ON TRANSFERENCE OF MULTIPLIERS ON MATRIX WEIGHTED L_p-SPACES

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ABSTRACT. We consider a periodic matrix weight W defined on $\mathbb{R}^d$ and taking values in the $N \times N$ positive-definite matrices. For such weights, we prove a transference results between multiplier operators on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$, $1 < p < \infty$, respectively. As a specific application, we study transference results for homogeneous multipliers of degree zero.

1. INTRODUCTION

A matrix weight is a locally integrable function $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ taking values in the set of positive definite Hermitian forms. The associated weighted space $L_p(\mathbb{R}^d;W)$, $1 \leq p < \infty$, is the set of measurable (vector-)functions $f : \mathbb{R}^d \to \mathbb{C}^N$ satisfying

$$\|f\|^p_{L_p(\mathbb{R}^d;W)} := \int_{\mathbb{R}^d} |W^{1/p} f|^p \, dx < \infty. \tag{1.1}$$

For periodic weights, i.e., $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$, we define the associated weighted space $L_p(\mathbb{T}^d;W)$, $1 \leq p < \infty$, as the set of measurable periodic (vector-)functions $f : \mathbb{T}^d \to \mathbb{C}^N$ satisfying

$$\|f\|^p_{L_p(\mathbb{T}^d;W)} := \int_{\mathbb{T}^d} |W^{1/p} f|^p \, dx < \infty. \tag{1.2}$$

In this paper, we study transference results for multiplier operators on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$ for periodic weights W. By a multiplier operator on a weighted vector-valued space, we mean a scalar multiplier that acts coordinate-wise. More precisely, for a scalar multiplier operator T on $\mathbb{R}^d$ (or $\mathbb{T}^d$), we lift T to an operator on functions f taking values in $\mathbb{C}^N$ by letting it act separately on each coordinate function,

$$(Tf)_j = Tf_j, \qquad j = 1,2,\ldots N. \tag{1.3}$$

We mention that there are applications where multiplier operators on matrix weighted L_p-spaces appear naturally. The present author used multiplier operators on $L_p(\mathbb{T};W)$ to study stability and Schauder basis properties of finitely generated shift-invariant systems in [8].

It is well-known that in the scalar case, there is a close connection between bounded L_p multipliers on the line and on the torus, and it turns out that such results can be considered in the matrix weighted case as well. Transference can thus reduce the workload needed to prove L_p-boundedness for multipliers on e.g. the torus; one only needs to consider the corresponding multiplier on the line (or vice-versa).

Scalar transference results for scalar L_p-multipliers were first established by de Leeuw [4]. A systematic treatment of transference for multipliers and maximal multiplier operators was given by Coifman and Weiss [3]. More recent developments can be found in [1,2,9].

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A number of authors have studied boundedness of multipliers on $L_p(\mathbb{R}^d;W)$. In their seminal papers [7, 11], Treĭl′ and Volberg proved that the Hilbert transform is bounded if and only if the weight W belongs to an appropriate matrix Muckenhoupt A_p class. This result was extended by Goldberg [5] who proved that boundedness of standard multipliers on $L_p(\mathbb{R}^d;W)$ are closely related to the matrix Muckenhoupt A_p condition on the weight W. The transference results obtained here allow us to obtain similar conclusions for multiplier sequences on $L_p(\mathbb{T}^d;W)$.

This paper is organized as follows. Section 2 contains the main results on transference for multipliers between $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$. Our main application is presented in Section 3, where we consider multipliers on L_p spaces with weights satisfying a matrix Muckenhoupt A_p condition. Finally, in Section 4, we consider a family of examples provided by homogeneous multipliers of degree zero. In particular, the discrete Riesz transform is considered.

2. Main transference results

This section contains our main result. We give results in two directions. In Proposition 2.4, we transfer boundedness for multipliers on $L_p(\mathbb{R}^d;W)$ to boundedness for discrete multipliers on $L_p(\mathbb{T}^d;W)$, while in Proposition 2.5 we transfer in the other direction from $L_p(\mathbb{T}^d;W)$ to $L_p(\mathbb{R}^d;W)$.

Before we state the transference results, we need to define the classes of bounded multipliers on $L_p(\mathbb{R}^d;W)$ and $L_p(\mathbb{T}^d;W)$ that will be considered.

Definition 2.1. Let $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ be a matrix weight, and let $1\le p<\infty$. We denote by $\mathscr{M}_p(\mathbb{R}^d;W)$ the set of all bounded functions b on $\mathbb{R}^d$ such that the operator

$$T_b(f) := \big(b\hat{f}\big)^{\vee}$$

extends to a bounded operator on $L_p(\mathbb{R}^d;W)$. The norm $\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$ of an element $b\in\mathscr{M}_p(\mathbb{R}^d;W)$ is by definition the norm of the operator T_b on $L_p(\mathbb{R}^d;W)$.

Similarly, for $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ a periodic matrix weight, we denote by $\mathscr{M}_p(\mathbb{T}^d;W)$ the set of bounded sequences $\mathbf{a}=\{a_k\}_{k\in\mathbb{Z}^d}$ such that the operator

$$T_{\mathbf{a}}(f)(x) := \sum_{k\in\mathbb{Z}^d} a_k\hat{f}(k)e^{2\pi ik\cdot x}$$

extends to a bounded operator on $L_p(\mathbb{T}^d;W)$. The norm $\|\{a_k\}\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$ of an element $\mathbf{a}\in\mathscr{M}_p(\mathbb{R}^d;W)$ is defined to be the norm of the operator $T_{\mathbf{a}}$ on $L_p(\mathbb{T}^d;W)$.

2.1. **Multipliers in $\mathscr{M}_p(\mathbb{R}^d;W)$.** We now focus on multipliers b in $\mathscr{M}_p(\mathbb{R}^d;W)$. The basic idea of transference is to sample b on $\mathbb{Z}^d$ and thereby obtain a multiplier in $\mathscr{M}_p(\mathbb{T}^d;W)$. For this to work, b must be well-behaved point-wise. A very useful notion in the theory of (scalar) transference is that of a regulated function. Let us recall the definition of a regulated function.

Definition 2.2. Let $t_0\in\mathbb{R}^d$. A bounded measurable function b on $\mathbb{R}^d$ is called regulated at the point t_0 if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^d}\int_{|t|\le\varepsilon}\big(b(t_0-t)-b(t_0)\big)dt=0.$$

The function b is called regulated if it is regulated at every point $t_0\in\mathbb{R}^d$.

We now turn to vector-valued multipliers. Our idea is to use scalar transference combined with a duality argument. The dual space of $L_p(D;W)$, for $1<p<\infty$, and $D\in\{\mathbb{T}^d,\mathbb{R}^d\}$, can be identified with, $L_q(D;W^{-q/p})$, where q is the conjugate exponent to p given by $\frac{1}{p}+\frac{1}{q}=1$, see [11] for further details. The pairing of $L_p(D;W)$ and $L_p(D;W)^*=L_q(D;W^{-q/p})$ is given by the integral

$$\int_D \langle f(x),g(x)\rangle_{\ell_2(\mathbb{C}^N)}\,dx=\sum_{j=1}^N\int_D f_j(x)\overline{g_j(x)}\,dx. \tag{2.1}$$

The integrals on the right-hand side of (2.1) are ordinary scalar integrals, and in the proof of Proposition 2.4 below we use the following well-known lemma from (scalar) transference repeatedly.

Lemma 2.3 ([3,6]). *Let T be the operator on $\mathbb{R}^d$ whose multiplier is $b(\xi)$, and let S be the operator on $\mathbb{T}^d$ whose multiplier is the sequence $\{b(m)\}_{m\in\mathbb{Z}^d}$. Assume that $b(\xi)$ is regulated at every point $m\in\mathbb{Z}^d$. Let $L_\varepsilon(x)=e^{-\pi\varepsilon|x|^2}$ for $x\in\mathbb{R}^d$ and $\varepsilon>0$. For every pair of trigonometric polynomials P and Q on $\mathbb{R}^d$, and $\alpha,\beta>0$ with $\alpha+\beta=1$, we have the identity*

$$\lim_{\varepsilon\to 0^+}\varepsilon^{d/2}\int_{\mathbb{R}^d}T(PL_{\varepsilon\alpha})(x)\overline{Q(x)L_{\varepsilon\beta}(x)}dx=\int_{\mathbb{T}^d}S(P)(x)\overline{Q(x)}dx. \tag{2.2}$$

With the notation in place, we can now state the first part of our main result.

Proposition 2.4. *Let $1<p<\infty$, and let $W:\mathbb{R}^d\to\mathbb{C}^{N\times N}$ be a periodic matrix weight with $W,W^{-q/p}\in L_{1,loc}$, where $\frac{1}{p}+\frac{1}{q}=1$. Suppose that b is a regulated function on $\mathbb{R}^d$ that is contained in $\mathscr{M}_p(\mathbb{R}^d;W)$. Then $\{b(m)\}_{m\in\mathbb{Z}^d}$ is in $\mathscr{M}_p(\mathbb{T}^d;W)$. Moreover,*

$$\|\{b(m)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W)}\le\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}.$$

Proof. The idea of proof is to use scalar transference together with the fact that the dual space to $L_p(\mathbb{T}^d;W)$ is $L_q(\mathbb{T}^d;W^{-q/p})$, with q the conjugate exponent to p, see [5]. Let $\mathscr{P}^{d,N}$ be the family

$$\left\{\mathbf{P}(x)=[P_1(x),\dots P_N(x)]^T\right\}$$

of vectors of trigonometric polynomials on $\mathbb{R}^d$. Take any $\mathbf{P}\in\mathscr{P}^{d,N}$. We now use (2.1) and Lemma 2.3 to calculate the norm of $S(\mathbf{P})$ in $L_p(\mathbb{T}^d;W)$. We notice that $\mathscr{P}^{d,N}$ is dense in $L_q(\mathbb{T}^d;W^{-q/p})$ since $W^{-q/p}\in L_{1,\text{loc}}$, which implies that

$$\|S(\mathbf{P})\|_{L_p(\mathbb{T}^d;W)}=\sup_{\mathbf{Q}\in\mathscr{P}^{d,N}:\|\mathbf{Q}\|_{L_q(\mathbb{T}^d;W^{-q/p})}\le 1}\left|\int_{\mathbb{T}^d}\langle S(\mathbf{P})(x),\mathbf{Q}(x)\rangle_{\ell_2}\,dx\right|. \tag{2.3}$$

We now estimate the right hand side of (2.3). Define $L_\varepsilon(x) := e^{-\pi\varepsilon|x|^2}$ for $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Using the scalar transference result (2.2) of Lemma 2.3, we obtain

$$\begin{aligned}
\int_{\mathbb{T}^d} \langle S(\mathbf{P})(x), \mathbf{Q}(x)\rangle_{\ell_2}\, dx &= \sum_{i=1}^N \int_{\mathbb{T}^d} S(P_i)(x)\overline{Q_i(x)}\, dx \\
&= \sum_{i=1}^N \lim_{\varepsilon\to 0^+} \varepsilon^{d/2} \int_{\mathbb{R}^d} T(P_i L_{\varepsilon/p})(x)\overline{Q_i(x)L_{\varepsilon/q}(x)}\, dx \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{d/2} \int_{\mathbb{R}^d} \langle T(\mathbf{P}L_{\varepsilon/p})(x), \mathbf{Q}(x)L_{\varepsilon/q}(x)\rangle_{\ell_2}\, dx \\
&= \lim_{\varepsilon\to 0^+} \varepsilon^{d/2} \int_{\mathbb{R}^d} \langle W^{1/p}(x)T(\mathbf{P}L_{\varepsilon/p})(x), W^{-1/p}(x)\mathbf{Q}(x)L_{\varepsilon/q}(x)\rangle_{\ell_2}\, dx \\
&\le \limsup_{\varepsilon\to 0^+} \left(\int_{\mathbb{R}^d} \varepsilon^{d/2}|W^{1/p}(x)T(\mathbf{P}L_{\varepsilon/p})(x)|^p\, dx\right)^{1/p} \\
&\qquad \times \limsup_{\varepsilon\to 0^+} \left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{-1/p}(x)\mathbf{Q}(x)|^q\, dx\right)^{1/q} \\
&\le \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)} \limsup_{\varepsilon\to 0^+} \left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{1/p}(x)\mathbf{P}(x)|^p\, dx\right)^{1/p} \\
&\qquad \times \limsup_{\varepsilon\to 0^+} \left(\int_{\mathbb{R}^d} \varepsilon^{d/2}L_\varepsilon(x)|W^{-1/p}(x)\mathbf{Q}(x)|^q\, dx\right)^{1/q} \\
&= \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)} \left(\int_{\mathbb{T}^d} |W^{1/p}(x)\mathbf{P}(x)|^p\, dx\right)^{1/p} \\
&\qquad \times \left(\int_{\mathbb{T}^d} |W^{-1/p}(x)\mathbf{Q}(x)|^q\, dx\right)^{1/q}. \qquad (2.4)
\end{aligned}$$

In the last step, we have used that for any periodic function $f \in L_1(\mathbb{T}^d)$, using Poisson's summation formula,

$$\begin{aligned}
\varepsilon^{d/2}\int_{\mathbb{R}^d} f(x)L_\varepsilon(x)\, dx &= \varepsilon^{d/2}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^d} f(x-k)e^{-\pi\varepsilon|x-k|^2}\, dx \\
&= \int_{\mathbb{T}^d} f(x)\varepsilon^{d/2}\sum_{k\in\mathbb{Z}^d} e^{-\pi\varepsilon|x-k|^2}\, dx \\
&= \int_{\mathbb{T}^d} f(x)\sum_{k\in\mathbb{Z}^d} e^{-\pi|k|^2/\varepsilon}e^{2\pi i x\cdot k}\, dx \\
&= \int_{\mathbb{T}^d} f(x)\, dx + E_\varepsilon,
\end{aligned}$$

where

$$|E_\varepsilon| \le \|f\|_{L_1(\mathbb{T}^d)} \sum_{k\ne 0} e^{-\pi|k|^2/\varepsilon} \to 0,$$

as $\varepsilon \to 0$. We can now complete the proof. Using the estimate (2.4) in (2.3), we immediately see that

$$\|S(\mathbf{P})\|_{L_p(\mathbb{T}^d;W)\to L_p(\mathbb{T}^d;W)} \leq \|T\|_{L_p(\mathbb{R}^d;W)\to L_p(\mathbb{R}^d;W)}\|\mathbf{P}\|_{L_p(\mathbb{T}^d;W)}.$$

Moreover, S can be extended to a bounded operator on $L_p(\mathbb{T}^d;W)$ with the required norm estimate, since $\mathscr{P}^{d,N}$ is dense in $L_p(\mathbb{T}^d;W)$. Therefore, $\|\{b(m)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W)} \leq \|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)}$. □

2.2. **Multipliers in $\mathscr{M}_p(\mathbb{T}^d;W)$.** We now turn to a converse result to Proposition 2.4. At a first glance, the statement of Proposition 2.5 below may appear unnatural since it requires information about dilated versions of the weight W. However, as will be demonstrated in Section 4, the most interesting class of weights is the Muckenhoupt class A_p, which is actually dilation invariant making the statement appear more natural.

Proposition 2.5. *Let $1 < p < \infty$, and let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a periodic matrix weight with $W, W^{-q/p} \in L_{1,loc}$, where $\frac{1}{p}+\frac{1}{q}=1$. Suppose that b is a bounded continuous function on $\mathbb{R}^d$ with $\{b(m/M)\}_{m\in\mathbb{Z}^d} \in \mathscr{M}_p(\mathbb{T}^d;W(M\cdot))$ uniformly in $M \in \mathbb{N}$. Then b is in $\mathscr{M}_p(\mathbb{R}^d;W)$. Moreover,*

$$\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)} \leq C_p := \sup_{M\in\mathbb{N}} \|\{b(m/M)\}_m\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))}.$$

Proof. Let $\mathbf{F}(x) = [F_1(x),\dots,F_N(x)]^T$ and $\mathbf{G}(x) = [G_1(x),\dots,G_N(x)]^T$ be vectors of compactly supported smooth functions. There is an $M_0 \geq 1$ such that $M \geq M_0$ implies that $\mathbf{F}(Mx)$ and $\mathbf{G}(Mx)$ are supported in $[-1/2,1/2)^d$. Let $M \in \mathbb{N}$ with $M \geq M_0$, and define

$$\mathbf{F}_M(x) = \sum_{k\in\mathbb{Z}^d} \mathbf{F}(M(x-k)), \qquad \mathbf{G}_M(x) = \sum_{k\in\mathbb{Z}^d} \mathbf{G}(M(x-k)).$$

A straightforward calculation shows that the Fourier coefficients of $\mathbf{F}_M$ and $\mathbf{G}_M$ satisfy $\widehat{\mathbf{F}_M}(m) = M^{-d}\widehat{\mathbf{F}}(m/M)$ and $\widehat{\mathbf{G}_M}(m) = M^{-d}\widehat{\mathbf{G}}(m/M)$. We use these facts to obtain,

(2.5)
$$\begin{aligned}
&\left|\sum_{i=1}^N \sum_{m\in\mathbb{Z}^d} b(m/M)\hat{F}_i(m/M)\overline{\hat{G}_i(m/M)}\mathrm{Vol}\big(\big[\tfrac{m}{M},\tfrac{m+1}{M}\big]^d\big)\right| \\
&\qquad = \left|M^d \sum_{i=1}^N \sum_{m\in\mathbb{Z}^d} b(m/M)\widehat{F_{i,M}}(m)\overline{\widehat{G_{i,M}}(m)}\right| \\
&\qquad = \left|M^d \int_{\mathbb{T}^d} \sum_{i=1}^N \left(\sum_{m\in\mathbb{Z}^d} b(m/M)\widehat{F_{i,M}}(m)e^{2\pi i m\cdot x}\right)\overline{G_{i,M}(x)}\,dx\right|
\end{aligned}$$

$$
\begin{aligned}
&= M^d \int_{\mathbb{T}^d} \langle T_{\{b(m/M)\}} \mathbf{F}_M(x), \mathbf{G}_M(x)\rangle_{\ell_2}\,dx \\
&= M^d \int_{\mathbb{T}^d} \langle W^{1/p}(Mx) T_{\{b(m/M)\}} \mathbf{F}_M(x), W^{-1/p}(Mx)\mathbf{G}_M(x)\rangle_{\ell_2}\,dx \\
&\le M^d \left(\int_{\mathbb{T}^d} |W^{1/p}(Mx) T_{\{b(m/M)\}} \mathbf{F}_M(x)|^p\,dx \right)^{1/p} \\
&\qquad\qquad \times \left(\int_{\mathbb{T}^d} |W^{-1/p}(Mx)\mathbf{G}_M(x)|^q\,dx \right)^{1/q} \\
&\le M^d \|\{b(m/M)\}_m\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))} \|\mathbf{F}_M\|_{L_p(\mathbb{T}^d;W(M\cdot))} \| \cdot \|\mathbf{G}_M\|_{L_q(\mathbb{T}^d;W^{-q/p}(M\cdot))} \\
&\le C_p \|\mathbf{F}\|_{L_p(\mathbb{R}^d;W)} \|\mathbf{G}\|_{L_q(\mathbb{R}^d;W^{-q/p})}.
\end{aligned}
$$

The functions $b(\xi)\widehat{F}_i(\xi)\widehat{G}_i(\xi)$ are Riemann integrable on $\mathbb{R}^d$, so letting the integer $M \to \infty$ in (2.5), we obtain using Parseval's relation,

$$
\left| \sum_{i=1}^N \int_{\mathbb{R}^d} b(\xi)\hat{F}_i(\xi)\overline{\hat{G}_i(\xi)}\,d\xi \right| = \left| \int_{\mathbb{R}^d} \langle T_b\mathbf{F}(x), \mathbf{G}(x)\rangle_{\ell_2}\,dx \right| \le C_p \|\mathbf{F}\|_{L_p(\mathbb{R}^d;W)} \|\mathbf{G}\|_{L_q(\mathbb{R}^d;W^{-q/p})}.
$$

Notice that the family of vectors of compactly smooth functions are dense in both $L_p(\mathbb{R}^d;W)$ and $L_q(\mathbb{R}^d;W^{-q/p})$, respectively, since $W, W^{-q/p} \in L_{1,\mathrm{loc}}$. Therefore, it follows that $b \in \mathscr{M}_p(\mathbb{R}^d;W)$ with $\|b\|_{\mathscr{M}_p(\mathbb{R}^d;W)} \le C_p$. □

3. Muckenhoupt matrix weights

So far we have proved two transference results, Proposition 2.4 and Proposition 2.5. However, for these results to be useful we need to have interesting examples of bounded multipliers on $L_p(\mathbb{R}^d;W)$ and/or $L_p(\mathbb{T}^d;W)$ that can be used for the transfer process. This section contains an application of Proposition 2.4 to the case of a matrix weight W that satisfies the so-called A_p condition for matrices.

The Muckenhoupt A_p-condition for matrix weights was introduced by Nazarov, Treĭl′ and Volberg in [7, 11] to study boundedness properties of the vector-valued Hilbert transform. Here we follow Roudenko [10] and give an equivalent and more direct definition of matrix A_p weights. It is proved in [10] that the following definition is equivalent to the A_p condition considered in [7, 11]. We let $\mathscr{B}(d)$ denote the family of all Euclidean balls in $\mathbb{R}^d$.

Definition 3.1. Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a matrix weight. For $1 < p < \infty$, let q denote the conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We say that W belongs to the matrix Muckenhoupt class A_p provided

$$
A(p,W) := \sup_{B\in\mathscr{B}(d)} \int_B \left(\int_B \left\| W^{1/p}(x) W^{-1/p}(t) \right\|^q \frac{dt}{|B|} \right)^{p/q} \frac{dx}{|B|} < \infty. \tag{3.1}
$$

We notice that a simple change of variable in (3.1) reveals that A_p is dilation invariant. More precisely, for a matrix weight $W \in A_p$, and any $M > 0$, the dilated weight $W(M\cdot)$ is also in A_p with the same bound $A(p,W(M\cdot)) = A(p,W)$. This fact will be used in Section 4.

The importance of the Muckenhoupt A_p class is already apparent from the study of the Hilbert transform in [7, 11]. Later Goldberg [5] demonstrated that the Muckenhoupt A_p class is also useful for the study of general vector-valued multipliers.

Our main result Theorem 3.3 will rely on Goldberg's result, which we will state in detail. The setup is the following. We consider a singular integral operator T of convolution type associated with a kernel $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$. That is, for a compactly supported test function f,

$$Tf(x) = \int_{\mathbb{R}^d} K(x-y) f(y)\, dy,$$

for almost all x outside $\operatorname{supp}(f)$. The following is a standard regularity hypothesis for K that we will need below: there exists a constant C such that

$$|K(x)| \le C|x|^{-d} \quad \text{and} \quad |\nabla K(x)| \le C|x|^{-d-1}, \qquad x \in \mathbb{R}^d \backslash \{0\}. \tag{3.2}$$

For this type of operators, Goldberg proved the following general result.

Theorem 3.2 (Goldberg [5])**.** *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a matrix weight.*

(i) *Suppose $W \in A_p$ for some $1 < p < \infty$. Assume that $T : L_r(\mathbb{R}^d) \to L_r(\mathbb{R}^d)$ is a bounded convolution operator for some $1 < r < \infty$, with associated convolution kernel K satisfying* (3.2)*. Then T extends to a bounded operator on $L_p(\mathbb{R}^d;W)$ with an operator norm that only depends on C and on the A_p constant of W.*

(ii) *Conversely, suppose T is a convolution operator with kernel K that is bounded on $L_p(\mathbb{R}^d;W)$ for some $1 < p < \infty$. If the kernel K satisfies*

$$|\nabla K(x)| \le C|x|^{-d-1}, \qquad x \in \mathbb{R}^d,$$

for some $C > 0$, and there is a unit vector $\mathbf{u} \in \mathbb{S}^{d-1}$ and a constant $a > 0$, such that

$$|K(r\mathbf{u})| \ge a|r|^{-d}, \qquad r \in \mathbb{R}\backslash\{0\}, \tag{3.3}$$

then W is in A_p.

We mention that proving $L_p(\mathbb{R}^d)$ boundedness for a scalar multiplier operator with an associated integrable convolution kernel reduces to a simple application of Young's inequality. However, in the matrix weighted setting such a simplified appoach fails, and more sophisticated results like Theorem 3.2 are needed to obtain $L_p(\mathbb{R}^d;W)$-boundedness, even for multipliers associated with nice smooth localized convolution kernels.

We now combine Theorem 3.2 with Propositions 2.4 and 2.5 to obtain the main application of our transference results.

Theorem 3.3. *Let $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ be a periodic matrix weight.*

(i) *Let $W \in A_p$ for some $1 < p < \infty$, and suppose that for some $1 < r < \infty$, $T_b : L_r(\mathbb{R}^d) \to L_r(\mathbb{R}^d)$ is a bounded multiplier operator induced by a regulated multiplier $b : \mathbb{R}^d \to \mathbb{R}$. If the associated convolution kernel K satisfies* (3.2)*, then for $M \in \mathbb{N}$: $\{b(m/M)\}_m \in \mathscr{M}_p(\mathbb{T}^d;W(M\cdot))$, and $b(\cdot/M) \in \mathscr{M}_p(\mathbb{R}^d;W(M\cdot))$, with*

$$\|\{b(m/M)\}\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))} \le \|b(\cdot/M)\|_{\mathscr{M}_p(\mathbb{R}^d;W(M\cdot))}. \tag{3.4}$$

Moreover, the bound on $\|b(\cdot/M)\|_{\mathscr{M}_p(\mathbb{R}^d;W(M\cdot))}$ depends only on C in (3.2) *and on the A_p constant of W.*

(ii) *Conversely, suppose that b is a bounded continuous function on* $\mathbb{R}^d$ *with*

$$\{b(m/M)\}_{m\in\mathbb{Z}^d} \in \mathscr{M}_p(\mathbb{T}^d;W(M\cdot)), \quad \text{uniformly in } M \in \mathbb{N},$$

for some $1<p<\infty$, *and suppose* $W^{-p/q} \in L_{1,loc}$, $\frac{1}{p}+\frac{1}{q}=1$. *If the multiplier* T_b *is associated with a kernel* K *that satisfies* $|\nabla K(x)| \le C|x|^{-d-1}$, $x \in \mathbb{R}^d\backslash\{0\}$, *and the kernel also satisfies* (3.3), *then* $W \in A_p$.

Proof. First we prove (i). Let $M \in \mathbb{N}$ and notice that $W \in A_p$ implies $W(M\cdot) \in A_p$ with $A(p,W(M\cdot)) = A(p,W)$. Moreover, the multiplier $b(\cdot/M)$ is associated with the kernel $M^d K(M\cdot)$ that satisfies (3.2) with the same constant as for K.

We now use Theorem 3.3 to deduce that $T_b(\cdot/M)$ is a bounded operator on $L_p(\mathbb{R}^d;W(M\cdot))$, i.e., $b(\cdot/M) \in \mathscr{M}_p(\mathbb{R}^d;W(M\cdot))$ with a norm that depends only the constant for K in (3.2) and on the A_p constant of W. The fact that $W(M\cdot) \in A_p$ implies that $W^{-q/p}(M\cdot) \in L_{1,loc}$, where q is the conjugate exponent to p, see [10]. Hence, Proposition 2.4 applies to $b(\cdot/M)$ in $\mathscr{M}_p(\mathbb{R}^d;W(M\cdot))$, and we immediately obtain the norm estimate (3.4).

We turn to the proof of (ii). By Proposition 2.5, T_b is bounded on $L_p(\mathbb{R}^d;W)$. We can now use Theorem 3.2.(ii) to conclude that $W \in A_p$. □

We conclude this paper by presenting some applications of Theorem 3.3.

4. EXAMPLES

Here we consider a fairly general setup that will provide a number of examples, including multipliers related to the Riesz transform.

We consider a C^∞ function Ω_0 on $\mathbb{R}^d\backslash\{0\}$ that is homogeneous of degree zero, i.e., $\Omega_0(\lambda x) = \Omega_0(x)$ for all $\lambda > 0, x \in \mathbb{R}^d\backslash\{0\}$. Then it is well-known that the induced multiplier operator T_{Ω_0} is associated with a convolution kernel of the type

$$K(x) = \frac{\Omega(x/|x|)}{|x|^d}, \qquad x \in \mathbb{R}^d\backslash\{0\}, \tag{4.1}$$

with Ω a C^∞ function on the unit sphere $\mathbb{S}^{d-1}$ with mean value zero, see e.g. [6, Prop. 2.4.7]. Since Ω_0 is bounded on $\mathbb{R}^d\backslash\{0\}$, T_{Ω_0} clearly extends to a bounded operator on $L_2(\mathbb{R}^d)$. By (re)defining the value of Ω_0 at zero appropriately, we can also think of Ω_0 as a regulated multiplier. Notice that K is smooth away from the origin, and it is homogeneous of degree $-d$, so it is straightforward to verify that the conditions given by (3.2) hold.

Hence, Theorem 3.3 applies to this setup. We summarize our findings in the following Corollary.

Corollary 4.1. *Let* $W:\mathbb{R}^d \to \mathbb{C}^{N\times N}$ *be a periodic matrix weight, and let* $\Omega_0 : \mathbb{R}^d\backslash\{0\} \to \mathbb{C}$ *be* C^∞ *and homogeneous of degree zero. We suppose that* Ω_0 *has been regularized at zero.*

- *Suppose* $W \in A_p$ *for some* $1<p<\infty$, *then*

$$\sup_{M\in\mathbb{N}} \|\{\Omega_0(m)\}_{m\in\mathbb{Z}^d}\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))} < \infty. \tag{4.2}$$

- *Conversely, suppose* $W^{-q/p} \in L_{1,loc}$, *for some* $1<p<\infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *If* (4.2) *holds, and there exists a direction* $\mathbf{u} \in \mathbb{S}^{d-1}$ *such that* $\Omega(\mathbf{u})\Omega(-\mathbf{u}) \neq 0$, *with* Ω *defined by* (4.1), *then* $W \in A_p$.

Proof. First notice that $\{\Omega_0(m)\}_{m\in\mathbb{Z}^d} = \{\Omega_0(m/M)\}_{m\in\mathbb{Z}^d}$ for any $M \in \mathbb{N}$ by the homogeneity of Ω_0. The first part of the corollary now follows directly from Theorem 3.3.(i) since the conditions given by (3.2) hold for the kernel (4.1). For the second part, we notice that whenever $\mathbf{u} \in \mathbb{S}^{d-1}$ is such that $\Omega(\mathbf{u})\Omega(-\mathbf{u}) \neq 0$, then

$$|r|^d |K(r\mathbf{u})| \geq \min\{|\Omega(-\mathbf{u})|, |\Omega(\mathbf{u})|\} > 0,$$

for $r \in \mathbb{R}\backslash\{0\}$. Hence, the second claim follows directly from Theorem 3.3.(ii). □

In particular, Corollary 4.1 applies to each of the Riesz multipliers

$$m_j(x) = -i\frac{x_j}{|x|}, \qquad j = 1,2,\ldots,d, \quad x \in \mathbb{R}^d\backslash\{0\}.$$

It follows that for a periodic matrix weight $W : \mathbb{R}^d \to \mathbb{C}^{N\times N}$ with $W^{-q/p} \in L_{1,loc}$, for $1 < p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$, $W \in A_p$ if and only if

$$\sup_{M\in\mathbb{N}} \|\{m_j(k)\}_{k\in\mathbb{Z}^d}\|_{\mathscr{M}_p(\mathbb{T}^d;W(M\cdot))} < \infty.$$

Another, more general, example where Corollary 4.1 applies is given by the multiplier

$$m(x) = \frac{P(x)}{|x|^k}, \qquad x \in \mathbb{R}^d\backslash\{0\},$$

with P a non-trivial homogeneous polynomial of degree $k \in \mathbb{N}$.

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