Opgavesæt 17 (SIF)

5,1#3

Vis, at $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ er en egenvektor for $A = \begin{bmatrix} -5 & -4 \\ 8 & 7 \end{bmatrix}$,
dvs. vis, at der findes skalar λ, så

$$A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\begin{bmatrix} -5 & -4 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} -5 \\ 8 \end{bmatrix} - 2 \begin{bmatrix} -4 \\ 7 \end{bmatrix} = \begin{bmatrix} -5 + 8 \\ 8 - 14 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

vi ser, at $\lambda = 3$.

5,1#7

$$\begin{bmatrix} 4 & 6 & -5 \\ 9 & 7 & -11 \\ 8 & 8 & -11 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 4 \\ 9 \\ 8 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} + 1 \begin{bmatrix} -5 \\ -11 \\ -11 \end{bmatrix}$$

$$= \begin{bmatrix} -4 + 12 - 5 \\ -9 + 14 - 11 \\ -8 + 16 - 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

så $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ er egenvektor med egenværdi -3.

5,1#13

hvis $\lambda = 3$ skal være en egenværdi for

$A = \begin{bmatrix} 10 & 7 \\ -14 & -11 \end{bmatrix}$, så skal determinanten af $(A - 3I_2)$ være nul:

$$\det(A - 3I_2) = \det\left(\begin{bmatrix} 10 & 7 \\ -14 & -11 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 7 & 7 \\ -14 & -14 \end{bmatrix} \right)$$

$$= -7 \cdot 14 + 7 \cdot 14 = 0$$

så $\lambda = 3$ <u>er</u> egenværdi!

for at finde en basis for egenrummet $\left(= \mathrm{Null}\left(A - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\right)$

skal vi finde alle løsninger til $(A - 3I_2)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ (på parametriseret vektorform).

$A - 3I_2 = \begin{bmatrix}7 & 7\\-14 & -14\end{bmatrix}$, totalmatrix for er:

$$\left[\begin{array}{cc|c}7 & 7 & 0\\-14 & -14 & 0\end{array}\right] \overset{R_2 \sim R_2 + \frac{14}{7}R_1}{\sim} \left[\begin{array}{cc|c}7 & 7 & 0\\0 & 0 & 0\end{array}\right] \overset{R_1 \sim \frac{1}{7}R_1}{\sim} \left[\begin{array}{cc|c}1 & 1 & 0\\0 & 0 & 0\end{array}\right] = \left[R \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$$

$\left[R \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$ er totalmatrix for lign.syst. $x_1 + x_2 = 0$

(x_2 er fri variabel, sæt $x_2 = s$)

$$\begin{cases} x_1 = -x_2 = -s \\ x_2 = = s \end{cases} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = s\begin{bmatrix}-1\\1\end{bmatrix}, \quad s \in \mathbb{R}$$

Så egenrummet (løsningsrummet for $(A - 3I_2)\vec{v} = \vec{0}$) er lig $\mathrm{span}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) \Rightarrow \mathcal{B} = \left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$ er en basis for egen.rummet.

5,1 #21

$$A = \begin{bmatrix}-13 & -4 & 8\\24 & 7 & -16\\-12 & -4 & 7\end{bmatrix}, \quad \lambda = -1$$

En måde at vise, at λ er egenværdi for A, er at udregne $\det(A - \lambda I_3) = 0$, men i dette tilfælde er det nemmere direkte at vise, at den reducerede trappeform af $A - \lambda I_3 = A + \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ har en nulrække → så finder vi egenrum samtidigt.

$$A - \lambda I_3 = \begin{bmatrix}-12 & -4 & 8\\24 & 8 & -16\\-12 & -4 & 8\end{bmatrix} = B$$

$(A-\lambda I_3)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ har totalmatrix

$$\left[\begin{array}{ccc|c}-12&-4&8&0\\24&8&-16&0\\-12&-4&8&0\end{array}\right]\underset{\sim}{R_3\sim R_3-R_1}\left[\begin{array}{ccc|c}-12&-4&8&0\\24&8&-16&0\\0&0&0&0\end{array}\right]\leftarrow \text{nulrække!}$$

Nu ved vi, at -1 er egenværdi, fordi der kommer til at være en søjle uden pivot $\Rightarrow$ dvs. vi ender med mindst én fri variabel $\Rightarrow$ dvs der findes en ikke-triviel $(\vec{v}\neq 0)$ løsning, så $(A-(-1)I_3)\vec{v}=\vec{0}$.

Egenrum: $\begin{matrix}R_1\sim\frac{1}{4}R_1\\R_2\sim\frac{1}{8}R_2\\\sim\end{matrix}\left[\begin{array}{ccc|c}-3&-1&2&0\\3&1&-2&0\\0&0&0&0\end{array}\right]$

$$\underset{\sim}{R_2\sim R_2+R_1}\left[\begin{array}{ccc|c}-3&-1&2&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$$\underset{\sim}{R_1\sim-\frac{1}{3}R_1}\left[\begin{array}{ccc|c}1&\frac{1}{3}&\frac{2}{3}&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

pivot $\quad x_2$ fri $\quad x_3$ fri

$$\begin{cases}x_1=-\frac{1}{3}x_2-\frac{2}{3}x_3 & =s\left(-\frac{1}{3}\right)+t\left(-\frac{2}{3}\right)\\ x_2=s & =s\cdot 1+t\cdot 0 \qquad s\in\mathbb{R}\\ x_3=t & =s\cdot 0+t\cdot 1 \qquad t\in\mathbb{R}\end{cases}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix}+t\begin{bmatrix}-\frac{2}{3}\\0\\1\end{bmatrix}$$

$$\operatorname{span}\left(\begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{2}{3}\\0\\1\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}-1\\3\\0\end{bmatrix},\begin{bmatrix}-2\\0\\3\end{bmatrix}\right)$$

vi ganger begge med skalar 3, det ændrer ikke span

da $\begin{bmatrix}-1&-2\\3&6\\0&3\end{bmatrix}\sim\dots\sim\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$, er $\mathcal{B}=\left(\begin{bmatrix}-1\\3\\0\end{bmatrix},\begin{bmatrix}-2\\0\\3\end{bmatrix}\right)$

lin. uafh. og dermed en basis for egenrummet for $\lambda=-1$.

5.2 #1

$$A = \begin{bmatrix} 3 & -3 \\ 2 & 8 \end{bmatrix} \qquad \text{karakteristisk pol} = \det\left(A - t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 3-t & -3 \\ 2 & 8-t \end{bmatrix}\right) = (3-t)(8-t) - 2\cdot(-3)$$

$$= t^2 - (3+8)t + 24 + 6$$

$$= t^2 - 11t + 30$$

$$= (t-5)(t-6)$$

(rødderne er givet i opg.)

eigenværdier = løsninger til $(t-5)(t-6)=0$

$t_1 = 5, \quad t_2 = 6$

egenvektorer:

for $t_1 = 5$:

finder løsningsmgd. til $(A - 5I_2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A - 5I_2 = \begin{bmatrix} 3 & -3 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$$

totalmatrix:

$$\left[\begin{array}{cc|c} -2 & -3 & 0 \\ 2 & 3 & 0 \end{array}\right] \underset{\sim}{R_2 \sim R_2 + R_1} \left[\begin{array}{cc|c} -2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$R_1 \sim -\tfrac{1}{2}R_1 \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

pivot $\quad x_2$ fri

$x_1 + \frac{3}{2}x_2 = 0, \quad x_2 = s, \; s \in \mathbb{R}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s \\ 1\cdot s \end{bmatrix} = s\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}$$

$$\text{egenrum} = \operatorname{span}\left(\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$$

at gange med 2 ændrer ikke span!

basis for egenrum for $t_1 = 5$: $\mathcal{B}_1 = \left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$

$t_2 = 6$:

$$A - 6I_2 = \begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix}$$

totalmatrix: $\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 2 & 2 & 0 \end{array}\right] \underset{\sim}{R_2 \sim R_2 + \frac{2}{3}R_1} \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\underset{\sim}{R_1 \sim \frac{-1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

pivot x_2 fri

$$\begin{cases} x_1 = -x_2 = -s \\ x_2 = \text{fri} = s \end{cases}, \quad s \in \mathbb{R}$$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ egenrum: $\operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\Rightarrow \mathcal{B}_2 = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ er en basis for egenrummet hørende til $t_2 = 6$.

5,2#11

A 4x4 matrix, karakt. polyn. $= p(t) = (t-3)(t-4)(t-(-1))^2$

så egenværdierne er: $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = -1$

egenrum for $\lambda_1 = 3$

$$B = \begin{bmatrix} -1 & 4 & -4 & -4 \\ 5 & -2 & 1 & 6 \\ 0 & 0 & -1 & 0 \\ 5 & -5 & 5 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 & -4 & -4 \\ 5 & -5 & 1 & 6 \\ 0 & 0 & -4 & 0 \\ 5 & -5 & 5 & 6 \end{bmatrix}$$

$B\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ har total matrix

$$\left[\begin{array}{cccc|c} -4 & 4 & -4 & -4 & 0 \\ 5 & -5 & 1 & 6 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 5 & -5 & 5 & 6 & 0 \end{array}\right] \underset{\sim}{R_1 \sim \frac{-1}{4}R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 5 & -5 & 1 & 6 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 5 & -5 & 5 & 6 & 0 \end{array}\right]$$

$$R_2 \sim R_2 - 5R_1$$
$$R_4 \sim R_4 - 5R_1$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 \Leftrightarrow R_3$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & \end{array}\right]$$

$$R_3 \Leftrightarrow R_4$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 & 0 \end{array}\right]$$

$$R_4 \sim R_4 - R_2 \qquad \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_4 \sim R_4 - R_3$$
$$R_2 \sim -1/4\, R_2$$
$$R_1 \sim R_1 - R_3 \qquad \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \sim R_1 - R_2 \qquad \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} \uparrow & & \uparrow & \uparrow & \\ p & x_2 \text{ fri} & p & p & \end{array} \end{array}$$

som er tol. matrix for

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ x_2 \text{ fri} \end{cases} \Rightarrow \begin{array}{l} x_1 = x_2 = s \\ x_2 = = s \\ x_3 = 0 = 0 \cdot s \\ x_4 = 0 = 0 \cdot s \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{egenrum} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$