AN EXAMPLE OF AN ALMOST GREEDY UNIFORMLY BOUNDED ORTHONORMAL BASIS FOR $L_p([0,1])$

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Abstract. We construct a uniformly bounded orthonormal almost greedy basis for $L_p([0,1])$, $1 < p < \infty$. The example shows that it is not possible to extend Orlicz's theorem, stating that there are no uniformly bounded orthonormal unconditional bases for $L_p([0,1])$, $p \neq 2$, to the class of almost greedy bases.

1. Introduction

Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be a bounded Schauder basis for a Banach space X, i.e., a basis for which $0 < \inf_n \|e_n\|_X \leq \sup_n \|e_n\|_X < \infty$. An approximation algorithm associated with $\mathcal{B}$ is a sequence $\{A_n\}_{n=1}^\infty$ of (possibly nonlinear) maps $A_n : X \to X$ such that for $x \in X$, $A_n(x)$ is a linear combination of at most n elements from $\mathcal{B}$. We say that the algorithm is convergent if $\lim_{n\to\infty} \|x - A_n(x)\|_X = 0$ for every $x \in X$. For a Schauder basis there is a natural convergent approximation algorithm. Suppose the dual system to $\mathcal{B}$ is given by $\{e_k^*\}_{k\in\mathbb{N}}$. Then the linear approximation algorithm is given by the partial sums $S_n(x) = \sum_{k=1}^n e_k^*(x)e_k$.

Another quite natural approximation algorithm is the greedy approximation algorithm where the partial sums are obtained by thresholding the expansion coefficients. The algorithm is defined as follows. For each element $x \in X$ we define the greedy ordering of the coefficients as the map $\rho : \mathbb{N} \to \mathbb{N}$ with $\rho(\mathbb{N}) \supseteq \{j : e_j^*(x) \neq 0\}$ such that for $j < k$ we have either $|\, e_{\rho(k)}^*(x)| < |\, e_{\rho(j)}^*(x)|$ or $|\, e_{\rho(k)}^*(x)| = |\, e_{\rho(j)}^*(x)|$ and $\rho(k) > \rho(j)$. Then the greedy m-term approximant to x is given by $\mathcal{G}_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x)e_{\rho(j)}$. The question is whether the greedy algorithm is convergent. This is clearly the case for an *unconditional basis* where the expansion $x = \sum_{k=1}^\infty e_k^*(x)e_k$ converges regardless of the summation order. However, Temlyakov and Konyagin [4] showed that the greedy algorithm may also converge for certain conditional bases. This leads to the definition of a quasi-greedy basis.

Definition 1.1 ([4]). *A bounded Schauder basis for a Banach space X is called* quasi-greedy *if there exists a constant C such that for $x \in X$, $\|\mathcal{G}_m(x)\|_X \leq C\|x\|_X$ for $m \geq 1$.*

Wojtaszczyk proved the following result which gives a more intuitive interpretation of quasi-greedy bases.

Theorem 1.2 ([9]). *A bounded Schauder basis for a Banach space X is* quasi-greedy *if and only if $\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\|_X = 0$ for every element $x \in X$.*

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In this note we study quasi-greedy bases for $L_p := L_p([0,1])$, $1 < p < \infty$, with a particular structure. We are interested in uniformly bounded bases $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ such that $\mathcal{B}$ is an orthonormal basis for L_2. It is a well-known result by Orlicz that such a basis can be unconditional only for $p = 2$, so it is never trivially quasi-greedy except for $p = 2$.

It was proved by Temlyakov [8] that the trigonometric system in L_p, $1 \leq p \leq \infty$, $p \neq 2$, fails to be quasi-greedy. Independently, and using a completely different approach, Córdoba and Fernández [1] proved the same result in the range $1 \leq p < 2$. One can also verify that the Walsh system fails to be quasi greedy in L_p, $p \neq 2$. This leads to a natural question: are there any uniformly bounded orthonormal quasi-greedy bases for L_p?

A negative answer to this question would give a nice improvement of Orlicz's theorem to the class of quasi-greedy bases. However, such an improved result is not possible. Below we construct a uniformly bounded orthonormal *almost greedy* basis for L_p, $1 < p < \infty$. An almost greedy basis is a quasi-greedy basis with one additional property.

Definition 1.3. *A bounded Schauder basis $\{e_n\}_n$ for a Banach space X is almost greedy if there is a constant C such that for $x \in X$,*

$$\|x - \mathcal{G}_n(x)\|_X \leq C \inf \Big\{ \Big\| x - \sum_{j\in A} e_j^*(x) e_j \Big\| : A \subset \mathbb{N}, |A| = n \Big\}.$$

It was proved in [2] that a basis is almost greedy if and only if it is quasi-greedy and democratic. A Schauder basis $\{e_n\}_n$ is called democratic if there exists C such that for any finite sets $A, B \subset \mathbb{N}$ with $|A| = |B|$, we have

$$\Big\| \sum_{j\in A} e_j \Big\|_X \leq C \Big\| \sum_{j\in B} e_j \Big\|_X.$$

We can now state the main result of this note.

Theorem 1.4. *There exists a uniformly bounded orthonormal almost greedy basis for $L_p([0,1])$, $1 < p < \infty$.*

We should note that without the assumption that the system be uniformly bounded, one can obtain a stronger result. It is known that the Haar system on $[0,1]$, normalized in L_p, is an unconditional and democratic basis (a so-called greedy basis) for L_p, $1 < p < \infty$, see [7].

2. A uniformly bounded almost greedy ONB for L_p

Classical uniformly bounded orthonormal systems such as the trigonometric basis and the Walsh system fail to form quasi-greedy bases for $L_p := L_p([0,1])$, $1 < p < \infty$. The problem behind this failure is that such systems are very far from being democratic. This behavior is not representative for all uniformly bounded orthonormal bases. In this section we construct an example of a quasi greedy uniformly bounded system in L_p, $1 < p < \infty$. The example we present is a variation on a construction by Kostyukovsky and Olevskiĭ [5]. The example in [5] was used to study pointwise convergence a.e. of greedy approximants to L_2-functions. Later Wojtaszczyk [9] used the same type of construction to define quasi-greedy bases for $X \oplus \ell_2$, with X a quasi-Banach space with a Besselian basis.

Let us introduce some notation. The Rademacher functions are given by $r_k(t) = \operatorname{sign}(\sin(2^k \pi t))$ for $k \geq 1$. Khintchine's inequality will be essential for the estimates below. The inequality states that for $1 \leq p < \infty$ there exist A_p, B_p such that for any finite sequence $\{a_k\}_{k\geq 1}$,

$$A_p\Big(\sum_k |a_k|^2\Big)^{1/2} \leq \Big(\int_0^1 \Big|\sum_k a_k r_k(t)\Big|^p\,dt\Big)^{1/p} \leq B_p\Big(\sum_k |a_k|^2\Big)^{1/2}. \tag{1}$$

Khintchine's inequality shows that the Rademacher functions form a democratic system in L_p. However, the Rademacher system is far from complete so it cannot be used directly to obtain an almost greedy basis in L_p. In our example we use the fact that the Rademacher functions form a subsystem of a complete system, namely the Walsh system. The Walsh system $\mathcal{W} = \{W_n\}_{n=0}^\infty$ is defined as follows. We let $W_0(t) = 1$ and for $n = \sum_{j=1}^k \varepsilon_j 2^{j-1}$, the binary expansion of $n \in \mathbb{N}$, we let

$$W_n(t) = \prod_{j=1}^k r_j^{\varepsilon_j}(t). \tag{2}$$

The Walsh system forms a uniformly bounded orthonormal basis for L_2 and a Schauder basis for L_p, $1 < p < \infty$, see [3]. The idea is to reorder the Walsh system such that we obtain large dyadic blocks of Rademacher functions with the remaining Walsh functions placed in between the Rademacher blocks. Let us consider the details.

For $k = 1, 2, \ldots$, we define the $2^k \times 2^k$ Olevskiĭ matrix $A^k = (a_{ij}^{(k)})_{i,j=1}^{2^k}$ by the following formulas

$$a_{i1}^{(k)} = 2^{-k/2} \quad \text{for} \quad i = 1, 2, \ldots, 2^k,$$

and for $j = 2^s + \nu$, with $1 \leq \nu \leq 2^s$ and $s = 0, 1, \ldots, k-1$, we let

$$a_{ij}^{(k)} = \begin{cases} 2^{(s-k)/2} & \text{for} \quad (\nu-1)2^{k-s} < i \leq (2\nu-1)2^{k-s-1} \\ -2^{(s-k)/2} & \text{for} \quad (2\nu-1)2^{k-s-1} < i \leq \nu 2^{k-s} \\ 0 & \text{otherwise.} \end{cases}$$

One can check (see [6, Chapter IV]) that A^k are orthogonal matrices and there exists a finite constant C such that for all i, k we have

$$\sum_{j=1}^{2^k} |a_{ij}^{(k)}| \leq C. \tag{3}$$

Put $N_k = 2^{10^k}$ and define F_k such that $F_0 = 0$, $F_1 = N_1 - 1$ and $F_k - F_{k-1} = N_k - 1$, $k = 1, 2, \ldots$. We consider the Walsh system $\mathcal{W} = \{W_n\}_{n=0}^\infty$ on $[0,1]$. We split $\mathcal{W}$ into two subsystems. The first subsystem $\mathcal{W}_1 = \{r_k\}_{k=1}^\infty$ is the Rademacher functions with their natural ordering. The second subsystem $\mathcal{W}_2 = \{\phi_k\}_{k=1}^\infty$ is the collection of Walsh functions not in $\mathcal{W}_1$ with the ordering from $\mathcal{W}$. We now impose the ordering

$$\phi_1, r_1, r_2, \ldots, r_{F_1}, \phi_2, r_{F_1+1}, \ldots, r_{F_2}, \phi_3, r_{F_2+1}, \ldots, r_{F_3}, \phi_4, \ldots$$

The block $\mathcal{B}_k := \{\phi_k, r_{F_{k-1}+1}, \ldots, r_{F_k}\}$ has length N_k, and we apply A^{10^k} to $\mathcal{B}_k$ to obtain a new orthonormal system $\{\psi_i^{(k)}\}_{i=1}^{N_k}$ given by

$$\psi_i^{(k)} = \frac{\phi_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{(10^k)} r_{F_{k-1}+j-1}. \tag{4}$$

The system ordered $\psi_1^{(1)}, \ldots, \psi_{N_1}^{(1)}, \psi_1^{(2)}, \ldots, \psi_{N_2}^{(2)}, \ldots$ will be denoted $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$. It is easy to verify that $\mathcal{B}$ is an orthonormal basis for L_2 since each matrix A^{10^k} is orthogonal. The system is uniformly bounded which follows by (3) and the fact that $\mathcal{W}$ is uniformly bounded. The system $\mathcal{B}$ is our candidate for an almost greedy basis for L_p, $1 < p < \infty$. We split the proof of Theorem 1.4 into three parts. First we prove that $\mathcal{B}$ is democratic in L_p. Then we prove that the system forms a Schauder basis for L_p, and the final step is to prove that the system forms a quasi-greedy basis for L_p.

Lemma 2.1. *The system $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$ is democratic in L_p, $1 < p < \infty$, with*

$$\Big\| \sum_{k \in A} \psi_k \Big\|_p \asymp |A|^{1/2}.$$

Proof. Fix $2 < p < \infty$. Let $S = \sum_{k\in\Lambda} \psi_k$ with $|\Lambda| = N$. We write

$$S = \sum_{k=1}^{\infty} \sum_{j \in \Lambda_k} \psi_j^{(k)} = \sum_{k=1}^{\infty} \frac{|\Lambda_k|}{\sqrt{N_k}} \phi_k + \sum_{k=1}^{\infty} \sum_{i\in\Lambda_k} \sum_{j=2}^{N_k} a_{ij}^{(10^k)} r_{F_{k-1}+j-1} := S_1 + S_2,$$

with $\sum_k |\Lambda_k| = |\Lambda|$, and $|\Lambda_k| \le N_k$. Notice that the coefficients of $\sum_{j\in\Lambda_k} \psi_j^{(k)}$ relative to the block $\mathcal{B}_k$ has l_2-norm $|\Lambda_k|^{1/2}$ since A^{10^k} is orthogonal. Hence, by Khintchine's inequality,

$$\|S_2\|_p = \Big\| \sum_{k=1}^{\infty} \sum_{j=2}^{N_k} \Big(\sum_{i\in\Lambda_k} a_{ij}^{(10^k)} \Big) r_{F_{k-1}+j-1} \Big\|_p \le B_p \Big(\sum_k |\Lambda_k| \Big)^{1/2} = B_p N^{1/2}.$$

We now estimate S_1. Write

$$S_1 = \sum_{k=1}^{\infty} \frac{|\Lambda_k|}{\sqrt{N_k}} \phi_k = \sum_{k\in A} \frac{|\Lambda_k|}{\sqrt{N_k}} \phi_k + \sum_{k \in B} \frac{|\Lambda_k|}{\sqrt{N_k}} \phi_k := S_1^1 + S_1^2,$$

where $A = \{k : |\Lambda_k| \le (N_k)^{3/4}\}$ and $B = \{k : |\Lambda_k| > (N_k)^{3/4}\}$. Using the Cauchy-Schwartz inequality,

$$\|S_1^1\|_p \le \sum_{k\in A} \frac{|\Lambda_k|}{\sqrt{N_k}} \le \Big(\sum_{k\in A} |\Lambda_k| \Big)^{1/2} \Big(\sum_{k\in A} \frac{|\Lambda_k|}{N_k} \Big)^{1/2} \le N^{1/2} \Big(\sum_{k\in\mathbb{N}} N_k^{-1/4} \Big)^{1/2} = CN^{1/2}.$$

We turn to S_1^2. If B is empty, we are done. Otherwise, B is a finite set and we can define $L = \max B$. Using $|\Lambda_k| \le N_k$, we obtain

$$\sum_{k\in B; k<L} \frac{|\Lambda_k|}{\sqrt{N_k}} \le \sum_{k \in B; k<L} \sqrt{N_k} \le \sum_{j=1}^{10^{L-1}} 2^{j/2} \le 2 \cdot 2^{(10^{L-1}/2)} \le 2 \cdot 2^{(10^L/4)} \le 2 \frac{|\Lambda_L|}{\sqrt{N_L}}$$

Hence,

$$\|S_1^2\|_p \leq \sum_{k\in B} \frac{|\Lambda_k|}{\sqrt{N_k}} \leq 3\frac{|\Lambda_L|}{\sqrt{N_L}} \leq 3\sqrt{|\Lambda_L|} \leq 3\sqrt{|\Lambda|} = 3N^{1/2},$$

where we used that $|\Lambda_L| \leq N_L$. We conclude that $\|S\|_p \leq C' N^{1/2}$, with C' independent of Λ. Since $N^{1/2} = \|S\|_2 \leq \|S\|_p$ we deduce that $\mathcal{B}$ is democratic in L_p, $2 \leq p < \infty$. For $1 < q < 2$ we have $\|S\|_q \leq \|S\|_2 = N^{1/2}$. By Hölder's inequality,

$$N = \|S\|_2^2 \leq \|S\|_q \|S\|_p \leq C_p N^{1/2} \|S\|_q,$$

for $1/q + 1/p = 1$. Again, we conclude that $\|S\|_q \asymp N^{1/2}$, so $\mathcal{B}$ is democratic in L_q, $1 < q < 2$. $\square$

Next we prove that $\mathcal{B}$ is a basis for L_p.

Lemma 2.2. *The system $\mathcal{B} = \{\psi_k\}_{k=1}^\infty$ is a Schauder basis for L_p, $1 < p < \infty$.*

Proof. Notice that $\operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathcal{W})$ by construction, so $\operatorname{span}(\mathcal{B})$ is dense in L_p, $1 < p < \infty$, since $\mathcal{W}$ is a Schauder basis for L_p. Fix $2 < p < \infty$ and let $f \in L_p$. Let $S_n(f) = \sum_{k=1}^n \langle f, \psi_k\rangle \psi_k$ be the partial sum operator. We need to prove that the family of operators $\{S_n\}_n$ is uniformly bounded on L_p. Notice that $\{\langle f, \psi_k\rangle\}_k \in \ell_2(\mathbb{N})$ since $L_p \subset L_2$. For $n \in \mathbb{N}$, we can find $L \geq 1$ and $1 \leq m \leq N_L$ such that

$$S_n(f) = \sum_{k=1}^n \langle f, \psi_k\rangle \psi_k = \sum_{k=1}^{L-1} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)}\rangle \psi_k^{(k)} + \sum_{k=1}^{m} \langle f, \psi_k^{(L)}\rangle \psi_k^{(L)} := T_1 + T_2.$$

Let us estimate T_1. If $L = 1$ then $T_1 = 0$, so we may assume $L > 1$. The construction of $\mathcal{B}$ shows that T_1 is the orthogonal projection of f onto

$$\operatorname{span}\Big(\bigcup_{k=1}^{L-1} \bigcup_{j=1}^{N_k} \{\psi_k^{(k)}\}\Big) = \operatorname{span}\big\{\{W_0, W_1, \ldots, W_{L-2}\} \cup \{r_{\ell_0}, r_{\ell_0+1}, \ldots, r_{F_{L-1}}\}\big\},$$

with $\ell_0 = \lfloor \log_2(L) \rfloor$. It follows that we can rewrite T_1 as

$$T_1 = \sum_{k=0}^{L-2} \langle f, W_k\rangle W_k + P_R(f),$$

where $P_R(f)$ is the orthogonal projection of f onto $\operatorname{span}\{r_{\ell_0}, r_{\ell_0+1}, \ldots, r_{F_{L-1}}\}$. Thus, using Khintchine's inequality,

$$\|T_1\|_p \leq C_p \|f\|_p + B_p \|f\|_p,$$

where C_p is the basis constant for the Walsh system in L_p. Next we rewrite T_2 in the system $\{\phi_L, r_{F_{L-1}+1}, \ldots, r_{F_L}\}$,

$$T_2 = \sum_{k=1}^{m} \langle f, \psi_k^{(L)}\rangle \frac{\phi_L}{\sqrt{N_L}} + \sum_{j=2}^{N_L} \Big(\sum_{k=1}^{m} \langle f, \psi_k^{(L)}\rangle a_{ij}^{(10^L)} \Big) r_{F_{L-1}+j-1}.$$

By Khintchine's inequality, and the fact that A^{10^L} is orthogonal,

$$\Big\| \sum_{j=2}^{N_L} \Big(\sum_{k=1}^{m} \langle f, \psi_k^{(L)} \rangle a_{ij}^{(10^L)} \Big) r_{F_{L-1}+j-1} \Big\|_p \le B_p \| \{ \langle f, \psi_k^{(L)} \rangle \}_k \|_{\ell_2} < \infty.$$

Also,

$$\Big\| \sum_{k=1}^{m} \langle f, \psi_k^{(L)} \rangle \frac{\phi_L}{\sqrt{N_L}} \Big\|_p \le \sum_{k=1}^{m} |\langle f, \psi_k^{(L)} \rangle| \frac{1}{\sqrt{N_L}} \le \| \{ \langle f, \psi_k^{(L)} \rangle \}_k \|_{\ell_2} \sqrt{\frac{m}{N_L}} \le \| \{ \langle f, \psi_k^{(L)} \rangle \}_k \|_{\ell_2},$$

so $\|T_2\|_p < \infty$. The estimates of T_1 and T_2 are independent of n, and we obtain that $\sup_n \|S_n(f)\|_p < \infty$. Using the Banach-Steinhaus theorem we deduce that $\{S_n\}_n$ is a uniformly bounded family of linear operators on L_p. We conclude that $\mathcal{B}$ is a Schauder basis for L_p, $2 < p < \infty$, and the result for $1 < p < 2$ follows by a duality argument. □

We can now complete the proof of Theorem 1.4. Lemma 2.3 below together with Lemmas 2.1 and 2.2 immediately give Theorem 1.4.

Lemma 2.3. *The system $\mathcal{B} = \{\psi_k\}_{k=1}^{\infty}$ is a quasi-greedy basis for L_p, $1 < p < \infty$.*

Proof. First we consider $2 < p < \infty$. Let $f \in L_p \subset L_2$. Then we have the L_p-norm convergent expansion

$$f = \sum_{i=1}^{\infty} \langle f, \psi_i \rangle \psi_i, \tag{5}$$

with $\|\{\langle f, \psi_i \rangle\}_i\|_{\ell_2} \le \|f\|_2 \le \|f\|_p$. It suffices to prove that $\mathcal{G}_m(f)$ is convergent in L_p since $\mathcal{G}_m(f) \to f$ in L_2. We write (formally)

$$\begin{aligned} f &= \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)} \rangle \psi_k^{(k)} \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)} \rangle \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} \langle f, \psi_i^{(k)} \rangle \sum_{j=2}^{N_k} a_{ij}^{(10^k)} r_{F_{k-1}+j-1} \\ &= S^1 + S^2. \end{aligned}$$

Consider a sequence $\{\varepsilon_i^k\} \subset \{0,1\}$. By Khintchine's inequality, and the fact that each A^{10^k} is orthogonal,

$$\Big\| \sum_{k=1}^{\infty} \sum_{j=2}^{N_k} \Big(\sum_{i=1}^{N_k} \varepsilon_i^k \langle f, \psi_i^{(k)} \rangle a_{ij}^{(10^k)} \Big) r_{F_{k-1}+j-1} \Big\|_p \le B_p \Big(\sum_{k} \sum_{i=1}^{N_k} \varepsilon_i^k |\langle f, \psi_i^{(k)} \rangle|^2 \Big)^{1/2}.$$

It follows that S^2 is convergent and actually converges unconditionally in L_p. From this and the convergence of the series (5), we conclude that the partial sums for the series S^1,

$$S_n^1 = \sum_{k=1}^{L-1} \sum_{j=1}^{N_k} \langle f, \psi_j^{(k)} \rangle \frac{\phi_k}{\sqrt{N_k}} + \sum_{j=1}^{m} \langle f, \psi_j^{(L)} \rangle \frac{\phi_L}{\sqrt{N_L}}$$

converge in L_p.

The series defining S^2 converges unconditionally, so it suffices to prove that the series defining S^1 converges in L_p when the coefficients $\{\langle f, \psi_i\rangle\}_i$ are arranged in decreasing order. We define the sets

$$\begin{aligned}
\Lambda_k &= \left\{ j : \frac{1}{N_k} < |\langle f, \psi_j^{(k)}\rangle| < \frac{1}{N_k^{1/10}} \right\} \\
\Lambda_k' &= \left\{ j : |\langle f, \psi_j^{(k)}\rangle| \le \frac{1}{N_k} \right\} \\
\Lambda_k'' &= \left\{ j : |\langle f, \psi_j^{(k)}\rangle| \ge \frac{1}{N_k^{1/10}} \right\}.
\end{aligned} \tag{6}$$

Then (formally)

$$S^1 = \sum_{k=1}^{\infty} \sum_{j\in\Lambda_k} \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{j\in\Lambda_k'} \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} + \sum_{k=1}^{\infty} \sum_{j\in\Lambda_k''} \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} = T + T' + T''.$$

Notice that $\sum_{j\in\Lambda_k'} \|\langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}}\|_p \le \sum_{j\in\Lambda_k'} \frac{|\langle f, \psi_j^{(k)}\rangle|}{\sqrt{N_k}} \le 1/\sqrt{N_k}$, so the series defining T' converges absolutely in L_p. For T'' we notice that $|\Lambda_k''| \le \|f\|_2^2 N_k^{1/5}$, so

$$\sum_{j\in\Lambda_k''} \left\| \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} \right\|_p \le \sum_{j\in\Lambda_k''} \frac{|\langle f, \psi_j^{(k)}\rangle|}{\sqrt{N_k}} \le \frac{\|f\|_p \|f\|_2^2}{N_k^{3/10}},$$

and the series defining T'' converges absolutely in L_p.

The series defining S^1, T' and T'' converge in L_p, so we may conclude that the series defining T converges in L_p. From (6), we get

$$|\langle f, \psi_i^{(k)}\rangle| > \frac{1}{N_k} \ge \frac{1}{N_{k+1}^{1/10}} \ge |\langle f, \psi_j^{(k+1)}\rangle|, \qquad i \in \Lambda_k, j \in \Lambda_{k+1}; k = 1, 2, \ldots$$

so when we arrange T by decreasing order, the rearrangement can only take place inside the blocks. The estimate

$$\sum_{j\in\Lambda_k} \left\| \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} \right\|_p \le \left(\sum_{j\in\Lambda_k} |\langle f, \psi_j^{(k)}\rangle|^2 \right)^{1/2} \frac{|\Lambda_k|^{1/2}}{\sqrt{N_k}}, \quad k \ge 1,$$

shows that rearrangements inside blocks are well-behaved, and

$$\sum_{j\in\Lambda_k} \left\| \langle f, \psi_j^{(k)}\rangle \frac{\phi_k}{\sqrt{N_k}} \right\|_p \to 0 \quad \text{as} \quad k \to \infty.$$

We conclude that $\mathcal{G}_m(f)$ is convergent in L_p and consequently $\mathcal{B}$ is a quasi-greedy basis in L_p, $2 \le p < \infty$. Fix $1 < q < 2$ and let p be given by $1/q + 1/p = 1$. By Lemma 2.1, for any finite subset $A \subset \mathbb{N}$,

$$\left\| \sum_{k\in A} \psi_k \right\|_q \left\| \sum_{k\in A} \psi_k \right\|_p \le C|A|,$$

so $\mathcal{B}$ is a so-called bi-democratic system in L_p. It follows from [2, Theorem 5.4; (1) $\Rightarrow$ (2)] that $\mathcal{B}$ is a quasi-greedy basis for L_q. This completes the proof. $\square$

Remark 2.4. *To get a uniformly bounded quasi-greedy basis consisting of smooth functions, we can use the same construction based on the trigonometric system with any lacunary subsequence playing the role of the Rademacher system.*

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