Walsh Type Wavelet Packet Expansions

Morten Nielsen

Department of Mathematics, University of South Carolina, South Carolina 29208

E-mail: nielsen@math.sc.edu

We consider a family of basic non-stationary wavelet packets generated using the Haar filters except for a finite number of scales where we allow the use of arbitrary filters. Such a system, which we call a system of Walsh type wavelet packets, can be considered as a smooth generalization of the Walsh functions. We show that the basic Walsh type wavelet packets share a number of metric properties with the Walsh system. We prove that the system constitutes a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, and we construct an explicit function in $L^1(\mathbb{R})$ for which the expansion fails. Then we prove that expansions of $L^p(\mathbb{R})$-functions, $1 < p < \infty$, in the Walsh type wavelet packets converge pointwise a.e. Finally, we prove that the analogous results are true for periodic Walsh type wavelet packets in $L^p[0, 1)$.

Key Words: Wavelet analysis, non-stationary wavelet packets, Walsh functions, L^p-convergence, convergence a.e.

1. INTRODUCTION

Wavelet analysis has provided a new class of orthogonal expansions in $L^2(\mathbb{R})$ with good time-frequency and regularity-approximation properties which have been successfully applied to signal processing, numerical analysis and quantum mechanics. Wavelet packet analysis extends such an orthogonal wavelet expansion to a whole library of orthogonal expansions with different time-frequency properties which can be searched for the best expansion with respect to some predetermined criteria. This adaptive approach have advantages over both wavelet and short-time Fourier analysis in applications where both transient and stationary phenomena are present. However, unlike the wavelet case, the properties of such orthonormal bases in other spaces than $L^2(\mathbb{R})$ have not been studied extensively. The focus in this paper will be on basic wavelet packet expansions of L^p-functions defined on the real line and on the unit interval, respectively. The most elementary example of a system of basic stationary wavelet packets is the Walsh system which is known to be a Schauder basis for $L^p[0, 1)$, $1 < p < \infty$ ([9]). An even stronger result proved by Billard and Sjölin is that the Walsh expansion of a given $L^p[0, 1)$-function converges pointwise a.e. ([1, 10]). However, it turns out that such nice convergence results can fail for more complicated basic stationary wavelet packets. The author proves in [8] that the stationary wavelet packets generated by the Daubechies filers of length $4, 6, \dots,$

respectively, fail to be a Schauder basis for $L^p(\mathbb{R})$ for p large regardless of the ordering of the functions. Thus, smooth basic stationary wavelet packets are not the "right" generalization of the Walsh functions within the wavelet packet framework if we are interested in L^p-convergence.

In the present paper we introduce a family of smooth basic non-stationary wavelet packets for which the same type of L^p-convergence results are true as for the Walsh system. Each member of the family is generated using the Haar filters except for a finite number of steps where we allow arbitrary filters.

The type of non-stationary wavelet packets we will use was introduced in [4]. The structure in which non-stationary wavelet packets live is that of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ (for the definition and properties, see e.g. [2, 7]). To every multiresolution analysis we have an associated scaling function ϕ and a wavelet ψ with the properties that

$$V_j = \overline{\text{span}}\{2^{j/2}\phi(2^j\cdot -k)|k\in\mathbb{Z}\},$$

and

$$\{\psi_{j,k}\equiv 2^{j/2}\psi(2^j\cdot -k)|j,k\in\mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. We denote $W_j = \overline{\text{span}}\{2^{j/2}\psi(2^j\cdot -k)|k\in\mathbb{Z}\}$.

We let $(F_0^{(p)}, F_1^{(p)})$, $p\in\mathbb{N}$, be a family of bounded operators on $\ell^2(\mathbb{Z})$ of the form

$$(F_\varepsilon^{(p)}a)_k = \sum_{n\in\mathbb{Z}} a_n h_\varepsilon^{(p)}(n-2k), \qquad \varepsilon = 0,1$$

with $h_1^{(p)}(n) = (-1)^n h_0^{(p)}(1-n)$ a real-valued sequence in $\ell^1(\mathbb{Z})$ such that

$$\begin{aligned} F_0^{(p)*}F_0^{(p)} + F_1^{(p)*}F_1^{(p)} &= 1 \\ F_0^{(p)}F_1^{(p)*} &= 0. \end{aligned}$$

We define the family of functions $\{w_n\}_{n=0}^\infty$ recursively by letting $w_0=\phi$, $w_1=\psi$ and then for $n\in\mathbb{N}$

$$w_{2n}(x) = 2\sum_{q\in\mathbb{Z}} h_0^{(p)} w_n(2x-q) \tag{1}$$

$$w_{2n+1}(x) = 2\sum_{q\in\mathbb{Z}} h_1^{(p)}(q) w_n(2x-q), \tag{2}$$

where $2^p \le n < 2^{p+1}$. The family $\{w_n\}_{n=0}^\infty$ is our basic non-stationary wavelet packets. It is proved in [4] that

$$\{w_n(\cdot -k)|n\ge 0, k\in\mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Moreover,

$$\{w_n(\cdot -k)|2^j\le n<2^{j+1}, k\in\mathbb{Z}\}$$

is an orthonormal basis for $W_j = \overline{\text{span}}\{2^{j/2}w_1(2^j\cdot -k)|k\in\mathbb{Z}\}$.

Each pair $(F_0^{(p)}, F_1^{(p)})$ can be chosen as a pair of quadrature mirror filters associated with a multiresolution analysis, but this is not necessary. The trigonometric polynomials given by

$$m_0^{(p)}(\xi) = \frac{1}{\sqrt{2}} \sum_k h_0^{(p)}(k) e^{-ik\xi} \quad \text{and} \quad m_1^{(p)}(\xi) = \frac{1}{\sqrt{2}} \sum_k h_1^{(p)}(k) e^{-ik\xi}$$

are called the symbols of the filters. The Fourier transform of (1) is given by

$$\hat{w}_{2n}(\xi) = m_0^{(p)}\left(\frac{\xi}{2}\right) \hat{w}_n\left(\frac{\xi}{2}\right),$$

and (2) becomes

$$\hat{w}_{2n+1}(\xi) = m_1^{(p)}\left(\frac{\xi}{2}\right) \hat{w}_n\left(\frac{\xi}{2}\right).$$

The Haar low-pass quadrature mirror filter $\{h_0(k)\}_k$ is given by $h_0(0) = h_0(1) = 1/\sqrt{2}$, $h_0(k) = 0$ otherwise, and the associated high-pass filter $\{h_1(k)\}_k$ is given by $h_1(k) = (-1)^k h_0(1-k)$. We now give the definition of the family of non-stationary wavelet packets we will consider

DEFINITION 1.1. Let $\{w_n\}_{n\geq 0, k\in\mathbb{Z}}$ be a family of non-stationary wavelet packets constructed by using a family $\{h_0^{(p)}(n)\}_{p=1}^{\infty}$ of finite filters for which there is a constant $K \in \mathbb{N}$ such that $h_0^{(p)}(n)$ is the Haar filter for every $p \geq K$. If $w_1 \in C^1(\mathbb{R})$ and it has compact support then we call $\{w_n\}_{n\geq 0}$ a family of Walsh type wavelet packets.

We call such functions (basic) Walsh type wavelet packets since it turns out that they share a number of metric properties with the Walsh system and they can therefore be considered a smooth generalization of the Walsh system. In section 3 we prove that Walsh type wavelet packets do form a Schauder basis for $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover, we prove that the Walsh type wavelet packets are equivalent in $L^p(\mathbb{R})$, $1 < p < \infty$, to the integer translates of the Walsh system. In section 4 we prove that the Schauder basis property is no longer true for $p = 1$ by constructing an explicit counterexample. As mentioned above, Billard and Sjölin [1, 10] proved that the Walsh-Fourier expansion of any $f \in L^p[0,1)$, $1 < p < \infty$, converges a.e. to f. In section 5 we prove that the same is true for Walsh type wavelet packet expansions for $L^p(\mathbb{R})$-functions. The final sections are devoted to generalizing the results to periodic versions of the Walsh type wavelet packets.

Remark. One may wonder if it is possible to modify Definition 1.1 slightly and use only, say, the Daubechies filter of length 4 from scale K in place of the Haar filter and still obtain a Schauder basis for $L^p(\mathbb{R})$. This is unfortunately not so which follows from the results proved in [8]. Such functions are bound to fail as a basis for $L^p(\mathbb{R})$ for p large.

2. WALSH FUNCTIONS. DEFINITION AND PROPERTIES

This section contains a brief review of the properties of the Walsh system we need. All the results are well known and the proofs can be found in [3].

We need two equivalent definitions of the Walsh system on $[0, 1)$. The first one fits into the wavelet packet scheme

DEFINITION 2.1. The Walsh system $\{W_n\}_{n=0}^\infty$ is defined recursively on $[0,1)$ by letting $W_0 = \chi_{[0,1)}$ and

$$W_{2n}(x) = W_n(2x) + W_n(2x-1)$$
$$W_{2n+1}(x) = W_n(2x) - W_n(2x-1).$$

We note that the Walsh system is the family of wavelet packets obtained by letting $\phi = \chi_{[0,1)}$, $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, and using only the Haar filters in the definition.

It turns out that the Walsh system is closed under pointwise multiplication, but this is hard to verify using Definition 2.1. An alternative definition of the Walsh system can be given in terms of the Rademacher functions. Consider the function

$$r_0(x) = \begin{cases} 1 & \text{for } x \in [0,1/2), \\ -1 & \text{for } x \in [1/2,1). \end{cases}$$

Extend r_0 to the real line by periodizing with period 1 and define $r_n(x) = r_0(2^n x)$. Then the Walsh system can be obtained by taking all possible finite products of the Rademacher functions. More precisely, for $n = \sum_{i=0}^\infty n_i 2^i \in \mathbb{N}_0$ the binary expansion of $n \in \mathbb{N}_0$, we define

$$w_n(x) = \prod_{i=0}^\infty (r_i(x))^{n_i} \chi_{[0,1)}(x).$$

To see that the definitions agree, we just have to note that $w_0 = \chi_{[0,1)}$ and, using the properties of the Rademacher functions,

$$w_{2n}(x) = w_n(2x) + w_n(2x-1)$$
$$w_{2n+1}(x) = w_n(2x) - w_n(2x-1),$$

i.e $W_n \equiv w_n$ for $n \in \mathbb{N}_0$. Using the multiplicative definition, it follows easily that the Walsh system is closed under pointwise multiplication. In fact, define the binary operator $\dot{+} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by

$$m \dot{+} n = \sum_{i=0}^\infty |m_i - n_i| 2^i,$$

where $m = \sum_{i=0}^\infty m_i 2^i$ and $n = \sum_{i=0}^\infty n_i 2^i$. Then

$$W_m(x) W_n(x) = W_{m \dot{+} n}(x). \tag{3}$$

Moreover, (3) shows that the Walsh functions are characters for the group of all binary sequences (indexed by $\mathbb{N}_0$) under bitwise addition. We also note that the Walsh functions W_n with $n < 2^J$ are constant on intervals of the form $[k2^{-J}, (k+1)2^{-J})$, $0 \le k < 2^J$.

The specific results we need for Section 4 are:

THEOREM 2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be the Lebesgue constants for the Walsh system defined by*

$$L_n = \int_0^1 \left| \sum_{k=0}^{n-1} W_k(x) \right| dx.$$

Define n_k, $k \in \mathbb{N}$, *by* $n_{2s} = \sum_{i=0}^{s} 2^{2i}$, *and* $n_{2s+1} = \sum_{i=0}^{s} 2^{2i+1}$, *then for all* $k \in \mathbb{N}$

$$L_{n_k} > \frac{1}{2}\left(\frac{k}{2} + 1\right),$$

LEMMA 2.1. *Let*

$$D_n(x) = \sum_{k=0}^{n-1} W_k(x).$$

Then

$$D_{2^k}(x) = 2^k \chi_{[0,2^{-k})}(x),$$

and

LEMMA 2.2. *Let* $f_1 \in L^2(\mathbb{R})$, *and define* $\{f_n\}_{n\geq 2}$ *recursively by*

$$f_{2n+\varepsilon}(x) = f_n(2x) + (-1)^{\varepsilon} f_n(2x-1), \qquad \varepsilon = 0, 1.$$

Then for $n, J \in \mathbb{N}$, $2^J \leq n < 2^{J+1}$, *we have*

$$f_n(x) = \sum_{s=0}^{2^J-1} W_{n-2^J}(s2^{-J}) f_1(2^J x - s).$$

Proof. The proof is by induction on n. First, note that for $n = 2, 3$,

$$\begin{aligned} f_2(x) &= f_1(2x) + f_1(2x-1) = W_0(0) f_1(2x) + W_0(1/2) f_1(2x-1), \\ f_3(x) &= f_1(2x) - f_1(2x-1) = W_1(0) f_1(2x) + W_1(1/2) f_1(2x-1), \end{aligned}$$

and for the inductive step observe that

$$\begin{aligned} f_{2[1\varepsilon_{J-1}\cdots\varepsilon_1]_2+\varepsilon}(x) &= f_{[1\varepsilon_{J-1}\cdots\varepsilon_1]_2}(2x) + (-1)^{\varepsilon} f_{[1\varepsilon_{J-1}\cdots\varepsilon_1]_2}(2x-1) \\ &= \sum_{s=0}^{2^{J-1}-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1]_2}(s2^{-(J-1)}) f_1(2^J x - s) \\ &\quad + (-1)^{\varepsilon} \sum_{s=0}^{2^{J-1}-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1]_2}(s2^{-(J-1)}) f_1(2^J x - 2^{J-1} - s), \end{aligned}$$

and using (2),

$$= \sum_{s=0}^{2^J-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1\varepsilon]}(s2^{-J}) f_1(2^J x - s).$$

∎

Remark. The matrix $H \in \mathbb{R}^{2^J \times 2^J}$ defined by

$$H_{i,j} = 2^{-J/2} W_i(j2^{-J}), \qquad i,j = 0,1,\ldots 2^J - 1,$$

is called the Hadamard Transform, and it follows from the previous lemma that the expansion coefficients of wavelet packets generated by Haar filters can be expressed in terms of this transform.

For the pointwise convergence of Walsh type wavelet packet expansions we need a theorem by Billard and Sjölin [1, 10]. We remind the reader that an operator T defined on $L^p(\mathbb{R})$ is of strong type (p,p) if it is sublinear and there is a constant C such that $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(\mathbb{R})$. The result by Billard and Sjölin is

THEOREM 2.2. *Let $f \in L^1[0,1)$ and define*

$$S_n(x,f) = \sum_{k=0}^{n} \int_0^1 f(t) W_k(t)\, dt\, W_k(x).$$

Then the operator G defined by

$$Gf(x) = \sup_n |S_n(x,f)|$$

is of strong type (p,p) for $1 < p < \infty$.

Remark. The operator G is often referred to as the Carleson operator for the Walsh system.

3. WALSH TYPE WAVELET PACKETS

Let $\{w_n\}$ be a family of Walsh type wavelet packets. We would like to prove that $\{w_n(\cdot - k) | n \geq 0, k \in \mathbb{Z}\}$ constitutes a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, that is equivalent to the Walsh basis $\{W_n(\cdot - k) | n \geq 0, k \in \mathbb{Z}\}$. To prove the equivalence with the Walsh system we need to generalize the following well known theorem [7]:

THEOREM 3.1. *Let $\psi \in C^1(\mathbb{R})$ be a compactly supported wavelet. Then there exists an isomorphism on $L^p(\mathbb{R})$ taking $\psi_{j,k}$ to $h_{j,k}$, with h the Haar wavelet.*

The generalization we need is the following

LEMMA 3.1. *Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets with K as in Definition 1.1, and let $\{W_n\}$ be the Walsh system. Let $f_{j,k}^n = 2^{j/2} w_n(2^j \cdot -k)$, and $g_{j,k}^n = 2^{j/2} W_n(2^j \cdot -k)$. Then there is an isomorphism $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for $1 < p < \infty$ such that*

$$Qf_{j,k}^n = g_{j,k}^n, \qquad j,k \in \mathbb{Z}, 2^K \leq n < 2^{K+1}.$$

Proof. Let $\{W_n^s\}_n$ be a family of non-stationary wavelet packets generated by taking any compactly supported $C^1(\mathbb{R})$ scaling function and associated wavelet (ϕ, ψ), and letting each $h^{(p)}$ be the Haar filter in definition of the non-stationary wavelet packets. Let $v_{j,k}^n = 2^{j/2} W_n^s(2^j \cdot -k)$. For each $n \geq 1, 2^j \leq n < 2^{j+1}$, we have

$$W_n = \sum_{s \in F} c_{n,s} h_{j,s}$$
$$W_n^s = \sum_{s \in F} c_{n,s} \psi_{j,s},$$

with F a finite set of indices. Then, for $2^K \leq n < 2^{K+1}$,

$$g_{p,k}^n = \sum_{s \in F} c_{n,s} h_{p+K, 2^K k + s}$$
$$v_{p,k}^n = \sum_{s \in F} c_{n,s} \psi_{p+K, 2^K k + s}.$$

Let $P : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the isomorphism defined by $Ph_{j,k} = \psi_{j,k}$. It follows that $Pg_{p,k}^n = v_{p,k}^n$. Hence, it suffices to find an isomorphism $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ such that $Qf_{j,k}^n = v_{j,k}^n$. Note that

$$\{f_{j,k}^n\}_{2^K \leq n < 2^{K+1}, j,k \in \mathbb{Z}}, \quad \text{and} \quad \{v_{j,k}^n\}_{2^K \leq n < 2^{K+1}, j,k \in \mathbb{Z}}$$

are both orthonormal bases for $L^2(\mathbb{R})$ (easy consequence of the multiresolution structure). Thus, Q defined by $Qf_{j,k}^n = v_{j,k}^n$, $2^K \leq n < 2^{K+1}, j, k \in \mathbb{Z}$, is unitary. The associated (Schwartz) kernel is given by

$$K(x, y) = \sum_{n=2^K}^{2^{K+1}-1} \sum_{j,k \in \mathbb{Z}} v_{j,k}^n(x) \, \overline{f_{j,k}^n(y)}.$$

We claim that K is a Calderón-Zygmund kernel. To verify this we choose $N \geq 1$ such that $\text{supp}(W_n^s), \text{supp}(w_n) \subset [-N, N]$ for $2^K \leq n < 2^{K+1}$. We have

$$|K(x, y)| \leq \sum_{n=2^K}^{2^{K+1}-1} \sum_{j,k \in \mathbb{Z}} 2^j |W_n^s(2^j x - k)| |w_n(2^j y - k)|.$$

Thus $(x, y) \in \text{supp}(K)$ implies that $|2^j x - k| \leq N$ and $|2^j y - k| \leq N$. Hence, $2^j |x - y| \leq 2N$ so

$$j \leq \log_2 \frac{2N}{|x - y|}.$$

Let

$$j_0 = \left\lfloor \log_2 \frac{2N}{|x - y|} \right\rfloor.$$

We have

$$\begin{aligned}|K(x,y)| &\le \sum_{n=2^K}^{2^{K+1}-1} \sum_{j\le j_0} 2^j\,(2N+1)\,\|W_n^s\|_\infty \|w_n\|_\infty\\ &\le C2^K\,(2N+1)\sum_{j\le j_0} 2^j \le \frac{2^{K+1}N(2N+1)C}{|x-y|}.\end{aligned}$$

Similar estimates give us

$$\begin{aligned}\Big|\frac{\partial}{\partial x}K(x,y)\Big| &\le \frac{C}{|x-y|^2}\\ \Big|\frac{\partial}{\partial y}K(x,y)\Big| &\le \frac{C}{|x-y|^2}.\end{aligned}$$

It follows that Q is a Calderón-Zygmund operator and thus bounded on $L^p(\mathbb{R})$, $1<p<\infty$ (see [6]). The same type of argument applies to Q^{-1} (the above estimates are symmetric in $f_{j,k}^n$ and $v_{j,k}^n$) and Q is therefore an isomorphism on $L^p(\mathbb{R})$. ■

We can now state and prove the main result of this section, the Walsh type wavelet packets do constitute a Schauder basis for $L^p(\mathbb{R})$ for $1<p<\infty$.

THEOREM 3.2. *Let $\{w_n\}_{n\ge 0}$ be a family of Walsh type wavelet packets with K as in Definition 1.1. Then $\{w_n(\cdot-k)\}_{n\ge0,k\in\mathbb{Z}}$ is a Schauder basis for $L^p(\mathbb{R})$, $1<p<\infty$.*

Proof. It is clear that the system

$$\{w_n(\cdot-k)\}_{n\ge0,k\in\mathbb{Z}}$$

is dense in $L^p(\mathbb{R})$ for $1<p<\infty$ since the associated wavelets $2^{j/2}\psi(2^jx-k)$, $j\ge0$, and the translates of the scaling function are all finite linear combinations of the wavelet packets. Hence, it suffices to prove that there exists a finite constant (depending on p) such that for any sequence $(c_{n,k})\subset\mathbb{C}$ and $M,N\ge1$ we have

$$\Big\|\sum_{0\le n\le N,|k|\le N} c_{n,k}w_n(\cdot-k)\Big\|_p \le C\Big\|\sum_{0\le n\le N+M,|k|\le M+N} c_{n,k}w_n(\cdot-k)\Big\|_p \tag{4}$$

We split the proof of (4) into several cases. We need to prove two intermediary results. First, we claim that the systems

$$\{w_n(\cdot-k)\}_{n\ge 2^{K+1},k\in\mathbb{Z}} \quad\text{and}\quad \{W_n(\cdot-k)\}_{n\ge 2^{K+1},k\in\mathbb{Z}} \tag{5}$$

are equivalent in $L^p(\mathbb{R})$ in the sense that there is an isomorphism Q on $L^p(\mathbb{R})$ mapping one system onto the other. Let $n\ge 2^{K+1}$. Note that

$$\hat{w}_n(\xi)=\prod_{j=1}^{K} m_{\varepsilon_j}(2^{-j}\xi)\cdot\hat{w}_{\tilde{n}}(2^{-K}\xi),$$

for some $2^K\le\tilde{n}<2^{K+1}$ and $K\ge1$. Thus

$$w_n(x-k)=\sum_{s\in F} c_{n,s} f_{K,s}^{\tilde{n}}(x-k), \tag{6}$$

with $f_{j,k}^{\tilde{n}} = 2^{j/2} w_{\tilde{n}}(2^j \cdot -k)$ and F a finite set (depending on n). The coefficients $c_{n,s}$ depend only on n and the Haar filter. Thus, W_n has the same expansion:

$$W_n(x-k) = \sum_{s \in F} c_{n,s} g_{K,s}^{\tilde{n}}(x-k), \tag{7}$$

with $g_{j,k}^{\tilde{n}} = 2^{j/2} W_{\tilde{n}}(2^j \cdot -k)$. Let $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the isomorphism defined by $Q f_{j,k}^n = g_{j,k}^n$. It follows from (6) and (7) that

$$Q w_n(\cdot - k) = W_n(\cdot - k), \qquad n \geq 2^{K+1}, k \in \mathbb{Z},$$

which proves (5). Secondly, let us check that

$$\{w_n(\cdot - k)\}_{0 \leq n < 2^{K+1}, k \in \mathbb{Z}}$$

is a Schauder basis for its closed linear span in $L^p(\mathbb{R})$. The kernel for the projection, $P_{N,M}$, onto

$$\{w_n(\cdot - k)\}_{0 \leq n \leq N < 2^{K+1}, |k| \leq M}$$

is given by

$$K_{N,M}(x,y) = \sum_{n=0}^{N} \sum_{|k| \leq M} w_n(x-k) \overline{w_n(y-k)}.$$

Fix K such that supp$(w_n) \subset [-K, K]$ for $0 \leq n < 2^{K+1}$. Then

$$\begin{aligned} |K_{N,M}(x,y)| &\leq \sum_{n=0}^{2^{K+1}-1} \sum_{k \in \mathbb{Z}} |w_n(x-k)||w_n(y-k)| \\ &\leq 2^{K+1}(2K+1) \max_{0 \leq n < 2^{K+1}} \{\|w_n\|_\infty^2\} \cdot \chi_{[0,2K]}(|x-y|). \end{aligned}$$

Hence, using Hölder's inequality and Fubini's Theorem, with $p^{-1} + q^{-1} = 1$,

$$\begin{aligned} \|P_{N,M} f\|_p^p &= \int \Big| \int K_{N,M}(x,y)\, f(y)\, dy \Big|^p dx \\ &\leq \int \Big(\int |f(y)||K_{N,M}(x,y)|^{1/p} \cdot |K_{N,M}(x,y)|^{1-1/p}\, dy \Big)^p dx \\ &= \int \int |f(y)|^p |K_{N,M}(x,y)|\, dy \Big(\int |K_{N,M}(x,y)|\, dy \Big)^{p/q} dx \\ &\leq C^{p/q} \int |f(y)|^p \int |K_{N,M}(x,y)|\, dx\, dy \\ &\leq C^{1+p/q} \|f\|_p^p, \end{aligned}$$

which proves the claim.

We now use the above results to show that (4) holds. We have already proved that whenever $M + N < 2^{K+1}$ then (4) holds. Suppose $N < 2^{K+1}$ and $M + N \geq 2^{K+1}$. The projection P_j onto V_j is bounded on $L^p(\mathbb{R})$, $1 \leq p < \infty$, provided that the scaling function has some decay at infinity, eg. $|\phi(x)| \leq C(1+|x|)^{-2}$ (see [7]), so it is certainly true for our compactly supported scaling function w_0. So

$$\begin{aligned}\Big\| \sum_{0\leq n\leq N, |k|\leq N} c_{n,k} w_n(\cdot - k)\Big\|_p &\leq C \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p \\ &\leq C\, C_1 \Big\| \sum_{0\leq n\leq N+M, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p.\end{aligned}$$

Finally, suppose $N \geq 2^{K+1}$. Then, using (5), the result for $N+M < 2^{K+1}$, the Schauder basis properties of the Walsh system, and $\|f\|_p \simeq \{\|P_{V_{K+1}} f\|_p + \|(1-P_{V_{K+1}})f\|_p\}$,

$$\begin{aligned}&\Big\| \sum_{0\leq n\leq N, |k|\leq N} c_{n,k} w_n(\cdot - k)\Big\|_p \\ &\leq \Big\{ \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq N} c_{n,k} w_n(\cdot - k)\Big\|_p + \Big\| \sum_{2^{K+1}\leq n\leq N, |k|\leq N} c_{n,k} w_n(\cdot - k)\Big\|_p \Big\} \\ &\leq C \Big\{ \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\ &\qquad\qquad \Big\| \sum_{2^{K+1}\leq n\leq N, |k|\leq N} c_{n,k} w_n(\cdot - k)\Big\|_p \Big\} \\ &\leq C \Big\{ \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\ &\qquad\qquad \Big\| \sum_{2^{K+1}\leq n\leq N, |k|\leq N} c_{n,k} W_n(\cdot - k)\Big\|_p \Big\} \\ &\leq C \Big\{ \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\ &\qquad\qquad \Big\| \sum_{2^{K+1}\leq n\leq N+M, |k|\leq M+N} c_{n,k} W_n(\cdot - k)\Big\|_p \Big\} \\ &\leq C \Big\{ \Big\| \sum_{0\leq n< 2^{K+1}, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\ &\qquad\qquad \Big\| \sum_{2^{K+1}\leq n\leq N+M, |k|\leq M+N} c_{n,k} w_n(\cdot - k)\Big\|_p \Big\} \\ &\leq C \Big\| \sum_{0\leq n\leq N+M, |k|\leq N+M} c_{n,k} w_n(\cdot - k)\Big\|_p.\end{aligned}$$

We conclude that (4) holds in general, and we are done. ■

We extract the following corollary from the above proof.

COROLLARY 3.1. *Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets with K as in Definition 1.1. If $w_1 \in C^1(\mathbb{R})$ then there exists an isomorphism $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$,*

$1 < p < \infty$, *such that*

$$Qw_n(\cdot - k) = W_n(\cdot - k), \qquad n \geq 0, k \in \mathbb{Z}.$$

Remark. The corollary shows that there exist positive constants c_p, C_p such that

$$c_p \leq \|w_n\|_p \leq C_p, \qquad n \in \mathbb{N},$$

so it follows from Orlicz's Theorem ([12, p. 60]) that the only $L^p(\mathbb{R})$-space where the Walsh type wavelet packets form an unconditional basis is $L^2(\mathbb{R})$.

4. A COUNTEREXAMPLE IN $L^1(\mathbb{R})$

It is interesting to know what happens in the "limit case" $L^1(\mathbb{R})$ of Theorem 3.2. It is well known that the exponentials $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ and the Walsh system fail to be a basis for $L^1[0,1)$ whereas the periodic wavelets do form a Schauder basis for $L^1[0,1)$ so it can go both ways. The next theorem provides an explicit function in $L^1(\mathbb{R})$ for which the expansion in the Walsh type wavelet packets fail. The construction is adapted from a counterexample for the Walsh system in [3].

THEOREM 4.1. *Let $\{w_n\}_n$ be a family of Walsh type wavelet packet system, and let K be defined as in Definition 1.1. Choose $L \in \mathbb{N}$ such that supp$(w_{2^K+1}) \subset [-L+1, L-1]$ and choose $M \in \mathbb{N}$ such that $2^M > 2L$. Let $N(k) = k^3 + M + 1$, and define $K : \mathbb{N} \to \mathbb{N}$ recursively by letting $K(1) = 2^K + 1$, $K(2n) = 2K(n)$, and $K(2n+1) = 2K(n) + 1$. Define f by*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} w_{K(n)}(x) \right).$$

Then $f \in L^1(\mathbb{R})$, but the wavelet packet expansion of f diverges in $L^1(\mathbb{R})$-norm.

Proof. Same as for the periodic case (Theorem 7.1). ■

5. POINTWISE CONVERGENCE FOR WALSH TYPE WAVELET PACKET EXPANSIONS

In this section, we prove that Walsh type wavelet packet expansions for $L^p(\mathbb{R})$-functions, $1 < p < \infty$, converge pointwise almost everywhere. Let $\{w_n\}_n$ be a family of Walsh type wavelet packets. We let

$$(Lf)(x) = \sup_{N \geq 1} \left| \sum_{n \leq N, |k| \leq N} \langle f, w_n(\cdot - k) \rangle w_n(x-k) \right|, \qquad f \in L^p(\mathbb{R}),\ 1 < p < \infty.$$

We call L the Carleson operator for the Walsh type wavelet packet system. The following result shows that the Carleson operator is well-behaved.

THEOREM 5.1. *The Carleson operator for any Walsh type wavelet packet system is of strong type (p,p), $1 < p < \infty$.*

Proof. Let us start by simplifying the problem. Choose $N \in \mathbb{N}$ such that $\mathrm{supp}(w_n) \subset [-N, N]$ for $n \geq 0$. Fix $p \in (1, \infty)$ and take any

$$f(x) = \sum_{n \geq 0, k \in \mathbb{Z}} c_{n,k} w_n(x-k) \in L^p(\mathbb{R}).$$

Define

$$f_k(x) = \sum_{n \geq 0} c_{n,k} w_n(x-k), \qquad g_k(x) = \sum_{n \geq 0} c_{n,k} W_n(x-k).$$

We have $\|f_k\|_p \simeq \|g_k\|_p$, with bounds independent of k by Corollary 3.1. Note that for $l \in \mathbb{Z}$

$$|\{x \in [l, l+1) : |Lf(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \sum_{k=l-N}^{l+1+N} \int |Lf_k(x)|^p \, dx$$

so (using the Marcinkiewicz interpolation theorem) it suffices to prove that $\|Lf_k\|_p \leq C\|f_k\|_p$, where C is a constant independent of k, since

$$\sum_{l \in \mathbb{Z}} \sum_{k=l-N}^{l+1+N} \|f_k\|_p^p \leq 2(N+1) \sum_{k \in \mathbb{Z}} \|f_k\|_p^p \leq 2C(N+1) \sum_{k \in \mathbb{Z}} \|g_k\|_p^p \leq \tilde{C} \|f\|_p^p.$$

We can, w.l.o.g., assume that $k = 0$. Let $K \in \mathbb{N}$ be the scale from which only the Haar filter is used to generate the wavelet packets $\{w_n\}_{n \geq 2^{K+1}}$. Let $m \in \mathbb{N}$ and suppose $2^J \leq m < 2^{J+1}$ for some $J > K+1$. Clearly, for each $x \in \mathbb{R}$,

$$\sum_{n=0}^{m} c_{n,0} w_n(x) = \sum_{n=0}^{2^{K+1}-1} c_{n,0} w_n(x) + \sum_{n=2^{K+1}}^{2^J-1} c_{n,0} w_n(x) + \sum_{n=2^J}^{m} c_{n,0} w_n(x),$$

so we have

$$\begin{aligned} \sup_{m \geq 1} \left| \sum_{n=0}^{m} c_{n,0} w_n(x) \right| &\leq \sup_{1 \leq m < 2^{K+1}} \left| \sum_{n=0}^{m} c_{n,0} w_n(x) \right| \\ &\quad + \sup_{J > K+1} \left| \sum_{n=2^{K+1}}^{2^J-1} c_{n,0} w_n(x) \right| + \sup_{J > K+1} (M_J f_0)(x), \end{aligned} \tag{8}$$

where

$$(M_J f_0)(x) = \sup_{2^J \leq m < 2^{J+1}} \left| \sum_{n=2^J}^{m} c_{n,0} w_n(x) \right|.$$

We use brute force to estimated the first term of (8)

$$\begin{aligned} \sup_{0 < m < 2^{K+1}} \left| \sum_{n=0}^{m} c_{n,0} w_n(x) \right| &\leq \sum_{n=0}^{2^{K+1}-1} |c_{n,0}| \|w_n(x)\|_\infty \chi_{[-N,N]}(x) \\ &\leq \|f_0\|_p \sum_{n=0}^{2^{K+1}-1} \|w_n\|_q \|w_n(x)\|_\infty \chi_{[-N,N]}(x). \end{aligned}$$

The second term of (8) satisfies

$$\Big\| \sup_{J>K+1} \Big| \sum_{n=2^{K+1}}^{2^J-1} c_{n,0} w_n(x) \Big| \Big\|_p \le C \|f_0\|_p$$

since the dyadic partial sums for the wavelet packet expansion for f_0 agree everywhere with the partial sums for the wavelet expansion for f_0 and the Carleson operator for the wavelet expansion is of strong type (p,p) (see [11]). The challenge is to prove that the third term is of type (p,p). Fix $x \in \mathbb{R} - \mathcal{D}$, where $\mathcal{D}$ is the family of dyadic rationals. Note that

$$(M_J f_0)(x) \le \sum_{j=0}^{2^K-1} (M_J^j f_0)(x),$$

where

$$(M_J^j f_0)(x) = \sup_{2^J+j2^{J-K} \le m < 2^J+(j+1)2^{J-K}} \Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big|,$$

so it suffices to prove that

$$\| \sup_{J>K+1} (M_J^j f_0) \|_p \le C \|f_0\|_p$$

for $j = 0, 1, \dots 2^K - 1$. Fix $J > K+1$, $0 \le j \le 2^K - 1$, and $2^J + j2^{J-K} \le m < 2^J + (j+1)2^{J-K}$. We have, using Lemma 2.2,

$$\begin{aligned} &\Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big| \\ &\qquad = \Big| \sum_{s=0}^{2^{J-K}-1} \Big\{ \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} W_{n-2^J-j2^{J-K}}(s2^{-J+K}) \Big\} w_{2^K+j}(2^{J-K}x - s) \Big|. \end{aligned}$$

Define

$$F_m(t) = \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} W_{n-2^J-j2^{J-K}}(t), \quad \text{and} \quad F(t) = \sup_{m<2^J+(j+1)2^{J-K}} |F_m(t)|.$$

Then

$$\Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big| \le \sum_{s=0}^{2^{J-K}-1} F(s2^{-J+K}) |w_{2^K+j}(2^{J-K}x - s)|,$$

and using the compact support of the wavelet packets,

$$\Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big| \le \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} F((\lfloor 2^{J-K}x \rfloor + l)2^{-J+K}).$$

Note that F is constant on dyadic intervals of type $[l2^{-J+K},(l+1)2^{-J+K})$ so by letting $\Delta_l=((\lfloor 2^{J-K}x\rfloor+l)2^{-J+K},(\lfloor 2^{J-K}x\rfloor+l+1)2^{-J+K})$ we obtain

$$\begin{aligned}\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| &\le \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} F((\lfloor 2^{J-K}x\rfloor+l)2^{-J+K})\\ &= \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1}\int_{\Delta_l} F(t)\,dt.\end{aligned}$$

We need an estimate of F that does not depend on J. Note that for k, $0\le k<2^{J-K}$, using (3),

$$W_{2^J+j2^{J-K}}(t)W_k(t)=W_{2^J+j2^{J-K}+k}(t)$$

since the binary expansions of 2^J+j2^{J-K} and of k have no 1's in common. Hence,

$$|F_m(t)|=|W_{2^J+j2^{J-K}}(t)F_m(t)|=\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}W_n(t)\Big|$$

so $F(t)\le 2(Gg_0)(t)$ with G the Carleson operator for the Walsh system. Thus,

$$\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| \le 2\|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1}\int_{\Delta_l} (Gg_0)(t)\,dt.$$

We let $\Delta_l^\star$ be the smallest dyadic interval containing Δ_l and x, and note that $|\Delta_l^\star|\le(N+1)|\Delta_l|$ since $x\in\Delta_0$ (here we use $x\notin\mathcal{D}$). We have

$$\begin{aligned}\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| &\le 2\|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1}\int_{\Delta_l^\star} (Gg_0)(t)\,dt\\ &\le 4\|w_{2^K+j}\|_\infty (N+1)^2(MGg_0)(x), \qquad (9)\end{aligned}$$

where M is the maximal operator of Hardy and Littlewood. The righthand side of (9) does not depend on m nor J so we may conclude that

$$\sup_{J>K+1}(M_J^j f_0)(x)\le 4\|w_{2^K+j}\|_\infty(N+1)^2(MGg_0)(x), \quad \text{a.e.,}$$

and thus, since M and G are both of strong type (p,p) (see [10]),

$$\|\sup_{J>K+1}(M_J^j f_0)\|_p\le C\|g_0\|_p\le C_1\|f_0\|_p, \qquad j=0,1,\dots 2^K-1,$$

and we are done. ∎

The pointwise convergence result now follows by a standard argument (see [3])

THEOREM 5.2. *The Walsh type wavelet packet expansion of any* $f\in L^p(\mathbb{R})$, $1<p<\infty$, *converges a.e.*

6. PERIODIC WALSH TYPE WAVELET PACKETS

Y. Meyer proves in [7] that by periodizing any (reasonable) orthonormal wavelet basis associated with a multiresolution analysis one obtains an orthonormal basis for $L^2[0,1)$. The periodization works equally well with non-stationary wavelet packets,

DEFINITION 6.1. Let $\{w_n\}_{n=0}^{\infty}$ be a family of non-stationary basic wavelet packets satisfying $|w_n(x)| \le C_n(1+|x|)^{-1-\varepsilon_n}$ for some $\varepsilon_n > 0$, $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$ we define the corresponding periodic wavelet packets $\widetilde{w_n}$ by

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} w_n(x-k).$$

Note that the hypothesis about the pointwise decay of the wavelet packets w_n ensures that the associated periodic wavelet packets are well defined functions contained in $L^p[0,1)$ for every $p \in [1,\infty]$. Moreover, the family $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[0,1)$ ([5]).

We can apply the periodization to any family of Walsh type wavelet packets. The next theorem shows that the periodic Walsh type wavelet packets do constitute a Schauder basis for $L^p[0,1)$, $1 < p < \infty$. Such functions can also be considered a generalization of the Walsh functions.

THEOREM 6.1. *Let $\{w_n\}_n$ be a family of Walsh type wavelet packets. Then the associated periodic system $\{\widetilde{w_n}\}_n$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$.*

Proof. We claim that the periodized system $\{\widetilde{w_n}\}_n$ is dense in $L^p[0,1)$. To verify the claim we let P_N be the projection onto the closed linear span of $\{\widetilde{w_n}\}_{n=0}^{N}$. By the construction of the periodic wavelet packets we have $P_{2^N-1} = P_{\widetilde{V}_N}$, where $\widetilde{V}_N$ is the periodized version of the multiresolution space V_N. But $P_{\widetilde{V}_N} f \to f$ for $f \in L^p[0,1)$ and the claim follows. So it suffices to prove that $\sup_N \|P_N\|_{L^p[0,1)\to L^p[0,1)} < \infty$. Suppose not. Note that each P_n is bounded on $L^p[0,1)$ since its kernel is bounded on $[0,1)^2$, so by the Banach-Steinhaus Theorem there exists $f \in L^p[0,1)$ such that

$$\sup_N \|P_N f\|_{L^p[0,1)} = \infty. \tag{10}$$

According to the proof of Theorem 3.2 there exists a constant C_p such that

$$\Big\| \sum_{n=0}^{N} \langle g, w_n(\cdot - k)\rangle w_n(\cdot - k) \Big\|_p \le C_p \|g\|_p \tag{11}$$

for every $N \ge 1$, $k \in \mathbb{Z}$, and every $g \in L^p(\mathbb{R})$. Fix K such that $\mathrm{supp}(w_n) \subset [-K,K]$ for $n \ge 0$. Then, for $x \in [0,1)$

$$\widetilde{w_n}(x) = \sum_{k=-K}^{K+1} w_n(x-k). \tag{12}$$

Choose N such that

$$\Big\| \sum_{n=0}^{N} \langle f, \widetilde{w_n} \rangle \widetilde{w_n} \Big\|_{L^p[0,1)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)}, \tag{13}$$

which is possible by (10). We substitute (12) into (13);

$$\Big\| \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\{ \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\} \Big\|_{L^p([0,1),\, dy\,)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)}.$$

By Minkowski's inequality

$$\begin{aligned} \Big\| \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\{ \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\} \Big\|_{L^p([0,1),\, dy\,)} \\ \leq \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\| \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p([0,1),\, dy\,)} \end{aligned}$$

so we can find k_1 and k_2 such that

$$\begin{aligned} C_p\|f\|_{L^p[0,1)} &= C_p \|\chi_{[0,1)} f\|_{L^p(\mathbb{R})} \\ &< \Big\| \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p([0,1),\, dy\,)} \\ &\leq \Big\| \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p(\mathbb{R},\, dy\,)} \\ &= \Big\| \sum_{n=0}^{N} \int_{\mathbb{R}} \{\chi_{[0,1)}(x) f(x)\}\overline{w_n(x-k_1)}\, dx\, w_n(y-k_1) \Big\|_{L^p(\mathbb{R},\, dy\,)}, \end{aligned}$$

which contradicts (11). Hence, our assumption that $\sup_N \|P_N\|_{L^p[0,1)\to L^p[0,1)} = \infty$ is wrong and we are done. ■

7. A COUNTEREXAMPLE IN $L^1[0,1)$

This section contains the analog to the counterexample of Theorem 4.1, the expansion in the periodic Walsh type wavelet packets fails in $L^1[0,1)$.

THEOREM 7.1. *Let $\{w_n\}_n$ be a family of Walsh type wavelet packets and let K be defined as in Definition 1.1. Choose $L \in \mathbb{N}$ such that supp$(w_{2^K+1}) \subset [-L+1, L-1]$ and choose $M \in \mathbb{N}$ such that $2^M > 2L$. Let $N(k) = k^3 + M + 1$, and define $K : \mathbb{N} \to \mathbb{N}$ recursively by letting $K(1) = 2^K + 1$, $K(2n) = 2K(n)$, and $K(2n+1) = 2K(n) + 1$. Define f by*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \Bigg(\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \widetilde{w_{K(n)}}(x) \Bigg).$$

Then $f \in L^1[0,1)$, but the periodic wavelet packet expansion of f diverges in $L^1[0,1)$-norm.

Proof. Let $1 \le 2^j \le n < 2^{j+1}$. It is clear from the construction of K and the recursive definition of the wavelet packets that $w_{K(n)} \in \text{span}\{w_{K(1)}(2^j x - k)\}_k$ and the expansion coefficients are given by the expansion coefficients of W_n is the Haar wavelets $h(2^j x - k)$. From this observation and Lemma 2.2

$$w_{K(n)}(x) = \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) w_{K(1)}(2^j x - s)$$

so

$$\begin{aligned}\widetilde{w_{K(n)}}(x) &= \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) \sum_{r\in\mathbb{Z}} w_{K(1)}(2^j x - 2^j r - s)\\ &\equiv \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) g_{j,s}(x).\end{aligned}$$

Lemma 2.1 shows that for $x \in [0,1)$

$$\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} W_{n-2^{N(k)}}(x) = \sum_{n=2^{k^3}}^{2^{k^3+1}-1} W_n(x) = W_{2^{k^3}}(x) 2^{k^3} \chi_{[0,2^{-k^3})}(x)$$

from which we obtain

$$\begin{aligned}&\int_0^1 \left| \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \widetilde{w_{K(n)}}(x) \right| dx\\ &\qquad = \int_0^1 \left| \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \left\{ \sum_{s=0}^{2^{N(k)}-1} W_{n-2^{N(k)}}(s2^{-N(k)}) g_{N(k),s}(x) \right\} \right| dx\\ &\qquad = 2^{k^3} \int_0^1 \left| \sum_{s=0}^{2^{N(k)-k^3}-1} W_{2^{k^3}}(s2^{-N(k)}) g_{N(k),s}(x) \right| dx\\ &\qquad \le 2^{k^3} \sum_{s=0}^{2^{N(k)-k^3}-1} \int_0^1 \left| W_{2^{k^3}}(s2^{-N(k)}) g_{N(k),s}(x) \right| dx\\ &\qquad \le 2^{k^3} 2^{N(k)-k^3} 2^{-N(k)} \|w_{K(1)}\|_{L^1(\mathbb{R})}\\ &\qquad = \|w_{K(1)}\|_{L^1(\mathbb{R})}.\end{aligned}$$

It follows that

$$\|f\|_{L^1[0,1)} \le \sum_{k=1}^{\infty} \frac{1}{k^2} \left\| \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \widetilde{w_{K(n)}} \right\|_{L^1[0,1)} \le \|w_{K(1)}\|_{L^1(\mathbb{R})} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

so $f \in L^1[0,1)$. We now have to show that the basic Walsh type wavelet packet expansion of f diverges in $L^1[0,1)$-norm. Define the sequence a_n by

$$a_n = \begin{cases} \frac{1}{k^2} & \text{for } 2^{N(k)} + 2^{k^3} \le n < 2^{N(k)} + 2^{k^3+1}, \quad k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{n=0}^{\infty} a_n \widetilde{w_{K(n)}}$. Define n_j by

$$n_{2j} = \sum_{i=0}^{j} 2^{2i} \text{ and } n_{2j+1} = \sum_{i=0}^{j} 2^{2i+1},$$

and note that $n_{k^3-2} < 2^{k^3}$ for $k \in \mathbb{N}$. Let us estimate the difference between the following two partial sums of f

$$\begin{aligned}
&\int_0^1 \left| \sum_{n=0}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} a_n \widetilde{w_{K(n)}}(x) - \sum_{n=0}^{2^{N(k)}+2^{k^3}-1} a_n \widetilde{w_{K(n)}}(x) \right| dx \\
&= \frac{1}{k^2} \int_0^1 \left| \sum_{s=0}^{2^{N(k)}-1} \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} W_{n-2^{N(k)}}(s2^{-N(k)}) g_{N(k),s}(x) \right| dx \\
&= \frac{1}{k^2} \int_0^1 \left| \sum_{s=0}^{2^{N(k)}-1} W_{2^{k^3}}(s2^{-N(k)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(s2^{-N(k)}) g_{N(k),s}(x) \right| dx,
\end{aligned}$$

where we used (3) to get the last equality. Using the fact that W_n is constant on $[p2^{-(k^3+1)}, (p+1)2^{-(k^3+1)})$ for $n \le 2^{k^3}$,

$$\begin{aligned}
= \frac{1}{k^2} \int_0^1 \Bigg| \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \\
\sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x) \Bigg| dx
\end{aligned}$$

Define $\{I_p\}_{p=0}^{2^{k^3+1}-1} \subset [0,1)$ by

$$\begin{aligned}
I_p &= \{x | 2^{N(k)}x \in [p2^{((N(k)-k^3-1))} + L, (p+1)2^{(N(k)-k^3-1)} - L]\} \\
&= [p2^{-(k^3+1)} + 2^{-N(k)}L, (p+1)2^{-(k^3+1)} - 2^{-N(k)}L].
\end{aligned}$$

Suppose $x \in I_l$. Consider

$$\sum_{r\in\mathbb{Z}} \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} w_{K(1)}(2^{N(k)}x - 2^{N(k)}r - p2^{(N(k)-k^3-1)} - s). \tag{14}$$

Note that

$$\begin{aligned}
&2^{N(k)}x - 2^{N(k)}r - p2^{(N(k)-k^3-1)} - s \\
&\in [(l-p)2^{(N(k)-k^3-1)} + L - r2^{N(k)} - s, (l+1-p)2^{(N(k)-k^3-1)} - L - r2^{N(k)} - s].
\end{aligned}$$

Using that $p \in [0, 2^{k^3+1} - 1]$ and $s \in [0, 2^{N(k)-k^3-1} - 1]$ we get the bounds

$$\begin{aligned}(l-p)2^{(N(k)-k^3-1)} + L - r2^{N(k)} - s &\geq -2^{N(k)} - r2^{N(k)} + L + 1\\ (l+1-p)2^{(N(k)-k^3-1)} - L - r2^{N(k)} - s &\leq 2^{N(k)} - L - r2^{N(k)}\end{aligned}$$

from which we deduce that it is only the terms with $r = 0$ that contribute to (14) since $\text{supp}(w_{K(1)}) \subset [-L+1, L-1]$. A similar argument using the definition of I_l, the fact that $2^{(N(k)-k^3-1)} = 2^M > 2L$, and the compact support of $w_{K(1)}$, shows that for $x \in I_l$

$$\sum_{r\in\mathbb{Z}} \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} w_{K(1)}(2^{N(k)}x - 2^{N(k)}r - p2^{N(k)-k^3-1} - s) \\ = \begin{cases} \sum_{r\in\mathbb{Z}} w_{K(1)}(2^{N(k)}x - r) & \text{for } p = l \\ 0 & \text{for } p \neq l. \end{cases}$$

Hence,

$$\begin{aligned}&\int_{I_l} \Bigg| \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \\ &\qquad \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x)\Bigg|\,dx \\ &\qquad = \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)})\Bigg| \int_{I_l} \Bigg| \sum_{r\in\mathbb{Z}} w_{K(1)}(2^{N(k)}x - r)\Bigg|\,dx \\ &\qquad = \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)})\Bigg| 2^{-N(k)} \int_{2^{N(k)}I_l} \Big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r)\Big|\,dx \\ &\qquad = \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)})\Bigg| 2^{-N(k)} \int_{l2^M+L}^{(l+1)2^M-L} \Big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r)\Big|\,dx \\ &\qquad = \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)})\Bigg| 2^{-N(k)}(2^M - 2L) \int_0^1 \Big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r)\Big|\,dx\end{aligned}$$

Finally, we use the following fact about the Lebesgue constants for the Walsh system (see Theorem 2.1)

$$\int_0^1 \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(x)\Bigg|\,dx = 2^{-(k^3+1)} \sum_{l=0}^{2^{k^3+1}-1} \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)})\Bigg| > \frac{1}{2}\left(\frac{k^3-2}{2}+1\right)$$

to obtain the estimate we want

$$
\begin{aligned}
\int_0^1 \Bigg| \sum_{n=0}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} a_n \widetilde{w_{K(n)}}(x) - \sum_{n=0}^{2^{N(k)}+2^{k^3}-1} a_n \widetilde{w_{K(n)}}(x) \Bigg| \, dx \\
&\geq \frac{1}{k^2} \sum_{l=0}^{2^{k^3+1}-1} \int_{I_l} \Bigg| \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \cdot \\
&\qquad \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x) \Bigg| \, dx \\
&= \frac{1}{k^2} \sum_{l=0}^{2^{k^3+1}-1} \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Bigg| 2^{-N(k)}(2^M - 2L) \int_0^1 \Bigg| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \Bigg| \, dx \\
&= \frac{1}{k^2}(2^M - 2L) \int_0^1 \Big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \Big| \, dx \, 2^{-M} \int_0^1 \Bigg| \sum_{n=0}^{n_{k^3-2}-1} W_n(x) \Bigg| \, dx \\
&> \frac{1}{k^2}(2^M - 2L) \int_0^1 \Bigg| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \Bigg| \, dx \, 2^{-M} \frac{1}{2} \left(\frac{k^3-2}{2} + 1 \right) \\
&\geq Ck
\end{aligned}
$$

for some $C > 0$. We conclude that the sequence of partial sums

$$P_{K(2^{N(k)}+2^{k^3}+n_{k^3-2}-1)} f - P_{K(2^{N(k)}+2^{k^3}-1)} f$$

diverges in $L^1[0,1)$ as $k \to \infty$. This proves the Theorem. ■

8. POINTWISE CONVERGENCE FOR PERIODIC WALSH TYPE WAVELET PACKET EXPANSIONS

We have the following consequence of the proof of Theorem 5.1.

THEOREM 8.1. *The periodic Walsh type wavelet packet expansion of any $f \in L^p[0,1)$, $1 < p < \infty$, converges a.e.*

Proof. Let $f \in L^p[0,1)$, and define N as in the proof of Theorem 5.1. Note that

$$\sum_{n=0}^{m} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x) = \sum_{k_1=-N}^{N+1} \sum_{k_2=-N}^{N+1} \left\{ \sum_{n=0}^{m} \int_0^1 f(y) \overline{w_n(y-k_1)} \, dy w_n(x-k_2) \right\},$$

so it follows at once from the proof of Theorem 5.1 that the Carleson operator associated with the periodic Walsh type wavelet packets is of strong type (p,p) for $1 < p < \infty$. ■

ACKNOWLEDGMENT

The author would like to acknowledge that this work was done under the direction of M. Victor Wickerhauser. The author would also like to thank three anonymous referees for several useful remarks.

REFERENCES

1. P. Billard. Sur la convergence presque partout des séries de Fourier-Walsh des fontions de l'espace $L^2(0,1)$. *Studia Math.*, 28:363–388, 1967.
2. I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.
3. B. Golubov, A. Efimov, and V. Skvortsov. *Walsh Series and Transforms*. Kluwer Academic Publishers, 1991.
4. N. Hess-Nielsen. Control of frequency spreading of wavelet packets. *Appl. and Comp. Harmonic Anal.*, 1:157–168, 1994.
5. N. Hess-Nielsen and M. V. Wickerhauser. Wavelets and time-frequency analysis. *Proceedings of the IEEE*, 84(4):523–540, 1996.
6. Y. Meyer. *Ondelettes et Opérateurs II: Opérateurs de Calderón-Zygmund*. Hermann, 1990.
7. Y. Meyer. *Wavelets and Operators*. Cambridge University Press, 1992.
8. M. Nielsen. *Size Properties of Wavelet Packets*. PhD thesis, Washington University, St. Louis, 1999.
9. R.E.A.C. Paley. A remarkable system of orthogonal functions. *Proc. Lond. Math. Soc.*, 34:241–279, 1932.
10. P. Sjölin. An inequality of Paley and convergence a.e. of Walsh-Fourier series. *Arkiv för Matematik*, 7:551–570, 1969.
11. T. Tao. On the almost everywhere convergence of wavelet summation methods. *Appl. and Comp. Harmonic Anal.*, 3:384–387, 1996.
12. P. Wojtaszczyk. *Banach spaces for analysts*. Cambridge University Press, 1991.