represents a line in the plane, and the coordinates of any point on the line give a solution. For example, we can let $x_1 = -2$ and $x_2 = 4$. In this case, we have

$$\begin{bmatrix}-4\\-2\end{bmatrix} = (-2)\begin{bmatrix}6\\3\end{bmatrix} + 4\begin{bmatrix}2\\1\end{bmatrix}.$$

There are infinitely many solutions. (See Figure 1.9.)

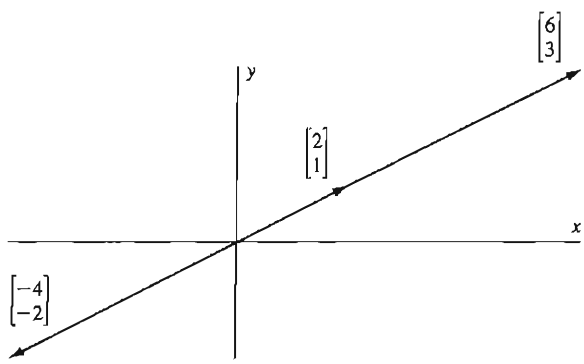


Figure 1.9 The vector $\begin{bmatrix}-4\\-2\end{bmatrix}$ is a linear combination of $\begin{bmatrix}6\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$.

(c) To determine if $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$, we must solve the system of equations

$$\begin{aligned} 3x_1 + 6x_2 &= 3 \\ 2x_1 + 4x_2 &= 4. \end{aligned}$$

If we add $-\frac{2}{3}$ times the first equation to the second, we obtain $0 = 2$, an equation with no solutions. Indeed, the two original equations represent parallel lines in the plane, so the original system has no solutions. We conclude that $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$. (See Figure 1.10.)

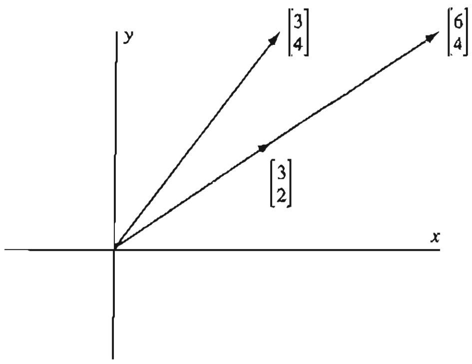


Figure 1.10 The vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is *not* a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Example 2 Given vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, show that the sum of any two linear combinations of these vectors is also a linear combination of these vectors.

Solution Suppose that $\mathbf{w}$ and $\mathbf{z}$ are linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Then we may write

$$\mathbf{w} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 \qquad \text{and} \qquad \mathbf{z} = a'\mathbf{u}_1 + b'\mathbf{u}_2 + c'\mathbf{u}_3,$$

where a, b, c, a', b', c' are scalars. So

$$\mathbf{w} + \mathbf{z} = (a + a')\mathbf{u}_1 + (b + b')\mathbf{u}_2 + (c + c')\mathbf{u}_3,$$

which is also a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

STANDARD VECTORS

We can write any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathcal{R}^2$ as a linear combination of the two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as follows:

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are called the *standard vectors* of $\mathcal{R}^2$. Similarly, we can write any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathcal{R}^3$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as follows:

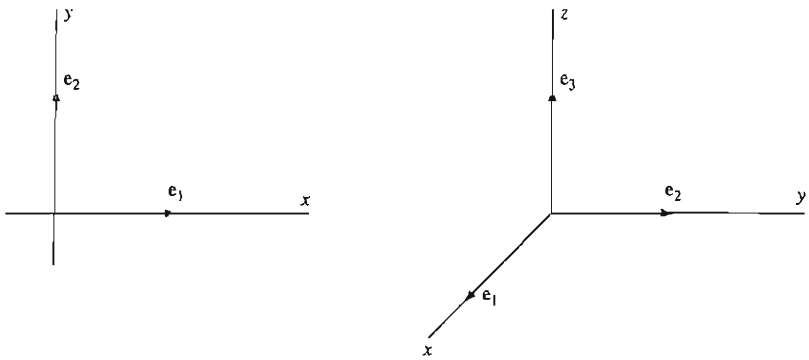
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are called the *standard vectors* of $\mathcal{R}^3$.

In general, we define the **standard vectors** of $\mathcal{R}^n$ by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

(See Figure 1.11.)



The standard vectors of $\mathcal{R}^2$ The standard vectors of $\mathcal{R}^3$

Figure 1.11

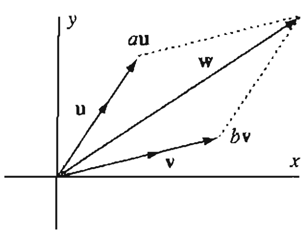


Figure 1.12 The vector **w** is a linear combination of the nonparallel vectors **u** and **v**.

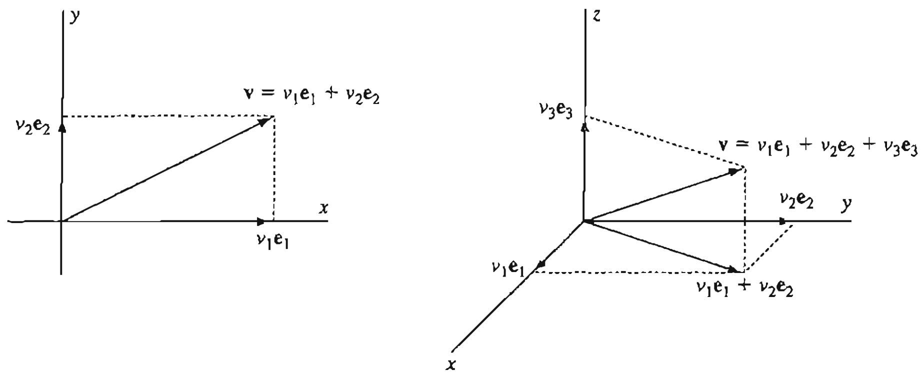
From the preceding equations, it is easy to see that every vector in $\mathcal{R}^n$ is a linear combination of the standard vectors of $\mathcal{R}^n$. In fact, for any vector $\mathbf{v}$ in $\mathcal{R}^n$,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

(See Figure 1.13.)

Now let $\mathbf{u}$ and $\mathbf{v}$ be nonparallel vectors, and let $\mathbf{w}$ be any vector in $\mathcal{R}^2$. Begin with the endpoint of $\mathbf{w}$ and create a parallelogram with sides $a\mathbf{u}$ and $b\mathbf{v}$, so that $\mathbf{w}$ is its diagonal. It follows that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$; that is, $\mathbf{w}$ is a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$. (See Figure 1.12.) More generally, the following statement is true:

> If $\mathbf{u}$ and $\mathbf{v}$ are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.



The vector **v** is a linear combination of standard vectors in $\mathcal{R}^2$.

The vector **v** is a linear combination of standard vectors in $\mathcal{R}^3$.

Figure 1.13

Practice Problem 1 ▶ Let $\mathbf{w} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$ and $\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$.

(a) Without doing any calculations, explain why **w** can be written as a linear combination of the vectors in $\mathcal{S}$.

(b) Express **w** as a linear combination of the vectors in $\mathcal{S}$. ◀

Suppose that a garden supply store sells three mixtures of grass seed. The deluxe mixture is 80% bluegrass and 20% rye, the standard mixture is 60% bluegrass and 40% rye, and the economy mixture is 40% bluegrass and 60% rye. One way to record this information is with the following 2×3 matrix:

$$\begin{array}{cc} & \begin{array}{ccc} \text{deluxe} & \text{standard} & \text{economy} \end{array} \\ B = & \begin{bmatrix} .80 & .60 & .40 \\ .20 & .40 & .60 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{bluegrass} \\ \text{rye} \end{array}$$

A customer wants to purchase a blend of grass seed containing 5 lb of bluegrass and 3 lb of rye. There are two natural questions that arise:

1. Is it possible to combine the three mixtures of seed into a blend that has exactly the desired amounts of bluegrass and rye, with no surplus of either?
2. If so, how much of each mixture should the store clerk add to the blend?

Let x_1, x_2, and x_3 denote the number of pounds of deluxe, standard, and economy mixtures, respectively, to be used in the blend. Then we have

$$\begin{aligned} .80x_1 + .60x_2 + .40x_3 &= 5 \\ .20x_1 + .40x_2 + .60x_3 &= 3. \end{aligned}$$

This is a *system of two linear equations in three unknowns*. Finding a solution of this system is equivalent to answering our second question. The technique for solving general systems is explored in great detail in Sections 1.3 and 1.4.

Using matrix notation, we may rewrite these equations in the form

$$\begin{bmatrix} .80x_1 + .60x_2 + .40x_3 \\ .20x_1 + .40x_2 + .60x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Now we use matrix operations to rewrite this matrix equation, using the columns of B as

$$x_1\begin{bmatrix}.80\\.20\end{bmatrix}+x_2\begin{bmatrix}.60\\.40\end{bmatrix}+x_3\begin{bmatrix}.40\\.60\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}.$$

Thus we can rephrase the first question as follows: Is $\begin{bmatrix}5\\3\end{bmatrix}$ a linear combination of the columns $\begin{bmatrix}.80\\.20\end{bmatrix}$, $\begin{bmatrix}.60\\.40\end{bmatrix}$, and $\begin{bmatrix}.40\\.60\end{bmatrix}$ of B? The result in the box on page 17 provides an affirmative answer. Because no two of the three vectors are parallel, $\begin{bmatrix}5\\3\end{bmatrix}$ is a linear combination of any pair of these vectors.

MATRIX–VECTOR PRODUCTS

A convenient way to represent systems of linear equations is by *matrix–vector products*. For the preceding example, we represent the variables by the vector $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ and define the *matrix–vector product* $B\mathbf{x}$ to be the linear combination

$$B\mathbf{x}=\begin{bmatrix}.80&.60&.40\\.20&.40&.60\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_1\begin{bmatrix}.80\\.20\end{bmatrix}+x_2\begin{bmatrix}.60\\.40\end{bmatrix}+x_3\begin{bmatrix}.40\\.60\end{bmatrix}.$$

This definition provides another way to state the first question in the preceding example: Does the vector $\begin{bmatrix}5\\3\end{bmatrix}$ equal $B\mathbf{x}$ for some vector $\mathbf{x}$? Notice that for the matrix–vector product to make sense, the number of columns of B must equal the number of components in $\mathbf{x}$. The general definition of a matrix–vector product is given next.

Definition Let A be an $m \times n$ matrix and $\mathbf{v}$ be an $n \times 1$ vector. We define the **matrix–vector product** of A and $\mathbf{v}$, denoted by $A\mathbf{v}$, to be the linear combination of the columns of A whose coefficients are the corresponding components of $\mathbf{v}$. That is,

$$A\mathbf{v}=v_1\mathbf{a}_1+v_2\mathbf{a}_2+\cdots+v_n\mathbf{a}_n.$$

As we have noted, for $A\mathbf{v}$ to exist, the number of columns of A must equal the number of components of $\mathbf{v}$. For example, suppose that

$$A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\quad\text{and}\quad\mathbf{v}=\begin{bmatrix}7\\8\end{bmatrix}.$$

Notice that A has two columns and $\mathbf{v}$ has two components. Then

$$A\mathbf{v}=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix}=7\begin{bmatrix}1\\3\\5\end{bmatrix}+8\begin{bmatrix}2\\4\\6\end{bmatrix}=\begin{bmatrix}7\\21\\35\end{bmatrix}+\begin{bmatrix}16\\32\\48\end{bmatrix}=\begin{bmatrix}23\\53\\83\end{bmatrix}.$$

CHAPTER

6 ORTHOGONALITY

Until now, we have focused our attention on two operations with vectors, namely, addition and scalar multiplication. In this chapter, we consider such geometric concepts as *length* and *perpendicularity* of vectors. By combining the geometry of vectors with matrices and linear transformations, we obtain powerful techniques for solving a wide variety of problems. For example, we apply these new tools to such areas as least-squares approximation, the graphing of conic sections, computer graphics, and statistical analyses. The key to most of these solutions is the construction of a basis of perpendicular eigenvectors for a given matrix or linear transformation.

To do this, we show how to convert any basis for a subspace of $\mathcal{R}^n$ into one in which all of the vectors are perpendicular to each other. Once this is done, we determine conditions that guarantee that there is a basis for $\mathcal{R}^n$ consisting of perpendicular eigenvectors of a matrix or a linear transformation. Surprisingly, for a matrix, a necessary and sufficient condition that such a basis exists is that the matrix be symmetric.

6.1 THE GEOMETRY OF VECTORS

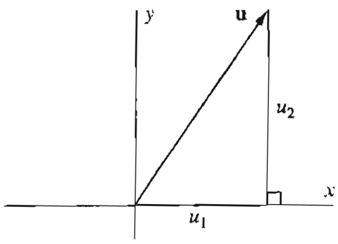
In this section, we introduce the concepts of length and perpendicularity of vectors in $\mathcal{R}^n$. Many familiar geometric properties seen in earlier courses extend to this more general space. In particular, the Pythagorean theorem, which relates the squared lengths of sides of a right triangle, also holds in $\mathcal{R}^n$. To show that many of these results hold in $\mathcal{R}^n$, we define and develop the notion of *dot product*. The dot product is fundamental in the sense that, from it, we can define length and perpendicularity.

Perhaps the most basic concept of geometry is length. In Figure 6.1(a), an application of the Pythagorean theorem suggests that we define the *length* of the vector $\mathbf{u}$ to be $\sqrt{u_1^2 + u_2^2}$.

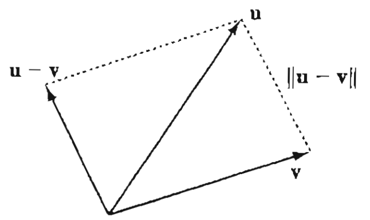
This definition easily extends to any vector $\mathbf{v}$ in $\mathcal{R}^n$ by defining its **norm (length)**, denoted by $\|\mathbf{v}\|$, by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

A vector whose norm is 1 is called a **unit vector**. Using the definition of vector norm, we can now define the **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$ as $\|\mathbf{u} - \mathbf{v}\|$. (See Figure 6.1(b).)



(a) The length of a vector $\mathbf{u}$ in $\mathcal{R}^2$



(b) The distance between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$

Figure 6.1

Example 1 Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and the distance between $\mathbf{u}$ and $\mathbf{v}$ if

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

Solution By definition,

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \quad \|\mathbf{v}\| = \sqrt{2^2 + (-3)^2 + 0^2} = \sqrt{13},$$

and the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(1-2)^2 + (2-(-3))^2 + (3-0)^2} = \sqrt{35}.$$

Practice Problem 1 ► Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(b) Determine the distance between $\mathbf{u}$ and $\mathbf{v}$.

(c) Show that both $\dfrac{1}{\|\mathbf{u}\|}\mathbf{u}$ and $\dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ are unit vectors. ◄

Just as we used the Pythagorean theorem in $\mathcal{R}^2$ to motivate the definition of the norm of a vector, we use this theorem again to examine what it means for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^2$ to be perpendicular. According to the Pythagorean theorem (see Figure 6.2), we see that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if

$$\begin{aligned} \|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \\ (v_1 - u_1)^2 + (v_2 - u_2)^2 &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\ v_1^2 - 2u_1v_1 + u_1^2 + v_2^2 - 2u_2v_2 + u_2^2 &= u_1^2 + u_2^2 + v_1^2 + v_2^2 \\ -2u_1v_1 - 2u_2v_2 &= 0 \\ u_1v_1 + u_2v_2 &= 0. \end{aligned}$$

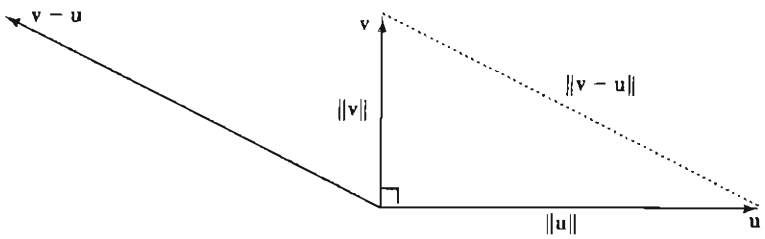


Figure 6.2 The Pythagorean theorem

The expression $u_1v_1 + u_2v_2$ in the last equation is called the *dot product* of **u** and **v**, and is denoted by $\mathbf{u} \cdot \mathbf{v}$. So **u** and **v** are perpendicular if and only if their dot product equals zero.

Using this observation, we define the **dot product** of vectors **u** and **v** in $\mathcal{R}^n$ by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

We say that **u** and **v** are **orthogonal (perpendicular)** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Notice that, in $\mathcal{R}^n$, the dot product of two vectors is a scalar, and the dot product of **0** with every vector is zero. Hence **0** is orthogonal to every vector in $\mathcal{R}^n$. Also, as noted, the property of being orthogonal in $\mathcal{R}^2$ and $\mathcal{R}^3$ is equivalent to the usual geometric definition of perpendicularity.

Example 2 Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -8 \\ 3 \\ 2 \end{bmatrix}.$$

Determine which pairs of these vectors are orthogonal.

Solution We need only check which pairs have dot products equal to zero.

$$\mathbf{u} \cdot \mathbf{v} = (2)(1) + (-1)(4) + (3)(-2) = -8$$
$$\mathbf{u} \cdot \mathbf{w} = (2)(-8) + (-1)(3) + (3)(2) = -13$$
$$\mathbf{v} \cdot \mathbf{w} = (1)(-8) + (4)(3) + (-2)(2) = 0$$

We see that **v** and **w** are the only orthogonal vectors.

Practice Problem 2 ▶ Determine which pairs of the vectors

$$\mathbf{u} = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

are orthogonal. ◀

It is useful to observe that the dot product of $\mathbf{u}$ and $\mathbf{v}$ can also be represented as the matrix product $\mathbf{u}^T\mathbf{v}$.

$$\mathbf{u}^T\mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \mathbf{u}\cdot\mathbf{v}$$

Notice that we have treated the 1×1 matrix $\mathbf{u}^T\mathbf{v}$ as a scalar by writing it as $u_1v_1 + u_2v_2 + \cdots + u_nv_n$ instead of $[u_1v_1 + u_2v_2 + \cdots + u_nv_n]$.

One useful consequence of identifying a dot product as a matrix product is that it enables us to "move" a matrix from one side of a dot product to the other. More precisely, if A is an $m \times n$ matrix, $\mathbf{u}$ is in $\mathcal{R}^n$, and $\mathbf{v}$ is in $\mathcal{R}^m$, then

$$A\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot A^T\mathbf{v}.$$

This follows because

$$A\mathbf{u}\cdot\mathbf{v} = (A\mathbf{u})^T\mathbf{v} = (\mathbf{u}^T A^T)\mathbf{v} = \mathbf{u}^T(A^T\mathbf{v}) = \mathbf{u}\cdot A^T\mathbf{v}.$$

Just as there are arithmetic properties of vector addition and scalar multiplication, there are arithmetic properties for the dot product and norm.

THEOREM 6.1

For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^n$ and every scalar c,

(a) $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$.

(b) $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(c) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.

(d) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$.

(e) $(\mathbf{v}+\mathbf{w})\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{u} + \mathbf{w}\cdot\mathbf{u}$.

(f) $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$.

(g) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$.

PROOF We prove parts (d) and (g) and leave the rest as exercises.

(d) Using matrix properties, we have

$$\begin{aligned} \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}^T(\mathbf{v}+\mathbf{w}) \\ &= \mathbf{u}^T\mathbf{v} + \mathbf{u}^T\mathbf{w} \\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}. \end{aligned}$$

(g) By (a) and (f), we have

$$\begin{aligned} \|c\mathbf{u}\|^2 &= (c\mathbf{u})\cdot(c\mathbf{u}) \\ &= c^2\mathbf{u}\cdot\mathbf{u} \\ &= c^2\|\mathbf{u}\|^2. \end{aligned}$$

By taking the square root of both sides and using $\sqrt{c^2} = |c|$, we obtain $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$. ∎

Because of Theorem 6.1(f), there is no ambiguity in writing $c\mathbf{u}\cdot\mathbf{v}$ for any of the three expressions in (f).

Note that, by Theorem 6.1(g), any nonzero vector $\mathbf{v}$ can be **normalized**, that is, transformed into a unit vector by multiplying it by the scalar $\dfrac{1}{\|\mathbf{v}\|}$. For if $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$, then

$$\|\mathbf{u}\| = \left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \left|\frac{1}{\|\mathbf{v}\|}\right|\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

This theorem allows us to treat expressions with dot products and norms just as we would algebraic expressions. For example, compare the similarity of the algebraic result

$$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$$

with

$$\|2\mathbf{u} + 3\mathbf{v}\|^2 = 4\|\mathbf{u}\|^2 + 12\mathbf{u}\cdot\mathbf{v} + 9\|\mathbf{v}\|^2.$$

The proof of the preceding equality relies heavily on Theorem 6.1:

$$\begin{aligned}
\|2\mathbf{u} + 3\mathbf{v}\|^2 &= (2\mathbf{u} + 3\mathbf{v})\cdot(2\mathbf{u} + 3\mathbf{v}) && \text{by (a)}\\
&= (2\mathbf{u})\cdot(2\mathbf{u} + 3\mathbf{v}) + (3\mathbf{v})\cdot(2\mathbf{u} + 3\mathbf{v}) && \text{by (e)}\\
&= (2\mathbf{u})\cdot(2\mathbf{u}) + (2\mathbf{u})\cdot(3\mathbf{v}) + (3\mathbf{v})\cdot(2\mathbf{u}) + (3\mathbf{v})\cdot(3\mathbf{v}) && \text{by (d)}\\
&= 4(\mathbf{u}\cdot\mathbf{u}) + 6(\mathbf{u}\cdot\mathbf{v}) + 6(\mathbf{v}\cdot\mathbf{u}) + 9(\mathbf{v}\cdot\mathbf{v}) && \text{by (f)}\\
&= 4\|\mathbf{u}\|^2 + 6(\mathbf{u}\cdot\mathbf{v}) + 6(\mathbf{u}\cdot\mathbf{v}) + 9\|\mathbf{v}\|^2 && \text{by (a) and (c)}\\
&= 4\|\mathbf{u}\|^2 + 12(\mathbf{u}\cdot\mathbf{v}) + 9\|\mathbf{v}\|^2
\end{aligned}$$

As noted earlier, we can write the last expression as $4\|\mathbf{u}\|^2 + 12\mathbf{u}\cdot\mathbf{v} + 9\|\mathbf{v}\|^2$. From now on, we will omit these steps when computing with dot products and norms.

! *CAUTION* Expressions such as $\mathbf{u}^2$ and $\mathbf{uv}$ are *not* defined.

It is easy to extend (d) and (e) of Theorem 6.1 to linear combinations, namely,

$$\mathbf{u}\cdot(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1\mathbf{u}\cdot\mathbf{v}_1 + c_2\mathbf{u}\cdot\mathbf{v}_2 + \cdots + c_p\mathbf{u}\cdot\mathbf{v}_p$$

and

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p)\cdot\mathbf{u} = c_1\mathbf{v}_1\cdot\mathbf{u} + c_2\mathbf{v}_2\cdot\mathbf{u} + \cdots + c_p\mathbf{v}_p\cdot\mathbf{u}.$$

As an application of these arithmetic properties, we show that the Pythagorean theorem holds in $\mathcal{R}^n$.

THEOREM 6.2

(Pythagorean Theorem in $\mathcal{R}^n$) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathcal{R}^n$. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

PROOF Applying the arithmetic of dot products and norms to the vectors $\mathbf{u}$ and $\mathbf{v}$, we have

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2.$$

Because $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = 0$, the result follows immediately. ■

ORTHOGONAL PROJECTION OF A VECTOR ON A LINE

Suppose we want to find the distance from a point P to the line $\mathcal{L}$ given in Figure 6.3. It is clear that if we can determine the vector $\mathbf{w}$, then the desired distance is given by $\|\mathbf{u}-\mathbf{w}\|$. The vector $\mathbf{w}$ is called the **orthogonal projection of u on** $\mathcal{L}$. To find $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathcal{L}$, let $\mathbf{v}$ be *any* nonzero vector along $\mathcal{L}$, and let $\mathbf{z} = \mathbf{u}-\mathbf{w}$. Then $\mathbf{w} = c\mathbf{v}$ for some scalar c. Notice that $\mathbf{z}$ and $\mathbf{v}$ are orthogonal; that is,

$$0 = \mathbf{z}\cdot\mathbf{v} = (\mathbf{u}-\mathbf{w})\cdot\mathbf{v} = (\mathbf{u}-c\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} - c\mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} - c\|\mathbf{v}\|^2.$$

So $c = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$, and thus $\mathbf{w} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Therefore the distance from P to $\mathcal{L}$ is given by

$$\|\mathbf{u}-\mathbf{w}\| = \left\|\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\|.$$

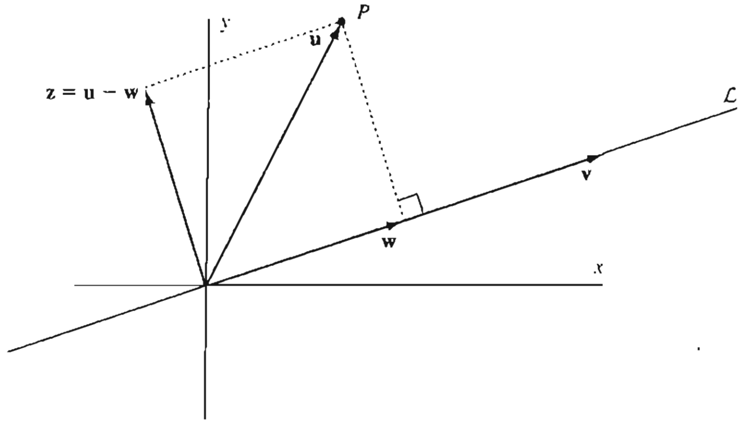


Figure 6.3 The vector **w** is the orthogonal projection of **u** on $\mathcal{L}$.

Example 3

Find the distance from the point $(4, 1)$ to the line whose equation is $y = \frac{1}{2}x$.

Solution Following our preceding derivation, we let

$$\mathbf{u} = \begin{bmatrix}4\\1\end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}, \qquad \text{and} \qquad \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{9}{5}\begin{bmatrix}2\\1\end{bmatrix}.$$

Then the desired distance is $\left\|\begin{bmatrix}4\\1\end{bmatrix} - \dfrac{9}{5}\begin{bmatrix}2\\1\end{bmatrix}\right\| = \dfrac{1}{5}\left\|\begin{bmatrix}2\\-4\end{bmatrix}\right\| = \dfrac{2}{5}\sqrt{5}.$

EXERCISES

In Exercises 1–8, two vectors **u** *and* **v** *are given. Compute the norms of the vectors and the distance d between them.*

1. $\mathbf{u} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

5. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

6. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

7. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

8. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

In Exercises 9–16, two vectors are given. Compute the dot product of the vectors, and determine whether the vectors are orthogonal.

9. $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

10. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

11. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

13. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

14. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}$

15. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

16. $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

In Exercises 17–24, two orthogonal vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|^2$, $\|\mathbf{v}\|^2$, *and* $\|\mathbf{u}+\mathbf{v}\|^2$. *Use your results to illustrate the Pythagorean theorem.*

17. $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

18. $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

19. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

20. $\mathbf{u} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$

21. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

23. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix}$

24. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$

In Exercises 25–32, two vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, *and* $\|\mathbf{u}+\mathbf{v}\|$. *Use your results to illustrate the triangle inequality.*

25. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$

26. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

27. $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

28. $\mathbf{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

29. $\mathbf{u} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

30. $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

31. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

32. $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$

In Exercises 33–40, two vectors **u** *and* **v** *are given. Compute the quantities* $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, *and* $\mathbf{u}\cdot\mathbf{v}$. *Use your results to illustrate the Cauchy–Schwarz inequality.*

33. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$
34. $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
35. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$
36. $\mathbf{u} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
37. $\mathbf{u} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$
38. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$
39. $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$
40. $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

In Exercises 41–48, a vector **u** *and a line* $\mathcal{L}$ *in* $\mathcal{R}^2$ *are given. Compute the orthogonal projection* **w** *of* **u** *on* $\mathcal{L}$, *and use it to compute the distance d from the endpoint of* **u** *to* $\mathcal{L}$.

41. $\mathbf{u} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $y = 0$
42. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $y = 2x$
43. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = -x$
44. $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $y = -2x$
45. $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $y = 3x$
46. $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $y = x$
47. $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $y = -3x$
48. $\mathbf{u} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $y = -4x$

For Exercises 49–54, suppose that **u**, **v**, *and* **w** *are vectors in* $\mathcal{R}^n$ *such that* $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 5$, $\mathbf{u}\cdot\mathbf{v} = -1$, $\mathbf{u}\cdot\mathbf{w} = 1$, *and* $\mathbf{v}\cdot\mathbf{w} = -4$.

49. Compute $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}$.
50. Compute $\|4\mathbf{w}\|$.
51. Compute $\|\mathbf{u}+\mathbf{v}\|^2$.
52. Compute $(\mathbf{u}+\mathbf{w})\cdot\mathbf{v}$.
53. Compute $\|\mathbf{v}-4\mathbf{w}\|^2$.
54. Compute $\|2\mathbf{u}+3\mathbf{v}\|^2$.

For Exercises 55–60, suppose that **u**, **v**, *and* **w** *are vectors in* $\mathcal{R}^n$ *such that* $\mathbf{u}\cdot\mathbf{u} = 14$, $\mathbf{u}\cdot\mathbf{v} = 7$, $\mathbf{u}\cdot\mathbf{w} = -20$, $\mathbf{v}\cdot\mathbf{v} = 21$, $\mathbf{v}\cdot\mathbf{w} = -5$, *and* $\mathbf{w}\cdot\mathbf{w} = 30$.

55. Compute $\|\mathbf{v}\|^2$.
56. Compute $\|3\mathbf{u}\|$.
57. Compute $\mathbf{v}\cdot\mathbf{u}$.
58. Compute $\mathbf{w}\cdot(\mathbf{u}+\mathbf{v})$.
59. Compute $\|2\mathbf{u}-\mathbf{v}\|^2$.
60. Compute $\|\mathbf{v}+3\mathbf{w}\|$.

T&F *In Exercises 61–80, determine whether the statements are true or false.*

61. Vectors must be of the same size for their dot product to be defined.
62. The dot product of two vectors in $\mathcal{R}^n$ is a vector in $\mathcal{R}^n$.
63. The norm of a vector equals the dot product of the vector with itself.
64. The norm of a multiple of a vector is the same multiple of the norm of the vector.
65. The norm of a sum of vectors is the sum of the norms of the vectors.
66. The squared norm of a sum of orthogonal vectors is the sum of the squared norms of the vectors.
67. The orthogonal projection of a vector on a line is a vector that lies along the line.
68. The norm of a vector is always a nonnegative real number.
69. If the norm of **v** equals 0, then $\mathbf{v} = \mathbf{0}$.
70. If $\mathbf{u}\cdot\mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
71. For all vectors **u** and **v** in $\mathcal{R}^n$, $|\mathbf{u}\cdot\mathbf{v}| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|$.
72. For all vectors **u** and **v** in $\mathcal{R}^n$, $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.
73. The distance between vectors **u** and **v** in $\mathcal{R}^n$ is $\|\mathbf{u}-\mathbf{v}\|$.
74. For all vectors **u** and **v** in $\mathcal{R}^n$ and every scalar c,
$$(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v}).$$
75. For all vectors **u**, **v**, and **w** in $\mathcal{R}^n$,
$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}.$$
76. If A is an $n\times n$ matrix and **u** and **v** are vectors in $\mathcal{R}^n$, then $A\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot A\mathbf{v}$.
77. For every vector **v** in $\mathcal{R}^n$, $\|\mathbf{v}\| = \|-\mathbf{v}\|$.
78. If **u** and **v** are orthogonal vectors in $\mathcal{R}^n$, then
$$\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|+\|\mathbf{v}\|.$$
79. If **w** is the orthogonal projection of **u** on a line through the origin of $\mathcal{R}^2$, then $\mathbf{u}-\mathbf{w}$ is orthogonal to every vector on the line.
80. If **w** is the orthogonal projection of **u** on a line through the origin of $\mathcal{R}^2$, then **w** is the vector on the line closest to **u**.

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81. Prove (a) of Theorem 6.1.
82. Prove (b) of Theorem 6.1.
83. Prove (c) of Theorem 6.1.
84. Prove (e) of Theorem 6.1.
85. Prove (f) of Theorem 6.1.
86. Prove that if **u** is orthogonal to both **v** and **w**, then **u** is orthogonal to every linear combination of **v** and **w**.
87. Let $\{\mathbf{v}, \mathbf{w}\}$ be a basis for a subspace W of $\mathcal{R}^n$, and define
$$\mathbf{z} = \mathbf{w} - \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$
Prove that $\{\mathbf{v}, \mathbf{z}\}$ is a basis for W consisting of orthogonal vectors.
88. Prove that the Cauchy–Schwarz inequality is an equality if and only if **u** is a multiple of **v** or **v** is a multiple of **u**.

89. Prove that the triangle inequality is an equality if and only if $\mathbf{u}$ is a nonnegative multiple of $\mathbf{v}$ or $\mathbf{v}$ is a nonnegative multiple of $\mathbf{u}$.

90. Use the triangle inequality to prove that $|\,\|\mathbf{v}\| - \|\mathbf{w}\|\,| \le \|\mathbf{v} - \mathbf{w}\|$ for all vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{R}^n$.

91. Prove $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$ for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^n$.

92. Let $\mathbf{z}$ be a vector in $\mathcal{R}^n$. Let $W = \{\mathbf{u} \in \mathcal{R}^n : \mathbf{u}\cdot\mathbf{z} = 0\}$. Prove that W is a subspace of $\mathcal{R}^n$.

93. Let $\mathcal{S}$ be a subset of $\mathcal{R}^n$ and

$$W = \{\mathbf{u} \in \mathcal{R}^n : \mathbf{u}\cdot\mathbf{z} = 0 \text{ for all } \mathbf{z} \text{ in } \mathcal{S}\}.$$

Prove that W is a subspace of $\mathcal{R}^n$.

94. Let W denote the set of all vectors that lie along the line with equation $y = 2x$. Find a vector $\mathbf{z}$ in $\mathcal{R}^2$ such that $W = \{\mathbf{u} \in \mathcal{R}^2 : \mathbf{u}\cdot\mathbf{z} = 0\}$. Justify your answer.

95. Prove the *parallelogram law* for vectors in $\mathcal{R}^n$:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

96. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal nonzero vectors in $\mathcal{R}^n$, then they are linearly independent.

97.[2] Let A be any $m \times n$ matrix.

(a) Prove that A^TA and A have the same null space. *Hint:* Let $\mathbf{v}$ be a vector in $\mathcal{R}^n$ such that $A^TA\mathbf{v} = \mathbf{0}$. Observe that $A^TA\mathbf{v}\cdot\mathbf{v} = A\mathbf{v}\cdot A\mathbf{v} = 0$.

(b) Use (a) to prove that $\operatorname{rank} A^TA = \operatorname{rank} A$.

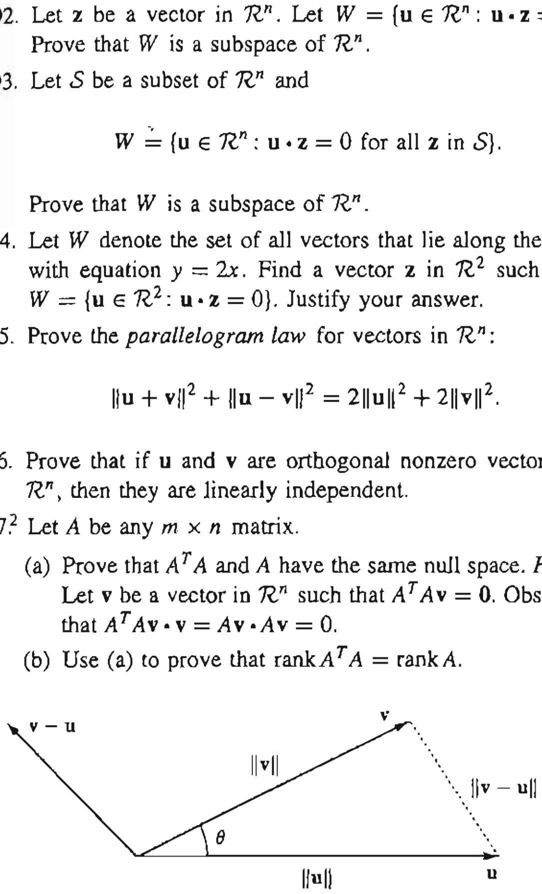

Figure 6.6

98.[3] Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathcal{R}^2$ or $\mathcal{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v}-\mathbf{u}$ determine a triangle. (See Figure 6.6.) The relationship between the lengths of the sides of this triangle and θ is called the *law of cosines*. It states that

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Use the law of cosines and Theorem 6.1 to derive the formula

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

In Exercises 99–106, use the formula in Exercise 98 to determine the angle between the vectors $\mathbf{u}$ *and* $\mathbf{v}$.

99. $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

100. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

101. $\mathbf{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

102. $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

103. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

104. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

105. $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

106. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathcal{R}^3$. *Define* $\mathbf{u}\times\mathbf{v}$ *to be the vector* $\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$, *which is called the* **cross product** *of* $\mathbf{u}$ *and* $\mathbf{v}$.

For Exercises 107–120, use the preceding definition of the cross product.

107. For every vector $\mathbf{u}$ in $\mathcal{R}^3$, prove that $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

108. Prove that $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$.

109. For every vector $\mathbf{u}$ in $\mathcal{R}^3$, prove that $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$.

110. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

111. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$ and all scalars c, prove that

$$c(\mathbf{u}\times\mathbf{v}) = c\mathbf{u}\times\mathbf{v} = \mathbf{u}\times c\mathbf{v}.$$

112. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}.$$

113. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}.$$

114. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

115. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}).$$

116. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.$$

[2] This exercise is used in Section 6.7 (on page 439).
[3] This exercise is used in Section 6.9 (on page 471).

117. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove that
$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}.$$

118. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that
$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

119. For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^3$, prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. *Hint:* Use Exercises 98 and 118.

120. For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathcal{R}^3$, prove the *Jacobi identity*:
$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$$

Exercises 121–124 refer to the application regarding the two methods of computing average class size given in this section. In Exercises 121–123, data are given for students enrolled in a three-section seminar course. Compute the average $\bar{v}$ determined by the supervisor and the average v^* determined by the investigator.

121. Section 1 contains 8 students, section 2 contains 12 students, and section 3 contains 6 students.

122. Section 1 contains 15 students, and each of sections 2 and 3 contains 30 students.

123. Each of the three sections contains 22 students.

124. Use Exercise 88 to prove that the two averaging methods for determining class size are equal if and only if all of the class sizes are equal.

In Exercise 125, use either a calculator with matrix capabilities or computer software such as MATLAB to solve the problem.

125. In every triangle, the length of any side is less than the sum of the lengths of the other two sides. When this observation is generalized to $\mathcal{R}^n$, we obtain the *triangle inequality* (Theorem 6.4), which states
$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$
for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$. Let
$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -8 \\ -6 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2.01 \\ 4.01 \\ 6.01 \\ 8.01 \end{bmatrix}, \quad \text{and}$$
$$\mathbf{v}_2 = \begin{bmatrix} 3.01 \\ 6.01 \\ 9.01 \\ 12.01 \end{bmatrix}.$$
(a) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}$.
(b) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}_1$.
(c) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}_2$.
(d) From what you have observed in (b) and (c), make a conjecture about when equality occurs in the triangle inequality.
(e) Interpret your conjecture in (d) geometrically in $\mathcal{R}^2$.

SOLUTIONS TO THE PRACTICE PROBLEMS

1. (a) We have $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$ and $\|\mathbf{v}\| = \sqrt{6^2 + 2^2 + 3^2} = 7$.

(b) We have $\|\mathbf{u} - \mathbf{v}\| = \left\| \begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix} \right\|$
$$= \sqrt{(-5)^2 + (-4)^2 + (-1)^2} = \sqrt{42}.$$

(c) We have
$$\left\| \frac{1}{\|\mathbf{u}\|}\mathbf{u} \right\| = \left\| \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$
and
$$\left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left\| \frac{1}{7} \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{6}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix} \right\| = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1.$$

2. Taking dot products, we obtain
$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (-5)(-1) + (3)(2) = 9$$
$$\mathbf{u} \cdot \mathbf{w} = (-2)(-3) + (-5)(1) + (3)(2) = 7$$
$$\mathbf{v} \cdot \mathbf{w} = (1)(-3) + (-1)(1) + (2)(2) = 0.$$
So $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, but $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal, and $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal.

3. Let $\mathbf{w}$ be the required orthogonal projection. Then
$$\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{(-2)(1) + (-5)(-1) + (3)(2)}{1^2 + (-1)^2 + 2^2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$
$$= \frac{3}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

6.2 ORTHOGONAL VECTORS

It is easy to extend the property of orthogonality to any set of vectors. We say that a subset of $\mathcal{R}^n$ is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal. The subset is called an **orthonormal set** if it is an orthogonal set consisting entirely of unit vectors.

ANSWERS TO SELECTED EXERCISES

Chapter 1

Section 1.1

1. $\begin{bmatrix} 8 & -4 & 20 \\ 12 & 16 & 4 \end{bmatrix}$ 3. $\begin{bmatrix} 6 & -4 & 24 \\ 8 & 10 & -4 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 4 \\ 0 & 6 \\ -4 & 8 \end{bmatrix}$ 7. $\begin{bmatrix} 3 & -1 & 3 \\ 5 & 7 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{bmatrix}$ 11. $\begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 2 & -4 \end{bmatrix}$

13. $\begin{bmatrix} -3 & 1 & -2 & -4 \\ -1 & -5 & 6 & 2 \end{bmatrix}$ 15. $\begin{bmatrix} -6 & 2 & -4 & -8 \\ -2 & -10 & 12 & 4 \end{bmatrix}$

17. not possible 19. $\begin{bmatrix} 7 & 1 \\ -3 & 0 \\ 3 & -3 \\ 4 & -4 \end{bmatrix}$

21. not possible 23. $\begin{bmatrix} -7 & -1 \\ 3 & 0 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$

25. -2 27. $\begin{bmatrix} 3 \\ 0 \\ 2\pi \end{bmatrix}$

29. $\begin{bmatrix} 2 \\ 2e \end{bmatrix}$ 31. $[2 \quad -3 \quad 0.4]$ 33. $\begin{bmatrix} 150 \\ 150\sqrt{3} \\ 10 \end{bmatrix}$ mph

35. (a) $\begin{bmatrix} 150\sqrt{2}+50 \\ 150\sqrt{2} \end{bmatrix}$ mph

(b) $50\sqrt{37+6\sqrt{2}} \approx 337.21$ mph

37. T	38. T	39. T	40. F	41. F
42. T	43. F	44. F	45. T	46. F
47. T	48. T	49. T	50. F	51. T
52. T	53. T	54. T	55. T	56. T

71. $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 4 \end{bmatrix}$

77. No. Consider $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 6 \\ 5 & 8 \end{bmatrix}$.

79. They must equal 0.

Section 1.2

1. $\begin{bmatrix} 12 \\ 14 \end{bmatrix}$ 3. $\begin{bmatrix} 9 \\ 0 \\ 10 \end{bmatrix}$ 5. $\begin{bmatrix} a \\ b \end{bmatrix}$ 7. $\begin{bmatrix} 22 \\ 5 \end{bmatrix}$

9. $\begin{bmatrix} sa \\ tb \\ uc \end{bmatrix}$ 11. $\begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix}$ 13. $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$ 15. $\begin{bmatrix} 21 \\ 13 \end{bmatrix}$

17. $\frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$

19. $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} 3-\sqrt{3} \\ 3\sqrt{3}+1 \end{bmatrix}$

21. $\frac{1}{2}\begin{bmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} \sqrt{3}-3 \\ 3\sqrt{3}+1 \end{bmatrix}$

23. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ 25. $\frac{1}{2}\begin{bmatrix} 3-\sqrt{3} \\ 3\sqrt{3}+1 \end{bmatrix}$

27. $\frac{1}{2}\begin{bmatrix} 3 \\ -3\sqrt{3} \end{bmatrix}$ 29. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

31. not possible 33. not possible

35. $\begin{bmatrix} -1 \\ 11 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

37. $\begin{bmatrix} 3 \\ 8 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -2 \\ -5 \end{bmatrix}$

39. not possible

41. $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$

43. $\begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

45. T	46. F	47. T	48. T	49. T
50. F	51. F	52. F	53. T	54. F
55. F	56. T	57. F	58. T	59. F
60. T	61. F	62. F	63. T	64. T

69. (a) 349,000 in the city and 351,000 in the suburbs
(b) 307,180 in the city and 392,820 in the suburbs

73. $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

91. (a) $\begin{bmatrix} 24.6 \\ 45.0 \\ 26.0 \\ -41.4 \end{bmatrix}$ (b) $\begin{bmatrix} 134.1 \\ 44.4 \\ 7.6 \\ 104.8 \end{bmatrix}$

5. (a) A basis does not exist because the sum of the multiplicities of the eigenvalues of the standard matrix of T is not 4.

(b) $\left\{\begin{bmatrix}-1\\-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}11\\10\\-3\\-13\\3\end{bmatrix}, \begin{bmatrix}15\\8\\-4\\-15\\1\end{bmatrix}, \begin{bmatrix}5\\10\\0\\-7\\1\end{bmatrix}\right\}$

6. (b) $T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) =$
$\begin{bmatrix}1.5x_1 - 3.5x_2 + 1.0x_3 + 0.5x_4\\ -3.0x_1 + 3.6x_2 - 0.2x_3 + 1.0x_4\\ -16.5x_1 + 22.3x_2 - 3.6x_3 + 5.5x_4\\ 4.5x_1 - 8.3x_2 + 1.6x_3 + 1.5x_4\end{bmatrix}.$

(c) T is not diagonalizable.

7. (a) $T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) =$
$\begin{bmatrix}11.5x_1 - 13.7x_2 + 3.4x_3 - 4.5x_4\\ 5.5x_1 - 5.9x_2 + 1.8x_3 - 2.5x_4\\ -6.0x_1 + 10.8x_2 - 1.6x_3\\ 5.0x_1 - 5.6x_2 + 1.2x_3 - 3.0x_4\end{bmatrix}$

(b) Answers are given correct to 4 places after the decimal point.
$\left\{\begin{bmatrix}3.0000\\2.0000\\1.0000\\1.0000\end{bmatrix}, \begin{bmatrix}2.0000\\1.0000\\-2.0000\\1.0000\end{bmatrix}, \begin{bmatrix}-2.4142\\-8.2426\\-24.7279\\1.0000\end{bmatrix}, \begin{bmatrix}0.4142\\0.2426\\0.7279\\1.0000\end{bmatrix}\right\}$

8. Answers are given correct to 4 places after the decimal point.

(b) $\begin{bmatrix}.2344\\.1934\\.1732\\.2325\\.1665\end{bmatrix},$

(c) $\begin{bmatrix}5.3\\5.2\\5.1\\6.1\\3.3\end{bmatrix}, \begin{bmatrix}5.8611\\4.8351\\4.3299\\5.8114\\4.1626\end{bmatrix}, \begin{bmatrix}5.8610\\4.8351\\4.3299\\5.8114\\4.1625\end{bmatrix}$

(d) $A^{100}\mathbf{p} \approx 25\mathbf{v}$

9. $r_n = (0.2)3^n - 2^n - (0.2)(-2)^n + 4 + 2(-1)^n$

Chapter 6

Section 6.1

1. $\|\mathbf{u}\| = \sqrt{34}$, $\|\mathbf{v}\| = \sqrt{20}$, and $d = \sqrt{58}$
3. $\|\mathbf{u}\| = \sqrt{2}$, $\|\mathbf{v}\| = \sqrt{5}$, and $d = \sqrt{5}$
5. $\|\mathbf{u}\| = \sqrt{11}$, $\|\mathbf{v}\| = \sqrt{5}$, and $d = \sqrt{14}$
7. $\|\mathbf{u}\| = \sqrt{7}$, $\|\mathbf{v}\| = \sqrt{15}$, and $d = \sqrt{26}$

9. 0, yes 11. 1, no 13. 0, yes 15. −2, no

17. $\|\mathbf{u}\|^2 = 20$, $\|\mathbf{v}\|^2 = 45$, $\|\mathbf{u}+\mathbf{v}\|^2 = 65$
19. $\|\mathbf{u}\|^2 = 13$, $\|\mathbf{v}\|^2 = 0$, $\|\mathbf{u}+\mathbf{v}\|^2 = 13$
21. $\|\mathbf{u}\|^2 = 14$, $\|\mathbf{v}\|^2 = 3$, $\|\mathbf{u}+\mathbf{v}\|^2 = 17$
23. $\|\mathbf{u}\|^2 = 14$, $\|\mathbf{v}\|^2 = 138$, $\|\mathbf{u}+\mathbf{v}\|^2 = 152$
25. $\|\mathbf{u}\| = \sqrt{13}$, $\|\mathbf{v}\| = \sqrt{44}$, $\|\mathbf{u}+\mathbf{v}\| = \sqrt{13}$
27. $\|\mathbf{u}\| = \sqrt{20}$, $\|\mathbf{v}\| = \sqrt{10}$, $\|\mathbf{u}+\mathbf{v}\| = \sqrt{50}$
29. $\|\mathbf{u}\| = \sqrt{21}$, $\|\mathbf{v}\| = \sqrt{11}$, $\|\mathbf{u}+\mathbf{v}\| = \sqrt{34}$
31. $\|\mathbf{u}\| = \sqrt{14}$, $\|\mathbf{v}\| = \sqrt{17}$, $\|\mathbf{u}+\mathbf{v}\| = \sqrt{53}$
33. $\|\mathbf{u}\| = \sqrt{13}$, $\|\mathbf{v}\| = \sqrt{34}$, $\mathbf{u}\cdot\mathbf{v} = -1$
35. $\|\mathbf{u}\| = \sqrt{17}$, $\|\mathbf{v}\| = 2$, $\mathbf{u}\cdot\mathbf{v} = -2$
37. $\|\mathbf{u}\| = \sqrt{41}$, $\|\mathbf{v}\| = \sqrt{18}$, $\mathbf{u}\cdot\mathbf{v} = 0$
39. $\|\mathbf{u}\| = \sqrt{21}$, $\|\mathbf{v}\| = \sqrt{6}$, $\mathbf{u}\cdot\mathbf{v} = 5$
41. $\mathbf{w} = \begin{bmatrix}5\\0\end{bmatrix}$ and $d = 0$
43. $\mathbf{w} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$ and $d = \frac{7\sqrt{2}}{2}$
45. $\mathbf{w} = \begin{bmatrix}0.7\\2.1\end{bmatrix}$ and $d = 1.1\sqrt{10}$

49. −3 51. 11 53. 441 55. 21
57. 7 59. 49
61. T 62. F 63. F 64. F 65. F
66. T 67. T 68. T 69. T 70. F
71. F 72. T 73. T 74. T 75. T
76. F 77. T 78. F 79. T 80. T
99. 135° 101. 180° 103. 60° 105. 150°

121. $\overline{v} = \frac{26}{3} \approx 8.6667$ and $v^* = \frac{244}{26} \approx 9.3846$
123. $\overline{v} = v^* = 22$

Section 6.2

1. no 3. no 5. no 7. yes

9. (a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-3\\0\end{bmatrix}\right\}$

(b) $\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$

11. (a) $\left\{\begin{bmatrix}1\\-2\\-1\end{bmatrix}, \begin{bmatrix}9\\3\\3\end{bmatrix}\right\}$

(b) $\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\end{bmatrix}, \frac{1}{\sqrt{11}}\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$