Kædereglen

Vi betragter funktionen

$$w = f(x_1, x_2, \ldots, x_m), \qquad \text{hvor} \qquad \begin{cases} x_1 &= g_1(t_1, t_2, \ldots, t_n) \\ x_2 &= g_2(t_1, t_2, \ldots, t_n) \\ \vdots & \\ x_m &= g_m(t_1, t_2, \ldots, t_n). \end{cases}$$

Forudsat alle ovennævnte funktioner har kontinuerte første ordens partielle afledede, gælder følgende (kædereglen):

$$\boxed{\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_m} \cdot \frac{\partial x_m}{\partial t_i},}$$

hvor $i = 1, 2, \ldots, n$.

Implicit funktionssætningen

Antag, at funktionen $F(x_1, x_2, \ldots, x_n, z)$ er kontinuert differentiabel i en omegn af punktet $(\mathbf{a}, b) = (a_1, a_2, \ldots, a_n, b)$, hvor

$$F(\mathbf{a}, b) = 0 \quad \text{og} \quad \left.\frac{\partial F}{\partial z}\right|_{(\mathbf{a},b)} \neq 0.$$

Så eksisterer der en kontinuert differentiabel funktion $z = g(x_1, x_2, \ldots, x_n)$ således $g(\mathbf{a}) = b$ og $F(\mathbf{x}, g(\mathbf{x})) = 0$ for $\mathbf{x}$ i en omegn af $\mathbf{a}$.

En anvendelse

Vi anvender nu kædereglen på:

$$F(x_1, x_2, \ldots, x_n, z) = 0, \qquad \text{hvor} \qquad \begin{cases} x_1 & = x_1 \\ x_2 & = x_2 \\ \vdots & \\ x_n & = x_n \\ z & = g(x_1, x_2, \ldots, x_n). \end{cases}$$

Vi får:

$$0 = \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_i} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x_i}.$$

Implicit differentiation

Fra identiteten

$$\frac{\partial x_j}{\partial x_i} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

ser vi, at

$$0 = \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_i} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x_i}.$$

$$\Downarrow$$

$$0 = \frac{\partial F}{\partial x_i} \cdot 1 + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x_i} \Rightarrow \frac{\partial z}{\partial x_i} = -\frac{\partial F / \partial x_i}{\partial F / \partial z}.$$

Dvs.

$$\boxed{\frac{\partial z}{\partial x_i} = -\frac{F_{x_i}}{F_z}.}$$