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On Stability of Finitely Generated Shift-Invariant Systems

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Abstract We consider the problem of completely characterizing when a system of integer translates in a finitely generated shift-invariant subspace of $L_2(\mathbb{R}^d)$ is stable in the sense that rectangular partial sums for the system are norm convergent. We prove that a system of integer translates is stable in $L_2(\mathbb{R}^d)$ precisely when its associated Gram matrix satisfies a suitable Muckenhoupt A_2 condition.

Keywords Shift-invariant space · Schauder basis · Integer translates · Vector Hunt-Muckenhoupt-Wheeden theorem · Muckenhoupt condition

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1 Introduction

A finitely generated shift-invariant (FSI) subspaces of $L_2(\mathbb{R}^d)$ is a subspace $S \subset L_2(\mathbb{R}^d)$ for which there exists a finite family Φ of $L_2(\mathbb{R}^d)$-functions such that

$$S = S(\Phi) := \overline{\operatorname{span}\{\varphi(\cdot - k) : \varphi \in \Phi, k \in \mathbb{Z}^d\}}.$$

FSI subspaces are used in several applications. Wavelets and other multi-scale methods are based on FSI subspaces [4, 6, 14], and FSI subspaces play an important role in multivariate approximation theory such as spline approximation [5] and approximation with radial basis functions [9, 19]. The fundamental structure of FSI spaces has been studied in a number of papers, see for example [1, 7, 8, 20]. Let us also mention the classical results on translates of functions by Kolmogoroff [16] and Helson [13].

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For many applications it is useful to have a stable generating set for S. Given the structure of S, it is natural to consider generating sets of integer translates. That is, a system with the following structure,

$$\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}, \tag{1.1}$$

where $\Psi = \{\psi_1, \dots, \psi_N\} \subset L_2(\mathbb{R}^d)$ is a finite subset. Often we take $\Psi = \Phi$, but Ψ may be different from Φ, and the two sets need not have the same cardinality but we always require that $S(\Psi) = S(\Phi)$.

We focus on the case where (1.1) has a unique bi-orthogonal system in S, i.e., there exist $\{g_k^{\psi}\} \subset S$ such that

$$\langle g_j^{\tilde{\psi}}, \psi(\cdot - k)\rangle = \delta_{\psi,\tilde{\psi}}\delta_{k,j}, \quad \psi, \tilde{\psi} \in \Psi;\, j, k \in \mathbb{Z}^d.$$

For such systems, we can define the "rectangular" partial sum operators by

$$T_{\mathbf{N}} f := \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}^d : |k_i| \le N_i} \langle f, g_k^{\psi}\rangle \psi(\cdot - k), \tag{1.2}$$

for $f \in S$ and $\mathbf{N} = (N_1, N_2, \dots, N_d) \in \mathbb{N}_0^d$, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The main result of this paper, which is stated in Theorem 1.2 below, completely characterizes when we have norm convergence

$$T_{\mathbf{N}} f \to f, \quad \text{as } \min_i N_i \to +\infty, \text{ for all } f \in S(\Psi). \tag{1.3}$$

For example, whenever (1.1) forms a Riesz basis for $S(\Psi)$, (1.3) clearly holds true. However, as Theorem 1.2 will show, we can have convergence in much more general cases where (1.1) fails to be a Riesz basis. It is known that Riesz basis properties of (1.1) can be completely characterized in terms of the Gram matrix for the system Ψ. In fact, (1.1) forms a Riesz basis for $S(\Psi)$ precisely when the spectrum of the Gram matrix for the system Ψ is bounded and bounded away from zero, see [8]. Therefore, it is only natural to expect that the convergence (1.3) can be characterized in terms of the Gram matrix for Ψ.

There is, in fact, one *very* restricted case where the convergence (1.3) has already been characterized. It was proved in [17] that in the univariate case (i.e., $d = 1$) with one generator (i.e., $N = 1$), (1.3) holds precisely when the Gram matrix (which is a scalar function in this case) is a Muckenhoupt A_2 weight. The characterization basically boils down to an application of the celebrated Hunt-Muckenhoupt-Wheeden Theorem [15]. The restricted case indicates that some type of Muckenhoupt condition on the Gram matrix is needed in order to obtain the wanted convergence characterization in the general case. We introduce the needed generalized Muckenhoupt condition in Definition 1.1 below.

Let us now state the main result of this paper precisely. First, we introduce some notation. We define the Fourier transform by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi}\, dx, \quad f \in L_2(\mathbb{R}^d). \tag{1.4}$$

The Gram matrix for $\Psi = \{\psi_1, \dots, \psi_N\}$ is the Hermitian positive semi-definite $N \times N$-matrix $W := W(\Psi)$ given by

$$W(\Psi) = \left(\sum_{k \in \mathbb{Z}^d} \hat{\psi}_i(\cdot - k), \overline{\hat{\psi}_j(\cdot - k)} \right)_{i,j=1}^{N}. \tag{1.5}$$

The Gram matrix is an example of a matrix weight. In general, we say that $W : \mathbb{T}^d \to \mathbb{C}^{N \times N}$, $\mathbb{T}^d = [-1/2, 1/2)^d$, is a matrix weight if it is a measurable function whose values are positive semi-definite $N \times N$-matrices.

To deal with the problem at hand, we introduce the following subclass of matrix weights. Some examples of such weights can be found in Sect. 5.

Definition 1.1 Let W be a $N \times N$ matrix weight on $\mathbb{T}^d$, i.e., a periodic measurable function defined on $\mathbb{T}^d$ whose values are positive semi-definite $N \times N$ matrices. We say that W satisfies the Muckenhoupt product A_2-matrix-condition provided that

$$\sup_R \left\| \left(\frac{1}{|R|} \int_R W d\xi \right)^{1/2} \left(\frac{1}{|R|} \int_R W^{-1} d\xi \right)^{1/2} \right\| < \infty, \tag{1.6}$$

where the sup is over all rectangles $R = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$. The collection of all such weights is denoted $\mathbb{PA}_2(d)$.

As far as we know, this is the first time that a product Muckenhoupt A_2-condition for matrix weights has been considered. However, non-product Muckenhoupt conditions have been considered by several authors in the matrix setting. The matrix A_2 condition on $\mathbb{T}$ was put into prominence by Treil and Volberg in their seminal papers [23, 24] where they generalized the Hunt-Muckenhoupt-Wheeden theorem to vector-valued functions. More recently, matrix Muckenhoupt conditions on $\mathbb{R}^d$, where the sup in (1.6) is taken over cubes and not rectangles, have been considered in [10, 11, 21] in order to study vector-valued singular integral operators and to construct vector-valued weighted Besov spaces. The product A_2-condition for scalar weights has a long history, see [3] and references therein. Schauder bases for Gabor systems were characterized in terms of scalar A_2 product conditions by Heil and Powell in [12].

We can now state the main result of this paper.

Theorem 1.2 *Let $S(\Psi)$ be a FSI space in $L_2(\mathbb{R}^d)$ for which Ψ has a bi-orthogonal system. Then the following conditions are equivalent.*

(a) $T_{\mathbf{N}} f \to f$, *as* $\min N_i \to +\infty$, *for all* $f \in S(\Psi)$
(b) $\{T_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}_0^d}$ *is a uniformly bounded family of operators on* S
(c) *The Gram matrix* $W(\Psi)$ *is in the Muckenhoupt class* $\mathbb{PA}_2(d)$.

It is completely straightforward to verify that conditions (a) and (b) are equivalent in Theorem 1.2. The main difficulty is to prove that (b) and (c) are equivalent. This will follow directly from Theorem 3.3, which will be proved in Sect. 3.

The structure of the paper is as follows. In Sect. 2 we explore the connection between FSI subspaces and weighted vector-valued L_2-spaces, and we characterize when (1.1) has a bi-orthogonal system in $S(\Psi)$. Section 3 is devoted to studying Fourier partial sum operators in the vector-valued setting, which through the Fourier transform gives an equivalent approach to Theorem 1.2. The main result of Sect. 3 is Theorem 3.3 that gives a vector-valued Hunt-Muckenhoupt-Wheeden type result for rectangular partial sums. In Sect. 4 we consider an application of Theorem 1.2 to the problem of obtaining Schauder bases for FSI spaces. It is proved that provided $W \in \mathbb{PA}_2(d)$, then we can find an enumeration of the system (1.1) that respects the rectangular partial sums considered in Theorem 1.2 and turns (1.1) into a Schauder basis for $S(\Psi)$. Section 5 contains a number of examples of $\mathbb{PA}_2(d)$ weights and related FSI subspaces.

2 Finitely Generated Shift Invariant Systems

In this section we explore the connection between FSI subspaces and weighted vector-valued L_2-spaces, and we characterize when (1.1) has a bi-orthogonal system in $S(\Psi)$. The main tool to study expansions in $S(\Psi)$ is the Fourier transform. As before, we assume that some ordering $\Psi = \{\psi_1, \psi_2, \dots, \psi_N\}$ has been imposed.

Following [8], we introduce the so-called bracket product given by

$$[f, g] : \mathbb{T}^d \to \mathbb{C} : x \to \sum_{k \in \mathbb{Z}^d} f(x+k)\overline{g(x+k)},$$

for $f, g \in L_2(\mathbb{R}^d)$. With this setup, we have the fundamental identity

$$\langle f, g\rangle_{L_2(\mathbb{R}^d)} = \langle \hat{f}, \hat{g}\rangle_{L_2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} [\hat{f}, \hat{g}]\, d\xi, \qquad f, g, \in L_2(\mathbb{R}^d). \tag{2.1}$$

Let us now consider a finite expansion in $S(\Psi)$

$$f = \sum_{\ell=1}^{N} \sum_{k \in \mathbb{Z}^d} c_{\ell,k} \psi_\ell(\cdot - k),$$

relative to the system (1.1). An application of the Fourier transform yields

$$\hat{f}(\xi) = \sum_{\ell=1}^{N} \left(\sum_{k \in \mathbb{Z}^d} c_{\ell,k} e^{-2\pi i k\cdot\xi} \right) \hat{\psi}_\ell := \sum_{\ell=1}^{N} \tau_\ell(\xi) \hat{\psi}_\ell. \tag{2.2}$$

The periodic functions τ_ℓ are not necessarily uniquely determined by f, but we can nevertheless calculate the norm of $\hat{f}$ using the bracket product and (2.1). We form the vector $\tau = [\tau_\ell]_{\ell=1}^N$, and we let τ^H denote the Hermitian transpose of τ. We obtain

$$\|\hat{f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} [\hat{f}, \hat{f}] = \int_{\mathbb{T}^d} \sum_{i=1}^{N} \sum_{j=1}^{N} [\hat{\psi}_i, \hat{\psi}_j] \tau_i(\xi) \overline{\tau_j(\xi)}\, d\xi$$

$$= \int_{\mathbb{T}^d} \tau(\xi)^H W(\xi) \tau(\xi)\, d\xi, \tag{2.3}$$

where $W := W(\Psi)$ is the Hermitian positive semi-definite $N \times N$-matrix given by

$$W(\Psi) = \big([\hat{\psi}_i, \hat{\psi}_j]\big)_{i,j=1}^N. \tag{2.4}$$

W is known as the Grammian matrix associated with Ψ. Notice that the Cauchy-Schwarz inequality shows that each entry in W is contained in $L_1(\mathbb{T}^d)$ since $\Psi \subset L_2(\mathbb{R}^d)$.

Let us introduce the vector-valued weighted space

$$L_2(\mathbb{T}^d; W) := \left\{ f : \mathbb{T}^d \to \mathbb{C}^N : \|f\|^2_{L_2(\mathbb{T}^d; W)} := \int_{\mathbb{T}^d} |W^{1/2}(\xi) f(\xi)|^2\, d\xi < \infty \right\}.$$

We need to factorize over $\mathcal{N} := \{g : \|g\|_{L_2(\mathbb{T}^d, W)} = 0\}$ in order to turn $L_2(\mathbb{T}^d; W)$ into a Hilbert space. However, it should be noted that we will mainly use this space in the case where W is positive a.e. For such weights, $\mathcal{N}$ only contains vector functions that vanish a.e.

The analysis so-far shows that the map $U : L_2(\mathbb{T}^d; W) \to S(\Psi)$ given by

$$U(\tau) := \left(\sum_{\ell=1}^N \tau_\ell(\xi) \hat{\psi}_\ell \right)^\vee \tag{2.5}$$

is an isometric isomorphism between $L_2(\mathbb{T}^d; W)$ and $S(\Psi)$.

Of special interest to our analysis is the trigonometric system

$$\{e^{-2\pi i k\cdot\xi} \mathbf{e}_j\}_{k \in \mathbb{Z}^d, j=1,\ldots,N},$$

where $\mathbf{e}_j$, $j = 1, \ldots, N$, is the standard basis for $\mathbb{C}^N$. Notice that $U(e^{-2\pi i k\cdot\xi} \mathbf{e}_j) = \psi_j(\cdot - k)$. Below we will use the isomorphism U to study metric properties of the shift invariant system (1.1) in terms of equivalent metric properties of $\{e^{-2\pi i k\cdot\xi} \mathbf{e}_j\}_{k \in \mathbb{Z}^d, j=1,\ldots,N}$ in $L_2(\mathbb{T}^d; W)$.

We begin by characterizing when (1.1) has a unique bi-orthogonal system in $S(\Psi)$. This turns out to be exactly when $W^{-1} \in L_1$, giving an extension of the scalar result obtained in [17]. We have the following proposition.

Proposition 2.1 *Let $S(\Psi)$, $\Psi = \{\psi_1, \ldots, \psi_N\}$, be a FSI space. The sequence*

$$\{\psi_j(\cdot - k) | k \in \mathbb{Z}^d, j = 1, \ldots, N\}$$

has a bi-orthogonal sequence in $S(\Psi)$ if and only if W is invertible a.e. and $W^{-1} \in L_1(\mathbb{T}^d; \mathbb{C}^{N\times N})$ (in particular, W is strictly positive definite a.e.). If this is the case, the unique dual element to $\psi_j(\cdot - k)$ is given by

$$U\big(e^{-2\pi i k\cdot\xi} W^{-1} \mathbf{e}_j\big), \quad k \in \mathbb{Z}^d, \tag{2.6}$$

where U is defined by (2.5).

Proof It suffices to study the system $\{e^{-2\pi ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d, j=1,\dots,N}$ in $L_2(\mathbb{T}^d;W)$. Suppose $W^{-1}\in L_1(\mathbb{T}^d)$. We claim that $e^{-2\pi ik\cdot\xi}W^{-1}\mathbf{e}_j$ is the dual functional to $e^{-2\pi ik\cdot\xi}\mathbf{e}_j$. Notice that

$$\begin{aligned}\|e^{-2\pi ik\cdot\xi}W^{-1}\mathbf{e}_j\|^2_{L_2(\mathbb{T}^d;W)} &= \int_{\mathbb{T}^d} |e^{-2\pi ik\cdot\xi}|^2\mathbf{e}_j^T W^{-1}(\xi)W(\xi)W^{-1}(\xi)\mathbf{e}_j\,d\xi\\ &= \int_{\mathbb{T}^d}(W^{-1})_{j,j}(\xi)<\infty,\end{aligned}$$

so $e^{-2\pi ik\cdot\xi}W^{-1}\mathbf{e}_j\in L_2(\mathbb{T}^d;W)$. Moreover, for $k,k'\in\mathbb{Z}^d$ and $j,j'\in\{1,2,\dots,N\}$,

$$\begin{aligned}\langle e^{2\pi ik\cdot\xi}\mathbf{e}_j, e^{2\pi ik'\cdot\xi}W^{-1}\mathbf{e}_{j'}\rangle_{L_2(\mathbb{T}^d;W)} &= \int_{\mathbb{T}^d} e^{2\pi i(k-k')\cdot\xi}\mathbf{e}_{j'}^T W^{-1}(\xi)W(\xi)\mathbf{e}_j\,d\xi\\ &= \int_{\mathbb{T}^d} e^{2\pi i(k-k')\cdot\xi}\mathbf{e}_{j'}^T\mathbf{e}_j\,d\xi\\ &= \delta_{j,j'}\delta_{k,k'}.\end{aligned}$$

Conversely, let $\{b_{j,k}\}\subset L_2(\mathbb{T}^d;W)$ be the unique dual system to $\{e^{-2\pi ik\xi}\mathbf{e}_j\}$. Thus,

$$\begin{aligned}\langle e^{-2\pi ik'\cdot\xi}\mathbf{e}_{j'}, b_{j,k}\rangle_{L_2(\mathbb{T}^d;W)} &= \int_{\mathbb{T}^d} b_{j,k}(\xi)^H W(\xi)\mathbf{e}_{j'}\,e^{-2\pi ik'\cdot\xi}d\xi\\ &= \delta_{j,j'}\delta_{k,k'}.\end{aligned}$$

Notice that $b_{j,k}(\xi)^H W(\xi)\mathbf{e}_{j'}\in L_1(\mathbb{T}^d)$ since $b_{j,k}(\xi)\in L_2(\mathbb{T}^d;W)$ and $W\in L_1(\mathbb{T}^d)$. The Fourier transform is injective on $L_1(\mathbb{T}^d)$ so we conclude that for a.a. $\xi\in\mathbb{T}^d$,

$$b_{j,k}(\xi)^H W(\xi) = e^{2\pi ik\cdot\xi}\mathbf{e}_j^T,\quad j=1,2,\dots,N.$$

It follows that W has full rank a.e., and we may solve for $b_{j,k}$ to get

$$b_{j,k}(\xi) = e^{-2\pi ik\cdot\xi}W^{-1}(\xi)\mathbf{e}_j.$$

We put $k=0$, and obtain

$$\infty > \|b_{j,0}\|^2_{L_2(\mathbb{T}^d;W)} = \int_{\mathbb{T}^d}\mathbf{e}_j^T W^{-1}(\xi)\mathbf{e}_j\,d\xi,\quad j=1,2,\dots,N.$$

Hence $\mathrm{trace}(W^{-1})\in L_1(\mathbb{T}^d)$. Recall that for a positive $N\times N$-matrix A, $\mathrm{trace}(A)\le N\|A\|\le N\cdot\mathrm{trace}(A)$, so

$$\int_{\mathbb{T}^d}\|W^{-1}\|(\xi)d\xi \asymp \int_{\mathbb{T}^d}\mathrm{trace}(W^{-1})(\xi)d\xi<\infty.$$ □

We conclude this section by using the map U to translate the problem of studying the rectangular partial sum operators given by (1.2) to an equivalent problem for rectangular trigonometric partial sums in $L_2(\mathbb{T}^d;W)$.

For any FSI subspace $S(\Psi)$, $\Psi = \{\psi_1, \dots, \psi_N\}$, for which $W(\Psi)^{-1} \in L_1(\mathbb{T}^d)$, we can define the partial sum operators, for $f \in L_2(\mathbb{T}^d; W)$,

$$\begin{aligned} S_{\mathbf{N}}\tau &= \sum_{j=1}^{N} \sum_{k:|k_i|\le N_i} \langle \tau, e^{-2\pi i k\cdot\xi} W^{-1}\mathbf{e}_j\rangle_{L_2(\mathbb{T}^d;W)} e^{-2\pi i k\cdot\xi}\mathbf{e}_j \\ &= \sum_{j=1}^{N} \sum_{k:|k_i|\le N_i} \langle \tau, e^{-2\pi i k\cdot\xi} \mathbf{e}_j\rangle_{L_2(\mathbb{T}^d;Id)} e^{-2\pi i k\cdot\xi}\mathbf{e}_j. \end{aligned} \tag{2.7}$$

We have the following corollary to Proposition 2.1.

Corollary 2.2 *Let $S(\Psi)$, $\Psi = \{\psi_1, \dots, \psi_N\}$, be a FSI space for which $W(\Psi)^{-1} \in L_1(\mathbb{T}^d)$. For any $\mathbf{N} \in \mathbb{N}_0^d$, we have*

$$\|S_{\mathbf{N}}\|_{L_2(\mathbb{T}^d;W)\to L_2(\mathbb{T}^d;W)} = \|T_{\mathbf{N}}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}. \tag{2.8}$$

Proof Let $f \in S(\Psi)$ with $\hat{f} = \sum_{j=1}^{N} \tau_j \hat{\psi}_j$. Recall that $U(\tau) = f$ and $U(e^{-2\pi i k\cdot\xi}\mathbf{e}_j) = \psi_j(\cdot - k)$. We now use (2.6) to obtain

$$\begin{aligned} U(S_{\mathbf{N}}\tau) &= \sum_{j=1}^{N} \sum_{k:|k_i|\le N_i} \langle \tau, e^{-2\pi i k\cdot\xi} W^{-1}\mathbf{e}_j\rangle_{L_2(\mathbb{T}^d;W)} U(e^{-2\pi i k\cdot\xi}\mathbf{e}_j) \\ &= \sum_{j=1}^{N} \sum_{k:|k_i|\le N_i} \langle U\tau, U(e^{-2\pi i k\cdot\xi} W^{-1}\mathbf{e}_j)\rangle_{L_2(\mathbb{R}^d)} U(e^{-2\pi i k\cdot\xi}\mathbf{e}_j) = T_{\mathbf{N}} f. \end{aligned}$$

This clearly implies that $\|S_{\mathbf{N}}\tau\|_{L_2(\mathbb{T}^d;W)} = \|T_{\mathbf{N}} f\|_{L_2(\mathbb{R}^d)}$, with $\|\tau\|_{L_2(\mathbb{T}^d;W)} = \|f\|_{L_2(\mathbb{R}^d)}$, so (2.8) follows. □

3 On a Vector Hunt-Muckenhoupt-Wheeden Product Theorem

In this section we study boundedness properties of linear operators on vector-valued spaces. In particular, we are interested in characterizing the matrix weights W such that the partial sum operators given by (2.7) on the space $L_2(\mathbb{T}^d; W)$ are uniformly bounded. A complete characterization of such matrix weights is given by Theorem 3.3 below.

Let us consider a linear operator T on $L_2(\mathbb{T}^d)$. We may apply T to functions f taking values in $\mathbb{C}^N$ by letting it act separately on each coordinate function, i.e.,

$$(Tf)_j = Tf_j, \quad j = 1, 2, \dots, N, \tag{3.1}$$

In case T is a (singular) integral operator with scalar kernel $S(x, y)$, the lifting of T to vector-valued functions simply corresponds to multiplying the kernel $S(x, y)$ by the $N \times N$-identity matrix.

A fundamental problem is to characterize the matrix weights $W:\mathbb{T}^d\to\mathbb{C}^{N\times N}$ for which certain families of singular integral operators extend to bounded operators on the weighted space $L_2(\mathbb{T}^d;W)$.

The vector-valued Hilbert transform was studied in the seminal paper by Treil and Volberg [24], and this was later generalized to other types of singular integral operators by Goldberg [11].

Let us state the result by Treil and Volberg in details, since it will be essential for the proof of Theorem 3.3. Let W be a $N\times N$ matrix weight on $\mathbb{T}$. We say that W satisfies the regular (periodic) Muckenhoupt A_2-condition if

$$\sup_I \left\| \left(\frac{1}{|I|}\int_I W\,d\xi\right)^{1/2}\left(\frac{1}{|I|}\int_I W^{-1}d\xi\right)^{1/2}\right\| < \infty, \tag{3.2}$$

where the sup is over all intervals $I\subset\mathbb{R}$. The collection of all such weights is denoted $\mathbb{A}_2(\mathbb{T})$. Also, notice that $\mathbb{A}_2(\mathbb{T})=\mathbb{P}\mathbb{A}_2(1)$.

The Hilbert transform H is defined on $L_2(\mathbb{T})$ by

$$H(f)(x) := \text{p.v.}\int_{\mathbb{T}} f(t)\cot(\pi(x-t))\,dt.$$

We lift H using (3.1) to a linear operator on $L_2(\mathbb{T};W)$ for any $N\times N$ matrix weight W on $\mathbb{T}$. The fundamental result by Treil and Volberg [24], see also [23], is the following.

Theorem 3.1 ([24]) *Let $W:\mathbb{T}\to\mathbb{C}^{N\times N}$ be a matrix weight. Then the Hilbert transform is bounded on $L_2(\mathbb{T};W)$ if and only if $W\in\mathbb{A}_2(\mathbb{T})$.*

We recall that the univariate Dirichlet kernel D_N is given by

$$D_N(t)=\frac{\sin 2\pi(N+1/2)t}{\sin\pi t},\qquad N\geq 1, \tag{3.3}$$

and for $f\in L_2(\mathbb{T})$,

$$S_N(f) := \sum_{k=-N}^{N}\hat{f}(k)e^{2\pi ik\cdot} = f * D_N := \int_{\mathbb{T}} f(t)D_N(\cdot - t)\,dt.$$

We have the following immediate corollary to Theorem 3.1.

Corollary 3.2 *Let $W:\mathbb{T}\to\mathbb{C}^{N\times N}$ be a matrix weight in $\mathbb{A}_2$. Then the partial sum operators $f\to f*D_N$ are uniformly bounded on $L_2(\mathbb{T};W)$.*

Proof We let $P_+=\frac{1}{2}(I+iH+S_0)$ denote the Riesz projection onto H^2 for $f\in L_2(\mathbb{T};W)$, where $S_0 f := \int_{\mathbb{T}} f(y)\,dy$ is the 0-order partial sum operator. It follows that P_+ is bounded on $L_2(\mathbb{T};W)$ since H is bounded according to Theorem 3.1, and S_0 is bounded according to [23, Lemma 1.5]. Notice that $f\to fe^{2\pi iM\cdot}$ is a unitary mapping on $L_2(\mathbb{T};W)$, just as in the scalar case. Then we observe that

$$f*D_N = e^{-2\pi iN\cdot}P_+(e^{2\pi iN\cdot}f) - e^{2\pi i(N+1)\cdot}P_+(e^{-2\pi i(N+1)\cdot}f),$$

and the result follows. □

We can now state the main result of this paper. Also notice that Corollary 2.2 and Theorem 3.3 give a direct proof of Theorem 1.2.

Theorem 3.3 *Let $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be a matrix weight with $W, W^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. Then the rectangular partial sum operators*

$$S_{\mathbf{N}} f(\xi) := \sum_{k\in\mathbb{Z}^d : |k_i|\le N_i} \hat{f}(k) e^{-2\pi i k\cdot\xi}, \quad \mathbf{N} \in \mathbb{N}_0^d,$$

are uniformly bounded on $L_2(\mathbb{T}^d; W)$ if and only if $W \in \mathbb{PA}_2(d)$.

The proof of Theorem 3.3 is based on Corollary 3.2 and the following two lemmata. Part (a) of Lemma 3.4 gives an equivalent formulation of the product $\mathbb{AP}(d)$ condition in terms of integrals of certain non-negative functions. This type of condition was first considered by S. Roudenko [21] for matrix A_p-weights on $\mathbb{R}^d$ associated with cubes (and not rectangles as is needed for our results).

Let $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be a matrix weight with $W, W^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. For convenience, we define the following quantity for any rectangle $R \subset \mathbb{R}^d$,

$$M(R, W) := \left\| \left(\frac{1}{|R|} \int_R W d\xi \right)^{1/2} \left(\frac{1}{|R|} \int_R W^{-1} d\xi \right)^{1/2} \right\|. \tag{3.4}$$

We have the following lemma.

Lemma 3.4 *Let $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be a matrix weight with $W, W^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. For a rectangle $R \subset \mathbb{R}^d$, we define $M(R, W)$ by (3.4). Then the following holds.*

(a) *We have uniformly in R,*

$$M(R, W)^2 \asymp \frac{1}{|R|^2} \int_R \int_R \|W^{1/2}(\xi) W^{-1/2}(\eta)\|^2 d\eta d\xi.$$

(b) *There exists a universal constant $c > 0$ such that for rectangles $R \subseteq \tilde{R} \subset \mathbb{R}^d$,*

$$M(R, W) \le c \frac{|\tilde{R}|}{|R|} M(\tilde{R}, W).$$

(c) *Suppose $W \in \mathbb{PA}(d)$, then the univariate weight $\xi_j \to W(\xi)$, obtained by fixing the variables ξ_k $(k \neq j)$, is uniformly in $\mathbb{A}_2(\mathbb{T})$ for a.e. $(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_d) \in \mathbb{T}^{d-1}$.*

Proof The proof of (a) can be found in [21] for the non-product case. The reader can easily check that the same proof works in the product case. For (b), we notice that

$$\begin{aligned} M(R, W)^2 &\asymp \frac{1}{|R|^2} \int_R \int_R \|W^{1/2}(\xi) W^{-1/2}(\eta)\|^2 d\eta d\xi \\ &\le \frac{|\tilde{R}|^2}{|R|^2} \frac{1}{|\tilde{R}|^2} \int_{\tilde{R}} \int_{\tilde{R}} \|W^{1/2}(\xi) W^{-1/2}(\eta)\|^2 d\eta d\xi \end{aligned}$$

$$\asymp \frac{|\tilde{R}|^2}{|R|^2} M(\tilde{R}, W)^2.$$

Now we turn to the proof of (c). It suffices to consider $\tilde{W}(t) := W(t, \xi_2, \dots, \xi_d)$ for $(\xi_2, \dots, \xi_d) \in \mathbb{T}^{d-1}$ fixed. Given an interval $I \subset \mathbb{R}$, we form $R_\varepsilon = I_\varepsilon(\xi_2) \times \cdots \times I_\varepsilon(\xi_d)$, where $I_\varepsilon(\xi_j)$ is an interval of length 2ε centered at ξ_j. Since $W \in \mathbb{PA}(d)$ there exists a constant C_W independent of $I \times R_\varepsilon$ such that

$$\begin{aligned} C_W^2 &\geq M(I \times R_\varepsilon, W)^2 \\ &\asymp \frac{1}{|R_\varepsilon|^2} \int_{R_\varepsilon} \int_{R_\varepsilon} \left(\frac{1}{|I|^2} \int_I \int_I \|W^{1/2}(t, \mathbf{u}) W^{-1/2}(w, \mathbf{v})\|^2 dt dw \right) d\mathbf{u} d\mathbf{v}. \end{aligned}$$

Hence, by Lebesgue's differentiation theorem, for almost every $(\xi_2, \dots, \xi_d) \in \mathbb{R}^{d-1}$,

$$C_W^2 \geq \lim_{\varepsilon \to 0^+} M(I \times R_\varepsilon, W)^2 = \frac{1}{|I|^2} \int_I \int_I \|\tilde{W}^{1/2}(t) \tilde{W}^{-1/2}(w)\|^2 dt dw \asymp M(I, \tilde{W})^2,$$

where the constants are independent of I and $(\xi_2, \dots, \xi_d)$. Hence $\tilde{W}$ is uniformly in $\mathbb{A}_2(\mathbb{T})$ for a.e. $(\xi_2, \dots, \xi_d) \in \mathbb{T}^{d-1}$. □

Lemma 3.5 estimates the norm of integral operators on $L_2(\mathbb{T}^d; W)$ with very localized kernels.

Lemma 3.5 *Suppose $Sf(\xi) = \int_{\mathbb{T}^d} S(\xi, \eta) f(\eta) d\eta$ is an integral operator with a scalar kernel $S(\xi, \eta)$ that satisfies $|S(\xi, \eta)| \leq \alpha |R|^{-1} \chi_{R \times R}$ for some bounded rectangle $R \subset \mathbb{R}^d$. Then the norm of S on $L_2(\mathbb{T}^d; W)$ is at most $d \cdot \alpha \cdot M(R, W)$, with $M(R, W)$ given by* (3.4). *Moreover, the kernel $|R|^{-1} \chi_R \times \chi_R$ induces an operator with norm exactly $M(R, W)$ on $L_2(\mathbb{T}^d; W)$.*

The proof of Lemma 3.5 for non-product A_p-weights can be found in Goldberg [11]. The reader can easily adapt the proof in [11] to the product case. We can now give a proof of Theorem 3.3.

Proof of Theorem 3.3 First we assume that $W \in \mathbb{PA}_2(d)$. The case $d = 1$ is exactly the conclusion of Corollary 3.2. Next we consider the case $d = 2$; the reader can easily verify that the argument below generalizes to any $d \geq 3$.

According to Lemma 3.4(c), $W_{\xi_1} := W(\xi_1, \cdot)$ and $W_{\xi_2} := W(\cdot, \xi_2)$ satisfy uniform Muckenhoupt $\mathbb{A}_2$-conditions on $\mathbb{T}$. Pick any $f \in L_2(\mathbb{T}^2, W)$. By Fubini's theorem, $f_{\xi_1} := f(\xi_1, \cdot) \in L_2(\mathbb{T}, W_{\xi_1})$ and $f_{\xi_2} := f(\cdot, \xi_2) \in L_2(\mathbb{T}, W_{\xi_2})$ for a.e. $[\xi_1]$ and $[\xi_2]$, respectively.

Let D_N be the univariate Dirichlet kernel given by (3.3), and we define

$$T_N^1 f := D_N * f_{\xi_2} := \int_{\mathbb{T}} f_{\xi_2}(t) D_N(\cdot - t)\, dt,$$

$$T_M^2 f := D_M * f_{\xi_1} := \int_{\mathbb{T}} f_{\xi_1}(t) D_M(\cdot - t)\, dt.$$

Notice that $T_{N,M} f = T_N^1 T_M^2 f$. We apply Corollary 3.2 to obtain

$$\int_{\mathbb{T}} |W_{\xi_1}^{1/2}(\xi_2) T_M^2 f_{\xi_1}(\xi_2)|^2 d\xi_2 \le C \int_{\mathbb{T}} |W_{\xi_1}^{1/2}(\xi_2) f_{\xi_1}(\xi_2)|^2 d\xi_2, \quad \text{a.e. } [\xi_1].$$

An integration yields,

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/2}(\xi_1, \xi_2) T_M^2 f(\xi_1, \xi_2)|^2 d\xi_2 d\xi_1 \\ &\quad \le C \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/2}(\xi_1, \xi_2) f(\xi_1, \xi_2)|^2 d\xi_2 d\xi_1. \end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned} \|T_{N,M} f\|_{L_2(\mathbb{T}^2, W)}^2 &= \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/2}(\xi_1, \xi_2) T_N^1 T_M^2 f(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \\ &\le C \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/2}(\xi_1, \xi_2) T_M^2 f(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 \\ &\le C^2 \int_{\mathbb{T}} \int_{\mathbb{T}} |W^{1/2}(\xi_1, \xi_2) f|^2 d\xi_1 d\xi_2. \end{aligned}$$

It follows that $\{T_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}_0^d}$ are uniformly bounded on $L_2(\mathbb{T}^2; W)$.

Now, let us assume that the operators $\{T_{\mathbf{N}}\}_{\mathbf{N} \in \mathbb{N}_0^d}$ are uniformly bounded on $L_2(\mathbb{T}^d; W)$. We have to prove that $M(R, W)$ given by (3.4) is uniformly bounded in R.

Let us first recall some elementary facts about the univariate Dirichlet kernel given by (3.3). The kernel D_N is real and $\|D_N\|_\infty = 2N + 1 = D_N(0)$. By Bernstein's inequality, $\|D_N'\|_\infty \le (2N+1)^2$. We can thus find an integer K (independent of N) such that for $t \in [-\frac{1}{KN}, \frac{1}{KN}]$ we have $D_N(t) \ge \left(1 - \frac{1}{2d}\right)^{1/d} \|D_N\|_\infty$.

Let a rectangle $R = I_1 \times I_2 \times \cdots \times I_d$ be given. For $j = 1, 2, \dots, d$, with $|I_j| > \frac{1}{2K}$, we define $N_j = 0$ and replace I_j with $[-1/2, 1/2)$, and obtain a possibly larger rectangle $\tilde{R}$. By Lemma 3.4(b) there is a universal constant c such that $M(\tilde{R}, W) \ge cM(R, W)$ since $|\tilde{R}| \le (2K)^d |R|$. Next, for each $j = 1, 2, \dots, d$ with $|I_j| \le \frac{1}{2K}$, we choose an integer $N_j \ge 1$ such that

$$\frac{1}{4K} \cdot \frac{1}{N_j} \le |I_j| \le \frac{1}{2K} \cdot \frac{1}{N_j}. \tag{3.6}$$

Notice that for $t, u \in I_j$, we have $t - u \in I_j - I_j \subset [-\frac{1}{KN_j}, \frac{1}{KN_j}]$ so

$$D_{N_j}(t - u) \ge \left(1 - \frac{1}{2d}\right)^{1/d} \|D_{N_j}\|_\infty. \tag{3.7}$$

For notational convenience we put $D_0 := 1$, and form the product kernel

$$D_{\mathbf{N}}(\xi) = \prod_{j=1}^{d} D_{N_j}(\xi_j).$$

The plan of attack is to use the simple fact that $f \to \chi_{\tilde{R}} T_{\mathbf{N}}(\chi_{\tilde{R}} f)$ is uniformly bounded in both R and $\mathbf{N} \in \mathbb{N}_0^d$. We notice that $f \to \chi_{\tilde{R}} T_{\mathbf{N}}(\chi_{\tilde{R}} f)$ has integral kernel

$$S_2(\xi, \eta) := \chi_{\tilde{R}}(\eta)\chi_{\tilde{R}}(\xi) D_{\mathbf{N}}(\eta - \xi).$$

We wish to estimate the operator norm of S_2 from below. For that purpose we first consider the operator with kernel

$$S(\xi, \eta) := S_1(\xi, \eta) - S_2(\xi, \eta) := \|D_{\mathbf{N}}\|_\infty \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) - \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) D_{\mathbf{N}}(\xi - \eta).$$

Notice that the estimate (3.7) implies the following size estimate

$$\begin{aligned}
|S(\xi, \eta)| &= \left| \|D_{\mathbf{N}}\|_\infty \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) - \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) D_{\mathbf{N}}(\xi - \eta) \right| \\
&\le \frac{\|D_{\mathbf{N}}\|_\infty}{2d} \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta) \\
&= \frac{|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty}{2d} |\tilde{R}|^{-1} \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta).
\end{aligned}$$

According to Lemma 3.5, the kernel S induces an operator of norm at most $\frac{1}{2}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R}, W)$ on $L_2(\mathbb{T}^d; W)$. At the same time, Lemma 3.5 shows that the operator with kernel $S_1(\xi, \eta) = |\tilde{R}| \|D_{\mathbf{N}}\|_\infty \cdot |\tilde{R}|^{-1} \chi_{\tilde{R}}(\xi)\chi_{\tilde{R}}(\eta)$ has norm exactly $|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R}, W)$ on $L_2(\mathbb{T}^d; W)$. The triangle inequality for operator norms implies that

$$\begin{aligned}
\frac{1}{2}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R}, W) &\ge \|S_1 - S_2\| \ge |\|S_1\| - \|S_2\|| \\
&\ge |\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R}, W) - \|S_2\|,
\end{aligned}$$

so $\|S_2\| \ge \frac{1}{2}|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty M(\tilde{R}, W)$. Moreover, by (3.6), we see that $|\tilde{R}| \cdot \|D_{\mathbf{N}}\|_\infty \ge (4K)^{-d}$, so we may conclude that

$$\begin{aligned}
M(R, W) &\le C M(\tilde{R}, W) \\
&\le 2C(4K)^d \|S_2\| \\
&= 2C(4K)^d \sup_{\|f\|_{L_2(\mathbb{T}^d; W)} = 1} \|\chi_{\tilde{R}} T_{\mathbf{N}}(\chi_{\tilde{R}} f)\|_{L_2(\mathbb{T}^d; W)} \\
&\le C' \sup_{\|f\|_{L_2(\mathbb{T}^d; W)} = 1} \|T_{\mathbf{N}} f\|_{L_2(\mathbb{T}^d; W)} \\
&\le C''
\end{aligned}$$

with constant C'' independent of R. Thus, we can finally conclude that $W \in \mathbb{PA}_2(d)$. □

Remark 3.6 As the reader may notice, modulo the complications added by the vector-valued setup, the "kernel localization" technique used to prove the only if part of Theorem 3.3 is in fact very similar to the original technique introduced by Hunt, Muckenhoupt, and Wheeden in [15].

4 Schauder Bases for FSI Spaces

In this section, we consider an application of Theorem 1.2 to the problem of obtaining Schauder bases for an FSI space $S(\Psi)$. Let us first recall some elementary facts about Schauder bases in a Hilbert space. We refer to [22] for more details.

A family $B = \{x_n : n \in \mathbb{N}\}$ of vectors in a Hilbert space H is a Schauder basis for H if for every $x \in H$ there exists a unique sequence $\{\alpha_n := \alpha_n(x) : n \in \mathbb{N}\}$ of scalars such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n = x$$

in the norm topology of H. The unique choice of scalars implies that $x \to \alpha_n(x)$ is a linear functional, for every $n \in \mathbb{N}$. Furthermore, for every $n \in \mathbb{N}$, there exists a unique vector y_n such that $\alpha_n(x) = \langle x, y_n\rangle$. It follows that

$$\langle x_m, y_n\rangle = \delta_{m,n}, \qquad m, n \in \mathbb{N}. \tag{4.1}$$

A pair of sequences $(\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}})$ in H is a *bi-orthogonal system* if $\langle u_m, v_n\rangle = \delta_{m,n}$, $m, n \in \mathbb{N}$. We say that $\{v_n\}_{n\in\mathbb{N}}$ is a *dual sequence* to $\{u_n\}_{n\in\mathbb{N}}$, and vice versa.

A dual sequence is not necessarily uniquely defined. In fact, it is unique if and only if the original sequence is complete in H (i.e., if the span of the original sequence is dense in H).

Suppose $B = \{x_n : n \in \mathbb{N}\}$ is complete, and has a unique dual sequence $\{y_n\}$. Then B is a Schauder basis for H if and only if the partial sum operators $S_N(x) = \sum_{n=1}^{N}\langle x, y_n\rangle x_n$ are uniformly bounded on H.

4.1 A Particular Enumeration of $\mathbb{Z}^2$

Expansions relative to Schauder bases need not converge unconditionally, and the ordering of the Schauder basis elements becomes crucial. To study Schauder basis properties of (1.1), we therefore first have to impose an ordering of the system (1.1) that is compatible with the result on rectangular partial sums given by Theorem 1.2.

Our starting point is to consider enumerations of $\mathbb{Z}^d$, i.e., bijective maps $\sigma : \mathbb{N} \to \mathbb{Z}^d$. In order not to cloud the picture by complicated notation, we restrict our attention to enumerations of $\mathbb{Z}$ and $\mathbb{Z}^2$. For $\mathbb{Z}$, we simply pick the enumeration σ^1 given by

$$0, 1, -1, 2, -2, \ldots.$$

Let $\Lambda(1) := \{\sigma^1\}$. For $d = 2$, we follow Heil and Powell [12], and consider the following class of enumerations.

Definition 4.1 Let $\Lambda(2)$ be the set containing all enumerations $\{(k_j, n_j)\}_{j=1}^{\infty}$ of $\mathbb{Z}^2$ defined in the following recursive manner.

1. The initial terms $(k_1, n_1) \dots (k_{J_1}, n_{J_1})$ are either

$$(0,0), (1,0), (-1,0), \dots, (A_1, 0), (-A_1, 0)$$

or

$$(0,0), (0,1), (0,-1), \dots, (0, B_1), (0, -B_1),$$

for some positive integers A_1 or B_1.

2. If $\{(k_j, n_j)\}_{j=1}^{J_k}$ has been constructed to be of the form $\{-A_k, \dots, A_k\} \times \{-B_k, \dots, B_k\}$ for some non-negative integers A_k, B_k, then terms are added to either the top and bottom or the left and right sides to obtain either the rectangle

$$\{-A_k, \dots, A_k\} \times \{-(B_k + 1), \dots, B_k + 1\}$$

or

$$\{-(A_k + 1), \dots, A_k + 1\} \times \{-B_k, \dots, B_k\}.$$

For example, terms would first be added to the left side ordered as

$$(-(A_k+1), 0), (-(A_k+1), 1), (-(A_k+1), -1), \dots, (-(A_k+1), B_k),$$
$$(-(A_k+1), -B_k),$$

and likewise for the right side. Top and bottom proceed analogously.

Remark 4.2 We leave it to the reader to verify that the above technique can be generalized to obtain admissible enumerations of $\mathbb{Z}^d$ recursively as follows. We always start at $\mathbf{0}$. Then at each step in the process where a rectangle

$$R = \{-N_1, \dots, N_1\} \times \{-N_2, \dots, N_2\} \times \cdots \times \{-N_d, \dots, N_d\}, \quad N_i \in \mathbb{N}_0,$$

has been reached, we proceed by only adding terms to two opposing "faces" of R. The terms are added to each of the two faces using an admissible enumeration of $\mathbb{Z}^{d-1}$.

4.2 A Characterization of Schauder Bases for FSI Subspaces

We consider $\Psi = \{\psi_1, \dots, \psi_N\} \subset L_2(\mathbb{R}^d)$, $d \in \{1, 2\}$, such that the system

$$F = \{\psi(\ell, k, \cdot) := \psi_\ell(\cdot - k)\}_{k \in \mathbb{Z}^d, \ell = 1, \dots, N} \tag{4.2}$$

has a unique dual system $\{g(\ell, k, \cdot)\}$ in $S(\Psi)$. Given $\sigma \in \Lambda(d)$, we lift σ to an enumeration $\tilde{\sigma}$ of $\{1, 2, \dots, N\} \times \mathbb{Z}^d$ defined as follows

$$(1, \sigma(1)), (2, \sigma(1)), \dots, (N, \sigma(1)), (1, \sigma(2)), \dots, (N, \sigma(2)), (1, \sigma(3)), \dots. \tag{4.3}$$

With this ordering, we define the partial sum operators

$$T_J^{\sigma} f := \sum_{j=1}^{J} \langle f, g(\tilde{\sigma}(j), \cdot)\rangle \psi(\tilde{\sigma}(j), \cdot), \quad f \in S(\Psi).$$

We also need to consider the associated partial sum operator in $L_2(\mathbb{T}^d; W)$. Put $e(\ell, k) := e^{-2\pi i k\cdot\xi}\mathbf{e}_\ell$, and $\tilde{e}(\ell, k) := U(e^{-2\pi i k\cdot\xi} W^{-1}(\xi)\mathbf{e}_\ell)$, with U defined by (2.5). Then

$$S_J^{\sigma}\tau := \sum_{j=1}^{J} \langle \tau, \tilde{e}(\tilde{\sigma}(j))\rangle_{L_2(\mathbb{T}^d;W)} e(\tilde{\sigma}(j)), \quad \tau \in L_2(\mathbb{T}^d; W),$$

satisfies $U(S_J^{\sigma}\tau) = T_J^{\sigma} f$ for $U(\tau) = f \in S(\Psi)$.

It is now immediate from our general discussion of Schauder bases that the following conditions are equivalent:

(i) The system F given by (4.2) is a Schauder basis for $S(\Psi)$ with the ordering induced by $\sigma \in \Lambda(d)$
(ii) The partial sum operators T_J^{σ} are uniformly bounded on $S(\Psi)$.

With the notation in place, we can state our main result on Schauder bases for FSI subspaces. The following result is a corollary of Theorem 1.2.

Corollary 4.3 *We consider a FSI subspace $S(\Psi)$ in $L_2(\mathbb{R}^d)$, with $d \in \{1, 2\}$, and $\Psi = \{\psi_1, \dots, \psi_N\}$. Assume that the system given by (4.2) has a unique dual system in $S(\Psi)$, and let $W(\Psi)$ be the Gram-matrix for Ψ. Then the following statements are equivalent*

(a) $\sup_{\sigma\in\Lambda(d)} \sup_J \|T_J^{\sigma}\| < \infty$.
(b) $W \in \mathbb{AP}(d)$.

Proof (a) $\Rightarrow$ (b): For $d = 1$, we notice that $T^{\sigma^1}_{(2J+1)N} = T_J$, with T_J given by (1.2), so $\sup_J \|T_J\| < \infty$, and $W \in \mathbb{A}_2(\mathbb{T})$ by Theorem 1.2. We turn to $d = 2$. Given a rectangle

$$R = \{-N_1, \dots, N_1\} \times \{-N_2, \dots, N_2\}, \quad N_1, N_2 \in \mathbb{N}_0,$$

we can use Definition 4.1 to construct an enumeration $\sigma \in \Lambda(2)$ such that $\sigma(\{1, \dots, J\}) = R$ for some $J \in \mathbb{N}$. Then $T_{N\cdot J}^{\sigma} = T_{(N_1,N_2)}$, and therefore

$$\sup_{N_1,N_2\geq 0} \|T_{(N_1,N_2)}\| < \infty.$$

Hence, $W \in \mathbb{AP}(2)$ by Theorem 1.2.

(b) $\Rightarrow$ (a): Assume that $d = 2$. Fix $f \in S(\Psi)$, and pick $\sigma \in \Lambda(2)$. For any J we let N_J be the largest integer $N_j \leq J$ for which $T_{N_j}^{\sigma} f = T_{L,K} f$, for some integers L, K. Now, by Theorem 1.2,

$$\begin{aligned}\|T_J^\sigma f\|_{L_2(\mathbb{R}^2)} &\le \|T_{L,K} f\|_{L_2(\mathbb{R}^2)} + \|(T_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R}^2)} \\ &\le C\|f\|_{L_2(\mathbb{R}^2)} + \|(T_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R}^2)}.\end{aligned}$$

Hence, it suffices to bound the norm of the term

$$(T_J^\sigma - T_{L,K})f = \sum_{j=N_J}^{J} \langle f, g(\tilde{\sigma}(j),\cdot)\rangle \psi(\tilde{\sigma}(j),\cdot). \tag{4.4}$$

According to Definition 4.1, the sum (4.4) contains terms that have been added to the top and bottom or left and right side of an rectangle. The cases are treated in a similar fashion. For definiteness, assume that (4.4) adds terms to the top of the rectangle.

We study the equivalent problem in $L_2(\mathbb{T}^d; W)$. Pick τ with $U(\tau) = f$, so $U(S_J^\sigma \tau) = T_J^\sigma f$. Notice that the ordering $\tilde{\sigma}$ given by (4.3) ensures that the sum $(S_J^\sigma - S_{L,K})\tau$ can be rewritten

$$(S_J^\sigma - S_{L,K})\tau = \sum_{j=1}^{N}\sum_{n=-M}^{M} \langle \tau, e^{2\pi i n\xi_1} e^{2\pi i (K+1)\xi_2}\mathbf{e}_j\rangle e^{2\pi i n\xi_1} e^{2\pi i (K+1)\xi_2}\mathbf{e}_j + E, \tag{4.5}$$

where the remainder E is a sum of at most $2N-1$ terms of the type $\langle \tau, e^{2\pi i k\cdot\xi}\mathbf{e}_\ell\rangle e^{2\pi i k\cdot\xi}\mathbf{e}_\ell$. We observe that, in general,

$$\|\langle \tau, e^{2\pi i k\cdot\xi}\mathbf{e}_\ell\rangle e^{2\pi i k\cdot\xi}\mathbf{e}_\ell\|_{L_2(\mathbb{T}^2;W)} \le \|W\|_{L_1(\mathbb{T}^2)}\|W^{-1}\|_{L_1(\mathbb{T}^2)}\|\tau\|_{L_2(\mathbb{T}^2;W)},$$

which follows from Hölder's inequality. We can thus use brute force to uniformly estimate the remainder E in (4.5) in terms of $\|\tau\|_{L_2(\mathbb{T}^2;W)}$. Next, we notice that,

$$\begin{aligned}&\left\|\sum_{j=1}^{N}\sum_{n=-M}^{M} \langle \tau e^{-2\pi i (K+1)\xi_2}, e^{2\pi i n\xi_1}\mathbf{e}_j\rangle e^{2\pi i n\xi_1}\mathbf{e}_j e^{2\pi i (K+1)\xi_2}\right\|_{L_2(\mathbb{T}^2;W)} \\ &\quad= \left\|\sum_{j=1}^{N}\sum_{n=-M}^{M} \langle \tau e^{-2\pi i (K+1)\xi_2}, e^{2\pi i n\xi_1}\mathbf{e}_j\rangle e^{2\pi i n\xi_1}\mathbf{e}_j\right\|_{L_2(\mathbb{T}^2;W)}.\end{aligned}$$

For notational convenience, we define the vector function

$$h(\xi_1) := \int_{\mathbb{T}} \tau(\xi_1,\xi_2)e^{-2\pi i (K+1)\xi_2}\,d\xi_2 = S_0(\tau(\xi_1,\cdot)e^{-2\pi i (K+1)\cdot}),$$

where S_0 is the 0-order partial sum operator. We recall that $W(\cdot,\xi_2)$ and $W(\xi_1,\cdot)$ are uniformly in $\mathbb{AP}(1)$ for a.e. ξ_2 and a.e. ξ_1, respectively. Hence, using Corollary 3.2 for the variable ξ_1,

$$\begin{aligned}&\left\|\sum_{j=1}^{N}\sum_{n=-M}^{M} \langle \tau e^{-2\pi i (K+1)\xi_2}, e^{2\pi i n\xi_1}\mathbf{e}_j\rangle e^{2\pi i n\xi_1}\mathbf{e}_j\right\|_{L_2(\mathbb{T}^2;W)}^2 \\ &\quad= \int_{\mathbb{T}}\int_{\mathbb{T}} |W^{1/2}(\xi_1,\xi_2)D_M * h|^2 d\xi_1 d\xi_2\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{\mathbb{T}}\int_{\mathbb{T}}|W^{1/2}(\xi_1,\xi_2)h|^2 d\xi_1 d\xi_2\\
&= C\int_{\mathbb{T}}\int_{\mathbb{T}}|W^{1/2}(\xi_1,\xi_2)S_0(\tau(\xi_1,\cdot)e^{-2\pi i(K+1)\cdot})|^2 d\xi_2 d\xi_1\\
&\le C'\int_{\mathbb{T}}\int_{\mathbb{T}}|W^{1/2}(\xi_1,\xi_2)\tau e^{-2\pi i(K+1)\xi_2}|^2 d\xi_2 d\xi_1\\
&= C'\|\tau\|^2_{L_2(\mathbb{T}^2;W)},
\end{aligned}
$$

where we also used that S_0 is bounded uniformly on $L_2(\mathbb{T};W(\xi_1,\cdot))$ for a.e. ξ_1. Collecting the estimates, we conclude that

$$
\|(T_J^\sigma - T_{L,K})f\|_{L_2(\mathbb{R}^2)} = \|(S_J^\sigma - S_{L,K})\tau\|_{L_2(\mathbb{T}^2;W)} \le C'\|\tau\|_{L_2(\mathbb{T}^2;W)} = C'\|f\|_{L_2(\mathbb{R}^2)},
$$

with C' independent of J. The proof in the case $d=1$ is similar. □

Remark 4.4 As in Remark 4.2, we leave it to the reader to verify that the proof of Corollary 4.3 can be generalized to arbitrary d by defining admissible enumerations of $\mathbb{Z}^d$ following the outline in Remark 4.2. Then (b) $\Rightarrow$ (a) in the proof of Corollary 4.3 can be used as the first step in an induction argument on d.

5 Some Examples

In this final section, we consider some examples of $\mathbb{PA}_2(d)$ weights, and some associated FSI subspaces.

5.1 The Case $N=1$

First, we consider the scalar case $N=1$ with d arbitrary. Our prime example in this case will be polynomials. Let $B=\{x\in\mathbb{R}^d : |x|\le 1\}$. Then for any polynomial $P(x)=\sum_\alpha c_\alpha x^\alpha$ of degree n on $\mathbb{R}^d$, we have the following estimate by Ricci and Stein [18],

$$
\int_B |P(x)|^{-\mu}dx \le c_{\mu,n}\left(\sum_\alpha |c_\alpha|\right)^{-\mu}, \tag{5.1}
$$

for $\mu n<1$. The constant $c_{\mu,n}$ is uniform for all polynomials of degree n. We observe that for the unit cube $R=\{x : |x_i|\le 1\}$, we have the trivial fact that $d^{-1/2}R\subset B$, so using (5.1),

$$
d^{-d/2}\int_R |P(x)|^{-\mu}dx = \int_{d^{-1/2}R}|P(d^{1/2}x)|^{-\mu}dx \le \tilde{c}_{\mu,n}\left(\sum_\alpha |c_\alpha|\right)^{-\mu}.
$$

Also, $\int_R |P(x)|dx$ and $\sum_\alpha |c_\alpha|$ are norms on the polynomials of degree n, and they are thus equivalent as norms on a finite dimensional space. Hence,

$$
\int_R |P(x)|^{-\mu}dx \le C_{\mu,n}\left(\int_R |P(x)|dx\right)^{-\mu}. \tag{5.2}
$$

Then we observe that the polynomials of degree n are invariant under affine transformations, so it follows that (5.2) holds for any rectangle R on $\mathbb{R}^d$. From this, we deduce that for any polynomial of degree n, $|P|^a$ is a (scalar) $\mathbb{PA}_2(d)$ weight provided $-1 < na < 1$.

5.2 Arbitrary N and d

Let us expand the example of Sect. 5.1 to arbitrary N. We notice that given polynomials $P_1, P_2, \dots, P_N$ on $\mathbb{R}^d$, the matrix

$$G := \operatorname{diag}(|P_1|^{a_1}, \dots, |P_N|^{a_N}) \in \mathbb{PA}_2(d)$$

provided $-1 < \deg(P_i)a_i < 1$ for $i = 1, \dots, N$. We let $Q = [-1/2, 1/2)^d$, and define $\Psi = \{\psi_1, \dots, \psi_N\}$ by

$$\hat{\psi}_i(\xi) = \chi_Q(\xi - k_i)\sqrt{|P_i|^{a_i}}, \quad i = 1, \dots, N,$$

where $\{k_i\}$ is a collection of distinct integers in $\mathbb{Z}^d$. An easy calculation shows that $W(\Psi) = G$. We thus have the norm convergence given by (1.3). However, the spectrum of $W(\Psi)$ is bounded away from zero precisely when all polynomials P_i have no roots on Q. Thus, for this example,

(i) We always have the norm convergence given by (1.3)
(ii) For $d = 1, 2$, we obtain Schauder bases for $S(\Psi)$ using Corollary 4.3
(iii) The system (1.1) is a Riesz basis for $S(\Psi)$ *only* when each P_i has no roots on Q.

That $W(\Psi)$ is diagonal is a reflection of the fact that the principal shift-invariant subspaces $S(\{\psi_i\})$, $i = 1, 2, \dots, N$, are pairwise orthogonal, and one can argue that the example does not truely belong in the matrix setting. We conclude this section with a more "genuine" matrix example for $d = 1$ and $N = 2$.

5.3 The Case $d = 1$ and $N = 2$

Let us consider the following example by Bownik [2]. For $t \in [-1/2, 1/2)$ we define

$$G(t) = U(t)^* \begin{bmatrix} 1 & 0 \\ 0 & b(t) \end{bmatrix} U(t), \qquad U(t) = \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix},$$

where $\alpha(t) = \operatorname{sign}(t)|t|^\delta$, $b(t) = |t|^\varepsilon$, with $-1 < \varepsilon < 1$, and δ satisfying $-2\delta \le \varepsilon \le 2\delta$. Then $G(t) \in \mathbb{AP}(1)$, see [2].

Define $\psi_1, \psi_2 \in L_2(\mathbb{R})$ by

$$\hat{\psi}_1(\xi) = \sqrt{G_{1,1}(t)}\chi_{[0,1)},$$
$$\hat{\psi}_2(\xi) = v_1(t)\chi_{[0,1)} + v_2(t)\chi_{[1,2)},$$

where

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \cos\beta(t) & -\sin\beta(t) \\ \sin\beta(t) & \cos\beta(t) \end{bmatrix} \begin{bmatrix} \sqrt{G_{2,2}(t)} \\ 0 \end{bmatrix},$$

with $\beta : \mathbb{T} \to [0, 2\pi)$ measurable such that $\sqrt{G_{1,1}(t)}v_1(t) = G_{1,2}(t) = G_{2,1}(t)$. We notice that this is always possible since $\det G(t) \geq 0$. Then a direct calculation shows that $\Psi = \{\psi_1, \psi_2\}$ satisfies $W(\Psi) = G(t)$.

The spectrum of $G(t)$ is not bounded away from zero, so Corollary 4.3 gives us an example of a conditional Schauder basis for $S(\Psi)$.

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