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TIGHT WAVELET FRAMES IN LEBESGUE AND SOBOLEV SPACES

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ABSTRACT. We study tight wavelet frame systems in $L_p(\mathbb{R}^d)$, and prove that such systems (under mild hypotheses) give atomic decompositions of $L_p(\mathbb{R}^d)$ for $1 < p < \infty$. We also characterize $L_p(\mathbb{R}^d)$ and Sobolev space norms by the analysis coefficients for the frame. We consider Jackson inequalities for best m-term approximation with the systems in $L_p(\mathbb{R}^d)$ and prove that such inequalities exist. Moreover, it is proved that the approximation rate given by the Jackson inequality can be realized by thresholding the frame coefficients. Finally, we show that in certain restricted cases, the approximation spaces, for best m-term approximation, associated with tight wavelet frames can be characterized in terms of (essentially) Besov spaces.

1. INTRODUCTION

A tight wavelet frame (TWF) for $L_2(\mathbb{R}^d)$ is a finite collection of functions $\Psi = \{\psi^\ell\}_{\ell\in E}$ in $L_2(\mathbb{R}^d)$, $E = \{1,2,\dots,L\}$, for which the system $X(\Psi) := \{2^{jd/2}\psi^\ell(2^j \cdot -k) | j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell \in E\}$ is a tight frame for $L_2(\mathbb{R}^d)$, i.e., there exists a constant $A > 0$ such that $\sum_{g\in X(\Psi)} |\langle f,g\rangle|^2 = A\|f\|_{L_2}^2$ for any $f \in L_2(\mathbb{R}^d)$. The functions Ψ are called the generators of the TWF. The construction and properties of TWFs in $L_2(\mathbb{R}^d)$ have been studied extensively by many authors (see e.g. [16, 17]). The purpose of this paper is to study such frames in spaces different from $L_2(\mathbb{R}^d)$.

In particular, we will study TWFs in $L_p(\mathbb{R}^d)$ and L_p-based Sobolev spaces. We prove that most reasonable TWFs give atomic decompositions of $L_p(\mathbb{R}^d)$, $1 < p < \infty$, and it is proved that we can characterize the $L_p(\mathbb{R}^d)$ and Sobolev norm by the analysis coefficients associated with the frame. An important consequence of the characterization is that there is a Jackson inequality for nonlinear approximation with TWFs, and moreover we will show that the rate of convergence given by the Jackson inequality can be reached simply by thresholding the analysis coefficients.

The structure of the paper is as follows. In Section 2 we review the most common method to construct TWFs, the so-called extension principles of Ron and Shen. The TWFs generated through an extension principle are based on a multiresolution analysis, and the generators are often called framelets. They have been studied extensively, see e.g. [2, 4, 15, 16, 17]. Section 3 contains the analysis of the properties of TWF expansions in $L_p(\mathbb{R}^d)$ and L_p-based Sobolev spaces. We give a complete characterization of the L_p-norm, $1 < p < \infty$, in terms of analysis coefficients associated with the frame, and prove that a TWF gives an atomic decomposition for

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$L_p(\mathbb{R}^d)$. The characterization has the same form as the classical characterization of the L_p-norm by wavelet coefficients, see e.g. [13]. In Section 3.3 the analysis is extended to L_p-based Sobolev spaces. In Section 4, we consider Jackson inequalities for best m-term approximation with TWFs in $L_p(\mathbb{R}^d)$, and we discuss some cases where a complete characterization of the approximation spaces – associated with best m-term approximation in $L_p(\mathbb{R}^d)$ with TWFs – in terms of (essentially) Besov spaces is possible. Two of the present authors have studied approximation with spline based framelets, defined on $\mathbb{R}$, in [11].

2. Tight wavelet frames

The most common methods to construct TWFs are the extension principles of Ron and Shen. Tight wavelet frames build through the extension principle are based on a multiresolution analysis and we will briefly touch upon some of the main ideas in the construction, see [4, 17, 16]. There is also the (significant) advantage with the MRA based constructions that there are fast associated algorithms.

For historical notes on this construction, we refer the reader to [4]. MRA based TWFs are called **framelets**. We begin by introducing some basic notation and general assumptions.

Let $\tau = (\tau^0, \tau^1, \ldots, \tau^L)$ be a vector of $2\pi\mathbb{Z}^d$-periodic measurable functions with τ^0 the mask of a refinable scaling function ϕ of a MRA $\{V_j\}_{j\in\mathbb{Z}}$. We assume that ϕ satisfies $\lim_{\xi\to 0}\widehat{\phi}(\xi) = 1$ and there exist $0 < c \le C < \infty$ such that $c \le \sum_{k\in\mathbb{Z}^d} |\widehat{\phi}(\cdot - 2\pi k)|^2 \le C$, i.e., ϕ generates a **Riesz basis** of the scaling space V_0 of the MRA. We associate the "wavelets" $\Psi = \{\psi^\ell\}_{\ell\in E}$ to τ by letting $\widehat{\psi^\ell}(2\xi) = \tau^\ell(\xi)\widehat{\phi}(\xi)$.

The following is the fundamental tool to construct framelets:

Theorem 2.1 (The Oblique Extension Principle (OEP) [4])**.** *Suppose there exists a $2\pi\mathbb{Z}^d$-periodic function Θ that is non-negative, essentially bounded, continuous at the origin with $\Theta(0) = 1$. If for every $\xi \in [-\pi,\pi]^d$ and $\nu \in \{0,\pi\}^d$,*

$$\Theta(2\xi)\tau^0(\xi)\overline{\tau^0(\xi+\nu)} + \sum_{\ell=1}^{L} \tau^\ell(\xi)\overline{\tau^\ell(\xi+\nu)} = \begin{cases} \Theta(\xi), & \nu = 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.1}$$

then the wavelet system $\{2^{jd/2}\psi^\ell(2^j\cdot - k) | j\in\mathbb{Z}, k\in\mathbb{Z}^d, \ell\in E\}$, defined by τ is a tight wavelet frame.

The system $X(\Psi)$ is usually called the **framelet system generated by Ψ**.

Remark 2.2. Theorem 2.1 can be stated in slightly more generality by introducing the notion of a spectrum for the scaling space V_0 and dropping the requirement that ϕ generates a Riesz basis, see [4].

Remark 2.3. For $\Theta \equiv 1$, Theorem 2.1 reduces to the Unitary Extension Principle (UEP) of Ron and Shen [17]. The advantage of the OEP compared to the UEP is that one can construct framelets with a high number of vanishing moments using the OEP. This is not possible with the UEP, where at least one of the generators has only one vanishing moment.

3. Tight wavelet frames in L_p and Sobolev Spaces

In this section we study TWFs in $L_p(\mathbb{R}^d)$, $1 < p < \infty$, (Section 3.1) and L_p-based Sobolev spaces (Section 3.3). In Section 3.2 we show that thresholding the analysis coefficients associated with a TWF is a bounded operation in $L_p(\mathbb{R}^d)$, $1 < p < \infty$. For notational convenience we let D denote the set of dyadic cubes $I = 2^{-j}([0,1]^d + k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, and write $\psi_I(x) := 2^{jd/2}\psi(2^j x - k)$.

3.1. **TWFs in $\mathbf{L_p}(\mathbb{R}^\mathbf{d})$.** Theorem 3.1 below will show that we can characterize the L_p-norms by the analysis coefficients associated with the TWF. Theorem 3.3 will show that there is a stable way to reconstruct L_p-functions using the TWF, and this leads to two results: TWFs form atomic decompositions of $L_p(\mathbb{R}^d)$ (see Corollary 3.6), and thresholding (or shrinkage of) the frame coefficients are stable operations in $L_p(\mathbb{R}^d)$ (see Section 3.2).

Theorem 3.1. *Let $\{\psi^\ell\}_{\ell\in E}$ be the generators of a tight wavelet frame for $L_2(\mathbb{R}^d)$. Suppose for all $\ell \in E$, some $\beta > 0$ and some $\varepsilon > 0$, $\psi^\ell \in C^\beta(\mathbb{R}^d)$,*

$$\int \psi^\ell(x)\,dx = 0 \quad \textit{and} \quad |\psi^\ell(x)| \le C(1+|x|)^{-d-\varepsilon}.$$

Then

$$\|f\|_p \asymp \left\| \left(\sum_{I\in D,\ell\in E} |\langle f,\psi_I^\ell\rangle|^2 |I|^{-1}\chi_I(x) \right)^{1/2} \right\|_p, \tag{3.1}$$

for $1 < p < \infty$, where χ_I denotes the indicator function for the subset $I \subset \mathbb{R}^d$.[1]

Proof. Let $\{\psi^{m,s}\}_{s=1}^{2^d-1}$ be the orthonormal Meyer wavelet(s) defined on $\mathbb{R}^d$. For each $\ell \in E$ we consider the integral kernel

$$K^\ell(x,y) := \sum_{I\in D} \psi_I^{m,1}(x)\overline{\psi_I^\ell(y)}.$$

Notice that the corresponding operator

$$T^\ell\colon f \mapsto \int_{\mathbb{R}^d} K^\ell(x,y) f(y)\,dy$$

is bounded on $L_2(\mathbb{R}^d)$ due to the fact that $\{\psi_I^\ell\}_{I\in D}$ is a subset of a frame. Also, standard estimates show that (see e.g. [3])

$$|K^\ell(x,y)| \le C|x-y|^{-d},$$

$$|K^\ell(x',y) - K^\ell(x,y)| \le C|x-x'|^\alpha |x-y|^{-d-\alpha},$$

and

$$|K^\ell(x,y') - K^\ell(x,y)| \le C|y-y'|^\alpha |x-y|^{-d-\alpha},$$

because of the smoothness and decay of ψ^ℓ. Thus T^ℓ is a Calderón-Zygmund operator and therefore bounded on $L_p(\mathbb{R}^d)$, $1 < p < \infty$. However $T^\ell f$ has a nice

[1] By $F \asymp G$ we mean that there exist two constants $0 < C_1 \le C_2 < \infty$ such that $C_1 F \le G \le C_2 F$.

expansion in the orthonormal Meyer wavelet, so using the L_p-characterization of such expansions we get

$$\Big\|\Big(\sum_{I\in D}|\langle f,\psi_I^\ell\rangle|^2|I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_p \asymp \|T^\ell f\|_p \le C\|f\|_p.$$

Using this estimate for $\ell=1,2,\ldots,L$, and the fact that $\ell_1\hookrightarrow\ell_2$ we get[2]

$$\begin{aligned}
\Big\|\Big(\sum_{I\in D,\ell\in E}&|\langle f,\psi_I^\ell\rangle|^2|I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_p\\
&=\Big\|\Big(\sum_{\ell\in E}\Big(\Big\{\sum_{I\in D}|\langle f,\psi_I^\ell\rangle|^2|I|^{-1}\chi_I(x)\Big\}^{1/2}\Big)^2\Big)^{1/2}\Big\|_p\\
&\le\Big\|\sum_{\ell\in E}\Big\{\sum_{I\in D}|\langle f,\psi_I^\ell\rangle|^2|I|^{-1}\chi_I(x)\Big\}^{1/2}\Big\|_p\\
&\le L\cdot C\|f\|_p.
\end{aligned}$$

Now we turn to the converse estimate. Notice that since we have a *tight* wavelet frame we have the identity

$$\langle f,g\rangle = A\sum_{I\in D,\ell\in E}\langle f,\psi_I^\ell\rangle\overline{\langle g,\psi_I^\ell\rangle},\quad f,g\in L_2(\mathbb{R}^d),$$

where $A>0$ is a constant depending only on the frame. Write

$$Wf(x)=\{|I|^{-1/2}\langle f,\psi_I^\ell\rangle\chi_I(x)\}_{I,\ell}$$

and notice that for $f\in L_2(\mathbb{R}^d)\cap L_p(\mathbb{R}^d)$ and $g\in L_2(\mathbb{R}^d)\cap L_{p'}(\mathbb{R}^d)$, with $p^{-1}+(p')^{-1}=1$,

$$\begin{aligned}
|\langle f,g\rangle| &= A\Big|\int\langle Wf(x),Wg(x)\rangle_{\ell_2}\,dx\Big|\\
&\le A\Big|\int\langle Wf(x),Wf(x)\rangle_{\ell_2}\langle Wg(x),Wg(x)\rangle_{\ell_2}\,dx\Big|\\
&\le A\|\langle Wf(x),Wf(x)\rangle_{\ell_2}\|_p\|\langle Wg(x),Wg(x)\rangle_{\ell_2}\|_{p'}\\
&\le AC\|\langle Wf(x),Wf(x)\rangle_{\ell_2}\|_p\|g\|_{p'}.
\end{aligned}$$

Taking the supremum of this estimate for $\{g\in L_2(\mathbb{R}^d)\cap L_{p'}(\mathbb{R}^d):\|g\|_{p'}\le 1\}$ we obtain

$$\|f\|_p\le\tilde{C}\|\langle Wf(x),Wf(x)\rangle_{\ell_2}\|_p.$$

This proves the result for $f\in L_2(\mathbb{R}^d)\cap L_p(\mathbb{R}^d)$. To complete the proof we just notice that from the first part of the proof it follows that $f\mapsto\langle Wf(x),Wf(x)\rangle_{\ell_2}$ is continuous on $L_p(\mathbb{R}^d)$. □

[2]The notation $V\hookrightarrow W$ means that the two (quasi)normed spaces V and W satisfy $V\subset W$ and there is a constant $C<\infty$ such that $\|\cdot\|_W\le C\|\cdot\|_V$.

From Theorem 3.1 we see that the following sequence space plays an important role.

Definition 3.2. *Let d_p denote the sequences $\{c_I^\ell\}_{I\in D,\ell\in E}$ for which*

$$|||\{c_I^\ell\}|||_p := \left\| \left(\sum_{I\in D,\ell\in E} |c_I^\ell|^2 |I|^{-1}\chi_I(x) \right)^{1/2} \right\|_p < \infty.$$

In fact, let us show that there is a stable reconstruction operator defined on d_p.

Theorem 3.3. *Let $\{\psi^\ell\}_{\ell\in E}$ be the generators of a tight wavelet frame for $L_2(\mathbb{R}^d)$. Suppose for all $\ell\in E$, some $\beta>0$ and some $\varepsilon>0$, $\psi^\ell\in C^\beta(\mathbb{R}^d)$, $\int \psi^\ell(x)\,dx=0$ and $|\psi^\ell(x)|\le C(1+|x|)^{-d-\varepsilon}$. Then the map $T:\, d_p\mapsto L_p(\mathbb{R}^d)$ defined by*

$$T\{c_I^\ell\} = \sum_{I\in D,\ell\in E} c_I^\ell \psi_I^\ell$$

is a bounded linear map. Moreover, the sum defining T converges unconditionally.

Proof. We consider the dual $(T^\ell)'$of the operator T^ℓ used in Theorem 3.1, i.e., the operator with kernel

$$\tilde{K}^\ell(x,y) := \sum_{I\in D} \psi_I^\ell(x)\overline{\psi_I^{m,1}(y)}.$$

By exactly the same arguments as given in the first part of the proof of Theorem 3.1, it can be shown that $(T^\ell)'$ is bounded on $L_p(\mathbb{R}^d)$. Take $\{c_I^\ell\}_{I\in D,\ell\in E}\in d_p$ and consider $f^\ell := \sum_{I\in D} c_I^\ell \psi_I^{m,1}$. This is a well-defined function in $L_p(\mathbb{R}^d)$ with

$$\|f^\ell\|_p \asymp \left\| \left(\sum_{I\in D} |c_I^\ell|^2 |I|^{-1}\chi_I(x) \right)^{1/2} \right\|_p,$$

where we used the characterization of $L_p(\mathbb{R}^d)$ using wavelets. Thus,

$$\begin{aligned}
\Big\| \sum_{I\in D,\ell\in E} c_I^\ell\psi_I^\ell \Big\|_p &\le \sum_{\ell\in E} \Big\| \sum_{I\in D} c_I^\ell\psi_I^\ell \Big\|_p = \sum_{\ell\in E} \big\| (T^\ell)' f^\ell \big\|_p \\
&\le C\sum_{\ell\in E} \|f^\ell\|_p \le \tilde{C} \sum_{\ell\in E} \left\| \left(\sum_{I\in D} |c_I^\ell|^2 |I|^{-1}\chi_I(x) \right)^{1/2} \right\|_p \\
&\le L\tilde{C} \left\| \left(\sum_{I\in D,\ell\in E} |c_I^\ell|^2 |I|^{-1}\chi_I(x) \right)^{1/2} \right\|_p,
\end{aligned}$$

and it follows that $T:\, d_p\mapsto L_p(\mathbb{R}^d)$ is bounded. Unconditionality follows easily from the observation that none of the above estimates depend on the sign of each c_I^ℓ. □

Recall the Lorentz space $\ell_{p,q}(\Lambda)$, $1\le p<\infty$, $0<q\le\infty$, for some countable set Λ, as the set of sequences $\{a_m\}_{m\in\Lambda}$ satisfying $\|\{a_m\}\|_{\ell_{p,q}}<\infty$, where

$$\|\{a_m\}\|_{\ell_{p,q}} = \begin{cases} \left(\sum_{j=0}^\infty \left(2^{j/\tau} a_{2^j}^*\right)^q \right)^{1/q}, & 0<q<\infty, \\ \sup_{j\ge 0} 2^{j/\tau} a_{2^j}^*, & q=\infty, \end{cases}$$

with $\{a_j^*\}_{j=0}^\infty$ a decreasing rearrangement of $\{a_m\}_{m\in\Lambda}$.

It is known from the orthonormal wavelet case [10, 11], that there exist constants $c, C > 0$ such that

$$c\|\{c_I\}\|_{\ell_{p,\infty}(D)} \leq \left\|\left(\sum_{I\in D}|c_I|^2|I|^{-2/p}\chi_I(x)\right)^{1/2}\right\|_p \leq C\|\{c_I\}\|_{\ell_{p,1}(D)},$$

for any $\{c_I\}\in\ell_{p,1}(D)$. Notice that for any $\{c_I^\ell\}\in\ell_{p,1}(D\times E)$,

$$|\{I,\ell : |c_I^\ell|>\varepsilon\}| = \sum_{\ell=1}^{L}|\{I : |c_I^\ell|>\varepsilon\}| \leq L\|S(\{c_I^\ell\}_{I,\ell})\|_p^p\varepsilon^{-p},$$

where $S(\{c_I^\ell\}_{I,\ell}) := \left(\sum_{I\in D,\ell\in E}|c_I^\ell|^2|I|^{-2/p}\chi_I(x)\right)^{1/2}$. Furthermore, since $\ell_1 \hookrightarrow \ell_2$ we have

$$\begin{aligned}\|S(\{c_I^\ell\}_{I,\ell})\|_p &\leq \sum_{\ell=1}^{L}\left\|\left(\sum_{I\in D}|c_I^\ell|^2|I|^{-2/p}\chi_I(x)\right)^{1/2}\right\|_p\\ &\leq C\sum_{\ell=1}^{L}\|\{c_I^\ell\}\|_{\ell_{p,1}(D)}\\ &\leq CL\|\{c_I^\ell\}\|_{\ell_{p,1}(D\times E)}.\end{aligned}$$

Combining these two estimates, we get that there exist constants $c, C > 0$ such that (3.2)

$$c\|\{c_I^\ell\}\|_{\ell_{p,\infty}(D\times E)} \leq \left\|\left(\sum_{I\in D,\ell\in E}|c_I^\ell|^2|I|^{-2/p}\chi_I(x)\right)^{1/2}\right\|_p \leq C\|\{c_I^\ell\}\|_{\ell_{p,1}(D\times E)},$$

for any $\{c_I^\ell\}\in\ell_{p,1}(D\times E)$.

Remark 3.4. We denote by $\psi_I^{\ell,p}$ the function ψ_I^ℓ normalized in $L_p(\mathbb{R}^d)$, i.e. $\|\psi_I^{\ell,p}\|_p \asymp |I|^{1/2-1/p}\|\psi_I^\ell\|_p$. Notice that by Theorem 3.3 and (3.2) the TWF system is $\ell_{\mathbf{p},\mathbf{1}}$**-hilbertian** in $L_p(\mathbb{R}^d)$, $1<p<\infty$, that is to say we have

$$\left\|\sum_{I\in D,\ell\in E}c_I^\ell\psi_I^{\ell,p}\right\|_p \leq C_p\|\{c_I^\ell\}\|_{\ell_{p,1}(D\times E)},$$

for any sequence $\{c_I^\ell\}\in\ell_{p,1}(D\times E)$.

We have in fact proved that any reasonable (in the sense of theorem 3.1) TWF system induces an atomic decomposition of $L_p(\mathbb{R}^d)$, $1<p<\infty$. Let us recall the definition of an atomic decomposition:

Definition 3.5. *Let X be a Banach space and X_d a Banach sequence space indexed by $\mathbb{N}$. Let $\{f_k\}\subset X$, $\{g_k\}\subset X^*$. Then $(\{f_k\},\{g_k\})$ is an* **atomic decomposition** *of X with respect to X_d if*

- *$\{g_k(f)\}\in X_d$ for all $f\in X$.*
- *For any $f\in X$ we have*

$$\|f\|_X \asymp \|\{g_k(f)\}\|_{X_d}.$$

- $f = \sum_{k=1}^{\infty} g_k(f) f_k$, $\forall f \in X$.

From this definition we read off the following:

Corollary 3.6. *Let $\{\psi^\ell\}_{\ell\in E}$ be the generators of a tight wavelet frame for $L_2(\mathbb{R}^d)$, with frame constant A. Suppose for all $\ell \in E$, some $\beta > 0$ and some $\varepsilon > 0$, $\psi^\ell \in C^\beta(\mathbb{R}^d)$, $\int \psi^\ell(x)\,dx = 0$, and $|\psi^\ell(x)| \le C(1+|x|)^{-d-\varepsilon}$. Then the system $(\{A^{-1}\psi_I^\ell\}_{I,\ell}, \{\psi_I^\ell\}_{I,\ell})$ is an atomic decomposition of $L_p(\mathbb{R}^d)$, $1 < p < \infty$, with respect to the sequence space d_p.*

3.2. **Thresholding the TWF analysis coefficients.** From a practical point of view, it is interesting to study different types of thresholding (or shrinkage) operators for the framelet system in L_p. Let $\delta\colon \mathbb{C}\times\mathbb{R}^+ \mapsto \mathbb{C}$ be a function for which there exists a constant C such that

$$|x - \delta(x,\lambda)| \le C \min(|x|, \lambda). \tag{3.3}$$

We call such a function δ a shrinkage rule, see e.g. [18]. The well-known notions of hard and soft thresholding are two of the prime examples of shrinkage rules. The expressions are given by $\delta(x,\lambda) = x\mathbf{1}(|x| > \lambda)$ and $\delta(x,\lambda) = x(1-\lambda/|x|)\mathbf{1}(|x|>\lambda)$, respectively.

We define the associated shrinkage operator T_λ^δ as

$$T_\lambda^\delta f = \sum_{I\in D, \ell\in E} \delta(\langle f, \psi_I^\ell\rangle, \lambda)\psi_I^\ell.$$

We claim that $T_\lambda^\delta f \to f$ in $L_p(\mathbb{R}^d)$, $1 < p < \infty$, as $\lambda \to 0$. To see this, we use the estimates given by Theorem 3.3,

$$\|f - T_\lambda^\delta f\|_p \le C \left\| \left(\sum_{I\in D,\ell\in E} |\langle f,\psi_I^\ell\rangle - \delta(\langle f,\psi_I^\ell\rangle,\lambda)|^2 |I|^{-1}\chi_I \right)^{1/2} \right\|_p.$$

By (3.3) and the dominated convergence theorem we see that $\|f - T_\lambda^\delta f\|_p \to 0$ as $\lambda \to 0$.

3.3. **Sobolev Spaces.** We now turn our attention to L_p-based Sobolev spaces. For $1 \le p \le \infty$ and $r \ge 0$ denote by $W^r(L_p(\mathbb{R}))$ the Sobolev space consisting of functions $f \in L_p(\mathbb{R}^d)$ satisfying

$$\|f\|_{W^r(L_p(\mathbb{R}^d))} := \|(I-\Delta)^{r/2} f\|_p < \infty,$$

with Δ the Laplace operator. We prove in this section that for TWFs with some smoothness and vanishing moments, we can actually characterize the Sobolev norm using the frame coefficients.

For a nonnegative integer N, we say that a function f belongs to the set $S^N(\mathbb{R}^d)$ if there exist constants $C, C_\alpha < \infty$ and $\varepsilon > 0$, such that

$$\begin{cases} \int x^\alpha f(x)dx = 0 & \text{for } \alpha\in\mathbb{N}^d,\ |\alpha|\le N, \\ |f(x)| \le C(1+|x|)^{-d-1-N-\varepsilon} & \text{for } x\in\mathbb{R}^d, \\ |\partial^\alpha f(x)| \le C_\alpha(1+|x|)^{-d-\varepsilon} & \text{for } x\in\mathbb{R}^d,\ \alpha\in\mathbb{N}^d,\ |\alpha|\le N+1. \end{cases} \tag{3.4}$$

Here, $|\alpha| := \sum_{k=1}^d \alpha_k$.

Remark 3.7. Given $N \in \mathbb{N}$, it is possible, using the oblique extension principle, to construct a generator Ψ of a framelet system such that $\psi \subset S^N(\mathbb{R}^d)$ (see e.g. [11]).

Theorem 3.8. *Given $1 < p < \infty$ and $r \geq 0$. Let $\{\psi^\ell\}_{\ell\in E}$ be the generators of a TWF for $L_2(\mathbb{R}^d)$ such that $\psi^\ell \in S^{\lceil r\rceil}(\mathbb{R}^d)$ for all $\ell \in E$. Then,*

$$\|f\|_{W^r(L_p(\mathbb{R}^d))} \asymp \Big\|\Big(\sum_{I\in D,\ell\in E} |\langle f,\psi_I^\ell\rangle|^2(1+|I|^{-2r/d})|I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_p, \tag{3.5}$$

for all $f \in W^r(L_p(\mathbb{R}^d))$, with equivalence depending only on p and r.

Proof. For notational convenience we write

$$\mathcal{W}^r f(x) := \Big(\sum_{I\in D,\ell\in E} |\langle f,\psi_I^\ell\rangle|^2(1+|I|^{-2r/d})|I|^{-1}\chi_I(x)\Big)^{1/2},$$

for any $f \in W^r(L_p(\mathbb{R}^d))$. Let us first consider the case $r \in \mathbb{N}$. According to Theorem 6.6.21 in [12], and the characterization of Sobolev functions using wavelet expansions, there exist constants C and C' depending only on r and p such that

$$\begin{aligned}\|\mathcal{W}^r f\|_p &\leq CL\Big\|\Big(\sum_{I\in D} |\langle f,\psi_I^m\rangle|^2(1+|I|^{-2r/d})|I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_p \\ &\leq C'L\|f\|_{W^r(L_p(\mathbb{R}^d))},\end{aligned}$$

for all $f \in W^r(L_p(\mathbb{R}^d))$. This gives us the lower bound in (3.5) for $r \in \mathbb{N}$.

To get the upper bound we recall that $\|f\|_{W^r(L_p(\mathbb{R}^d))} \asymp \|f\|_p + \|(-\Delta)^{r/2}f\|_p$, and since $\|f\|_p \leq C\|\mathcal{W}^0 f\|_p \leq C'\|\mathcal{W}^r f\|_p$ by Theorem 3.1, it suffices to show that $\|(-\Delta)^{r/2}f\|_p \leq C\|\mathcal{W}^r f\|_p$. Fix two functions $f \in W^r(L_p(\mathbb{R}^d)) \cap L_2(\mathbb{R}^d)$ and $g \in L_{p'}(\mathbb{R}^d) \cap W^r(L_2(\mathbb{R}^d))$, where $1 = 1/p + 1/p'$. Since $\{\psi^\ell\}_{\ell\in E}$ are generators of a TWF, we have

$$\begin{aligned}&\langle f,(-\Delta)^{r/2}g\rangle \\ &\quad= A\sum_{I\in D,\ell\in E}\langle f,\psi_I^\ell\rangle\overline{\langle(-\Delta)^{r/2}g,\psi_I^\ell\rangle} \\ &\quad= A\int\sum_{I\in D,\ell\in E}\langle f,\psi_I^\ell\rangle|I|^{-r/d-1/2}\overline{\langle(-\Delta)^{r/2}g,\psi_I^\ell\rangle|I|^{r/d-1/2}}\chi_I(x)\,dx.\end{aligned}$$

Thus, by the Cauchy-Schwartz inequality

$$\begin{aligned}|\langle(-\Delta)^{r/2}f,g\rangle| &= |\langle f,(-\Delta)^{r/2}g\rangle| \\ &\leq A\int \mathcal{W}^r f(x)\Big(\sum_{I\in D,\ell\in E}|\langle(-\Delta)^{r/2}g,\psi_I^\ell\rangle|^2|I|^{2r/d}|I|^{-1}\chi_I(x)\Big)^{1/2}dx \\ &= A\int \mathcal{W}^r f(x)\Big(\sum_{I\in D,\ell\in E}|\langle g,\tilde{\psi}_I^\ell\rangle|^2|I|^{-1}\chi_I(x)\Big)^{1/2}dx\end{aligned}$$

where $\tilde{\psi}^\ell := (-\Delta)^{r/2}\psi^\ell$ and we have used that $\langle(-\Delta)^{r/2}g,\psi_I^\ell\rangle|I|^{r/d} = \langle g,\tilde{\psi}_I^\ell\rangle$ in the last inequality. According to [13, p. 170], $(-\Delta)^{r/2}\colon S^r \mapsto S^0$. Thus, Theorem 6.4.9

in [12], and the characterization of Lebesgue functions using wavelet expansions gives,

$$
\begin{aligned}
|\langle(-\Delta)^{r/2}f,g\rangle| &\le A\|\mathcal{W}^r f\|_p \Big\| \Big(\sum_{I\in D,\ell\in E} |\langle g,\tilde{\psi}_I^{\ell}\rangle|^2 |I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_{p'} \\
&\le CL\|\mathcal{W}^r f\|_p \Big\| \Big(\sum_{I\in D} |\langle g,\psi_I^{m}\rangle|^2 |I|^{-1}\chi_I(x)\Big)^{1/2}\Big\|_{p'} \\
&\le C'\|\mathcal{W}^r f\|_p \|g\|_{p'}.
\end{aligned}
$$

Now, taking the supremum over all $g\in L_{p'}(\mathbb{R}^d)\cap L_2^r(\mathbb{R}^d)$ with $\|g\|_{p'}\le 1$, we obtain

$$
\|(-\Delta)^{r/2}f\|_p \le C'\|\mathcal{W}^r f\|_p,
$$

for $f\in W^r(L_p(\mathbb{R}^d))\cap L_2(\mathbb{R}^d)$ and thus for $f\in W^r(L_p(\mathbb{R}^d))$ since $f\mapsto \mathcal{W}^r f$ is continuous from $W^r(L_p(\mathbb{R}^d))$ to $L_p(\mathbb{R}^d)$. In order to conclude the theorem we need to prove (3.5) for a general $r>0$. Define for each $I\in D$, $\ell\in E$ and $x\in\mathbb{R}^d$ the discrete weight function $w_r := w_r(I,\ell) := (1+|I|^{-2r/d})|I|^{-1}\chi_I(x)$. Notice that

$$
\|\mathcal{W}^r f\|_{L_p} = \left(\int_{\mathbb{R}^d} \big\|\{\langle f,\psi_I^{\ell}\rangle\}\big\|_{\ell_2(w_r)}^p\,dx\right)^{1/p}, \qquad r>0. \tag{3.6}
$$

Define

$$
\mathcal{J}\colon W^N(L_p(\mathbb{R}^d)) \mapsto L_p(\ell_2(w_N)) \quad \text{by} \quad f\mapsto \{\langle f,\psi_I^{\ell}\rangle\}_{I\in D,\ell\in E}
$$

and define

$$
\mathcal{P}\colon L_p(\ell_2(w_N)) \mapsto W^N(L_p(\mathbb{R}^d)) \quad \text{by} \quad \{c_I^{\ell}\}_{I\in D,\ell\in E} \mapsto \sum_{I\in D,\ell\in E} c_I^{\ell}\psi_I^{\ell}.
$$

Then the arguments above show that $\mathcal{P}\circ\mathcal{J} = \mathrm{Id}_{W^N(L_p(\mathbb{R}^d))}$ for all $N\in\mathbb{N}_0$, in other words, $W^N(L_p(\mathbb{R}^d))$ is a retract of $L_p(\ell_2(w_N))$.

$$
\begin{tikzcd}
W^N(L_p(\mathbb{R}^d)) \arrow[rr, "\mathrm{Id}_{W^N(L_p(\mathbb{R}^d))}"] \arrow[dr, "\mathcal{J}"'] & & W^N(L_p(\mathbb{R}^d)) \\
& L_p(\ell_2(w_N)) \arrow[ur, "\mathcal{P}"'] &
\end{tikzcd}
$$

For a given $r>0$, $r\notin\mathbb{N}$, take $N\in\mathbb{N}_0$ such that $r=(1-\theta)N+\theta(N+1)$ for some $\theta\in(0,1)$. Notice that $w^r \asymp w_N^{(1-\theta)} w_{N+1}^{\theta}$. Now, according to Theorem 5.5.3 in [1] we have

$$
\ell_2(w_r) = \big(\ell_2(w_N),\ell_2(w_{N+1})\big)_{[\theta]},
$$

with equivalent norms. Here, $(X,Y)_{[\theta]}$ denotes complex interpolation between X and Y.[3] Furthermore, by Theorem 5.1.2 in [1],

$$
\big(L_p(\ell_2(w_N)),L_p(\ell_2(w_{N+1}))\big)_{[\theta]} = L_p\big(\big(\ell_2(w_N),\ell_2(w_{N+1})\big)_{[\theta]}\big) = L_p(\ell_2(w_r)),
$$

---

[3] We have adopted the notation for complex interpolation as used by Bergh and Löfström in [1]

with equivalent norms. Thus, $W^r(L_p(\mathbb{R}^d))$ is a retract of $L_p(\ell_2(w_r))$ for a general $r \geq 0$. $\square$

Remark 3.9. The reader will notice that the spaces studied so far, $L_p(\mathbb{R}^d)$ and L_p-based Sobolev spaces, belong to the Triebel-Lizorkin scale of function spaces. It can be verified (at the expense of "messy" estimates) that sufficiently nice TWFs also can be used to characterize the Triebel-Lizorkin norms. We leave the details for the reader.

4. Jackson inequalities for tight wavelet frames

We will now look at some of the implications that can be derived from the various characterizations given in the previous section. The main result will be a Jackson inequality that will give a certain rate for m-term approximation for "nice" functions. We consider two interpretations of the word "nice". When we do not assume any smoothness or vanishing moments for the TWF, we get the Jackson estimates for functions in a sparsity class defined in terms of the TWF. If we assume the generators for the TWF has some smoothness and vanishing moments (the OEP tells us that such nice generators do exist), then we can state the Jackson inequality in terms of smoothness measured on the Besov scale.

First we introduce some notions that will be used later. A dictionary $\mathcal{D} = \{g_k\}_{k\in\mathbb{N}}$ in $L_p(\mathbb{R}^d)$ is a countable collection of quasi-normalized elements from $L_p(\mathbb{R}^d)$. For $\mathcal{D}$ we consider the collection of all possible m-term expansions with elements from $\mathcal{D}$:

$$\Sigma_m(\mathcal{D}) := \Big\{ \sum_{i\in\Lambda} c_i g_i \,\Big|\, c_i \in \mathbb{C}, \operatorname{card}\Lambda \leq m \Big\}.$$

The error of the best m-term approximation to an element $f \in L_p(\mathbb{R}^d)$ is then

$$\sigma_m(f,\mathcal{D})_p := \inf_{f_m\in\Sigma_m(\mathcal{D})} \|f - f_m\|_{L_p(\mathbb{R}^d)}.$$

Definition 4.1 (Approximation spaces)**.** *The approximation space $\mathcal{A}_q^\gamma(L_p(\mathbb{R}^d),\mathcal{D})$ is defined by*

$$|f|_{\mathcal{A}_q^\gamma(L_p(\mathbb{R}^d),\mathcal{D})} := \left(\sum_{m=1}^{\infty} \big(m^\gamma \sigma_m(f,\mathcal{D})_p\big)^q \frac{1}{m} \right)^{1/q} < \infty,$$

and (quasi)normed by $\|f\|_{\mathcal{A}_q^\gamma(L_p(\mathbb{R}^d),\mathcal{D})} = \|f\|_p + |f|_{\mathcal{A}_q^\gamma(L_p(\mathbb{R}^d),\mathcal{D})}$, for $0 < q,\gamma < \infty$, with the ℓ_q norm replaced by the sup-norm when $q = \infty$.

It is well known that the main tool in the characterization of $\mathcal{A}_q^\gamma(L_p(\mathbb{R}^d),\mathcal{D})$ comes from the link between approximation theory and interpolation theory (see *e.g.* [7, Theorem 9.1, Chapter 7]). Let $X_p(\mathbb{R}^d)$ be a Banach space with semi-(quasi)norm $|\cdot|_{X_p}$ continuously embedded in $L_p(\mathbb{R}^d)$. Given $\alpha > 0$, the Jackson inequality

$$\sigma_m(f,\mathcal{D})_p \leq C m^{-\alpha} |f|_{X_p(\mathbb{R}^d)}, \qquad \forall f \in X_p(\mathbb{R}^d),\ \forall m \in \mathbb{N} \tag{4.1}$$

and the Bernstein inequality

$$|S|_{X_p(\mathbb{R}^d)} \leq C' m^{\alpha} \|S\|_p, \qquad \forall S \in \Sigma_m(\mathcal{D}) \tag{4.2}$$

(with constants C and C' independent of f, S and m) imply, respectively, the continuous embedding

$$\Big(L_p(\mathbb{R}^d), X_p(\mathbb{R}^d)\Big)_{\gamma/\alpha,q} \hookrightarrow \mathcal{A}_q^{\gamma}(L_p(\mathbb{R}^d), \mathcal{D})$$

and the converse embedding

$$\Big(L_p(\mathbb{R}^d), X_p(\mathbb{R}^d)\Big)_{\gamma/\alpha,q} \hookleftarrow \mathcal{A}_q^{\gamma}(L_p(\mathbb{R}^d), \mathcal{D})$$

for all $0 < \gamma < \alpha$ and $q \in (0, \infty]$.

We want to obtain a Jackson estimate for σ_m when $\mathcal{D}$ is any (reasonable) TWF. For this we need to define a class of "nice" and "smooth" functions. This will be the following class as introduced in [8] for Hilbert spaces.

Definition 4.2 (Sparsity class)**.** *Let $X(\Psi)$ be a TWF. For $p \in (1, \infty)$, $\tau \in (0, \infty)$ and $q \in (0, \infty]$, we let $\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi), M)$ denote the closure (in $L_p(\mathbb{R}^d)$) of the set*

$$\left\{ f \in L_p(\mathbb{R}^d) \,\middle|\, \exists \Lambda \subset \mathbb{N}, \operatorname{card}\Lambda < \infty,\, f = \sum_{(I,\ell)\in\Lambda} c_I^{\ell} \psi_I^{\ell,p},\, \|\{c_k\}\|_{\ell_{\tau,q}} \leq M \right\}.$$

Then we define

$$\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi)) := \bigcup_{M>0} \mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi), M),$$

with

$$|f|_{\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d),X(\Psi))} = \inf\{M : f \in \mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi), M)\}.$$

Remark 4.3. Since the TWF system is $\ell_{p,1}$-hilbertian (cf. Remark 3.4), Proposition 3 in [10] gives an equivalent definition of $\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$ for $p \in (1, \infty)$, $\tau < p$ and $q \in [1, \infty]$:

$$\mathcal{K}_{\tau,q}\big(L_p(\mathbb{R}^d), X(\Psi)\big) = \left\{ f \in L_p(\mathbb{R}^d) \,\middle|\, \exists \{c_I^{\ell}\}_{I,\ell} \in \ell_{\tau,q},\, f = \sum_{I\in D, \ell\in E} c_I^{\ell} \psi_I^{\ell,p} \right\},$$

and $|f|_{\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d),X(\Psi))}$ equals the smallest norm $\|\{c_I^{\ell}\}_{I,\ell}\|_{\ell_{\tau,q}}$ such that $f = \sum_{I,\ell} c_I^{\ell} \psi_I^{\ell,p}$.

4.1. **A general Jackson inequality.** For the sparsity class $\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$ we have the following rather general Jackson inequality.

Proposition 4.4. *Let $X(\Psi)$ be a TWF that satisfies the hypothesis of Theorem 3.3. Then for $1 < p < \infty$, $\tau < p$, and $\alpha = 1/\tau - 1/p$, we have the Jackson inequality*

$$\sigma_m(f, X(\Psi))_p \leq C m^{-\alpha} |f|_{\mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d),X(\Psi))}, \qquad \forall m \in \mathbb{N}, \tag{4.3}$$

for all $f \in \mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d), X(\Psi))$.

Proof. Given $f \in \mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d), X(\Psi))$, let $\{c_I^\ell\}_{I,\ell} \in \ell_{\tau,1}$ be a sequence satisfying $f = \sum_{I,\ell} c_I^\ell \psi_I^{\ell,p}$ and $|f|_{\mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d),X(\Psi))} = \|\{c_I^\ell\}_{I,\ell}\|_{\ell_{\tau,1}}$. Let $\Lambda \subset D \times E$ be a finite set, $\operatorname{card}\Lambda = m < \infty$, such that $\{c_I^\ell\}_{(I,\ell)\in\Lambda}$ is the m largest coefficients from the sequence $\{c_I^\ell\}_{I,\ell}$. Then,

$$\Big\| f - \sum_{(I,\ell)\in\Lambda} c_I^\ell \psi_I^{\ell,p} \Big\|_p = \Big\| \sum_{(I,\ell)\in\Lambda^c} c_I^\ell \psi_I^{\ell,p} \Big\|_p$$

$$\leq C_p \|\{c_I^\ell\}_{(I,\ell)\in\Lambda^c}\|_{\ell_{p,1}} \leq C' \sum_{k=\log_2(m)+1}^{\infty} 2^{j/p} |c_{2^j}^*|,$$

where $\{c_j^*\}_{j\in\mathbb{N}}$ is a decreasing rearrangement of $\{c_I^\ell\}_{I,\ell}$. Since $\alpha = 1/\tau - 1/p$ we get

$$\sum_{k=\log_2(m)+1}^{\infty} 2^{j/p} |c_{2^j}^*| \leq m^{-\alpha} \sum_{k=\log_2(m)+1}^{\infty} 2^{j/\tau} |c_{2^j}^*|$$

$$\leq m^{-\alpha} \|\{c_I^\ell\}_{I,\ell}\|_{\ell_{\tau,1}} = m^{-\alpha} |f|_{\mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d),X(\Psi))}.$$

□

Remark 4.5. By Proposition 4.4, $\mathcal{K}_{\tau,1}(L_p(\mathbb{R}^d), X(\Psi)) \hookrightarrow \mathcal{A}_\infty^\alpha(L_p(\mathbb{R}^d), \mathcal{D})$, $\alpha = 1/\tau + 1/p$. It is easy to verify that (see e.g. [11])

$$\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi)) \hookrightarrow \big(\mathcal{K}_{\tau_1,1}(L_p(\mathbb{R}^d), X(\Psi)), \mathcal{K}_{\tau_2,1}(L_p(\mathbb{R}^d), X(\Psi))\big)_{\theta,q},$$

for $\frac{1}{\tau} = \frac{\theta}{\tau_1} + \frac{1-\theta}{\tau_2}$, and thus,

$$\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi)) \hookrightarrow \mathcal{A}_q^\alpha(L_p(\mathbb{R}^d), \mathcal{D}), \qquad \alpha = 1/\tau - 1/p,$$

for a general $q \in [1,\infty]$.

It may not always be easy to check whether a function $f \in \mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$. Therefore, it is interesting to study the set of functions in $L_p(\mathbb{R}^d)$ depending only on the behavior of the coefficients $\langle f, \psi_I^{\ell,p'}\rangle$. For $p \in (1,\infty)$, $\tau \in (0,\infty)$ and $q \in (0,\infty]$, we let $\tilde{\mathcal{K}}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$ denote the set

$$\left\{ f \in L_p(\mathbb{R}^d) \,\middle|\, |f|_{\tilde{\mathcal{K}}_{\tau,q}(L_p(\mathbb{R}^d),X(\Psi))} := \|\{\langle f, \psi_I^{\ell,p'}\rangle\}\|_{\ell_{\tau,q}} < \infty, 1 = 1/p + 1/p' \right\}.$$

Lemma 4.6. *Given $p \in (1,\infty)$ and $\tau \in [1,p)$, let $N \in \mathbb{N}_0$ be a nonnegative integer such that $1 - \tau/p < (N+1)/d$. Suppose $\psi^\ell \in S^N(\mathbb{R}^d)$ for $\ell = 1,2,\ldots,L$. Then*

$$\tilde{\mathcal{K}}_{\tau,\tau}(L_p(\mathbb{R}^d), X(\Psi)) = \mathcal{K}_{\tau,\tau}(L_p(\mathbb{R}^d), X(\Psi)),$$

with equivalent norms.

Proof. By Remark 4.3 we have $\tilde{\mathcal{K}}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi)) \hookrightarrow \mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$. Thus, given $f \in L_p(\mathbb{R}^d)$, we only need to show that if $f = \sum c_I^\ell \psi_I^{\ell,p}$ with $\{c_I^\ell\} \in \ell_\tau$ then

$\{\langle f, \psi_I^{\ell,p'}\rangle\} \in \ell_\tau$, with $\|\{\langle f, \psi_I^{\ell,p'}\rangle\}\|_{\ell_\tau} \leq C\|\{c_I^\ell\}\|_{\ell_\tau}$. First, notice that

$$\langle f, \psi_I^{\ell,p'}\rangle = \sum_{I',\ell'} c_{I'}^{\ell'} \langle \psi_{I'}^{\ell',p}, \psi_I^{\ell,p'}\rangle.$$

Hence, it suffices to show that the double infinite matrix $(\langle \psi_{I'}^{\ell',p}, \psi_I^{\ell,p'}\rangle)_{I,I',\ell,\ell'}$ is bounded on ℓ_τ. Let us introduce the notation $\psi_{j,k}^\ell := \psi_I^\ell$ and $c_{j,k}^\ell := c_I^\ell$ for $I = [2^{-j}k, 2^{-j}(k+1)]$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$. By Proposition 6.6.20 in [12] we have for $j \leq j'$

$$|\langle \psi_{j',k'}^{\ell'}, \psi_{j,k}^\ell\rangle| \leq \frac{C2^{(j-j')(d/2+N+1)}}{(1+|k-2^{j-j'}k'|)^{d+\gamma}},$$

for some $\gamma > 0$. Since $\psi_{j,k}^{\ell,p'} \asymp 2^{jd(1/p'-1/2)}\psi_{j,k}^\ell$, this gives the bound

$$|\langle \psi_{j',k'}^{\ell',p}, \psi_{j,k}^{\ell,p'}\rangle| \leq C2^{-|j'-j|(N+1)} a_{j'-j;k,k'}, \tag{4.4}$$

where

$$a_{m;k,k'} := \begin{cases} 2^{-md/p'}(1+|k-2^{-m}k'|)^{-d-\gamma} & \text{for } m \geq 0, \\ 2^{md/p}(1+|2^m k - k'|)^{-d-\gamma} & \text{for } m < 0, \end{cases}$$

For notational convenience we suppress the index ℓ in the following. For fixed $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ we have

$$|\langle f, \psi_{j,k}^{p'}\rangle| \leq \sum_{j' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}^d} |\langle \psi_{j',k'}^p, \psi_{j,k}^{p'}\rangle| |c_{j',k'}|$$

Using the bound (4.4) and Hölders inequality for the sum over j', with $1 = 1/\tau + 1/\tau'$, we get

$$\begin{aligned} |\langle f, \psi_{j,k}^{p'}\rangle| &\leq C \sum_{j' \in \mathbb{Z}} 2^{-|j'-j|(N+1)} \Big(\sum_{k' \in \mathbb{Z}^d} a_{j'-j;k,k'} |c_{j',k'}|\Big) \\ &= C \sum_{j' \in \mathbb{Z}} 2^{-|j'-j|(N+1)(1/\tau'+1/\tau)} \Big(\sum_{k' \in \mathbb{Z}^d} a_{j'-j;k,k'} |c_{j',k'}|\Big) \\ &\leq C \Big(\sum_{j' \in \mathbb{Z}} 2^{-|j'-j|(N+1)}\Big)^{1/\tau'} \\ &\qquad \Bigg(\sum_{j' \in \mathbb{Z}} 2^{-|j'-j|(N+1)} \Big(\sum_{k' \in \mathbb{Z}^d} a_{j'-j;k,k'} |c_{j',k'}|\Big)^\tau\Bigg)^{1/\tau} \\ &\leq C' \Bigg(\sum_{m \in \mathbb{Z}} 2^{-|m|(N+1)} \Big(\sum_{k' \in \mathbb{Z}^d} a_{m;k,k'} |c_{m+j,k'}|\Big)^\tau\Bigg)^{1/\tau}. \end{aligned} \tag{4.5}$$

Lemma 8.10 in [14] implies for any $\{d_{k'}\}_{k'} \in \ell_\tau$, $1 \leq \tau < \infty$,

$$\sum_{k \in \mathbb{Z}^d} \Big(\sum_{k' \in \mathbb{Z}^d} a_{m;k,k'} |d_{k'}|\Big)^\tau \leq C2^{md(\tau/p-1)} \sum_{k' \in \mathbb{Z}^d} |d_{k'}|^\tau, \qquad \text{for } m \in \mathbb{Z}.$$

This estimate and (4.5) yields

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}|\langle f,\psi^{p'}_{j,k}\rangle|^\tau &\le C\sum_{m\in\mathbb{Z}}2^{-|m|(N+1)}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\Big(\sum_{k'\in\mathbb{Z}^d}a_{m;k,k'}|c_{m+j,k'}|\Big)^\tau\\ &\le C'\sum_{m\in\mathbb{Z}}2^{-|m|(N+1)}2^{md(\tau/p-1)}\sum_{j\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}^d}|c_{m+j,k'}|^\tau\\ &\le C'\Big(\sum_{m\in\mathbb{Z}}2^{-|m|\delta}\Big)\|\{c_{j,k'}\}\|^\tau_{\ell_\tau},\end{aligned}$$

where $\delta := N+1-d|\tau/p-1| > 0$. Thus, $\|\{\langle f,\psi^{p'}_{j,k}\rangle\}\|_{\ell_\tau}\le C\|\{c_{j,k'}\}\|_{\ell_\tau}$ and the lemma follows. $\square$

We now want to use Theorem 3.3 to show that it is possible to obtain the same asymptotic upper bound for $\sigma_m(f,X(\Psi))$ as in (4.3), just by including the m largest normalized framelet coefficients in the approximation. That is to say, we can obtain the approximation rate, associated with the general Jackson inequality, just by thresholding the TWF analysis coefficients.

The basic observation we need is the following: Let $1<p<\infty$ and $\Lambda\subset D\times E$ be a finite set. Since the TWF system is $\ell_{p,1}$-hilbertian, we have

$$\begin{aligned}\Big\|\sum_{(I,\ell)\in\Lambda}c^\ell_I\psi^{\ell,p}_I\Big\|_p &\le C\|\{c^\ell_I\}_{(I,\ell)\in\Lambda}\|_{\ell_{p,1}}\\ &\le C'\|\{c^\ell_I\}_{(I,\ell)\in\Lambda}\|_{\ell_\infty}\sum_{j=0}^{\log_2(\operatorname{card}\Lambda)}2^{j/p} \qquad (4.6)\\ &\le C''\|\{c^\ell_I\}_{(I,\ell)\in\Lambda}\|_{\ell_\infty}(\operatorname{card}\Lambda)^{1/p},\end{aligned}$$

Then we have the following thresholding version of the Jackson inequality.

Proposition 4.7. *Fix $1<p<\infty$, and $s>0$, and let $f\in L_p(\mathbb{R}^d)$. Given $m\in\mathbb{N}$, let $\Lambda\subset D\times E$ be a set of indices corresponding to the m largest coefficients $c^{\ell,p}_I := \langle f,\psi^{\ell,p'}_I\rangle$, $1=1/p+1/p'$. If $\|(c^{\ell,p}_I)_{I\in D,\ell}\|_{\ell_{\tau,\infty}}\le M<\infty$, $1/\tau = s+1/p$, then,*

$$\Big\|f-\sum_{(I,\ell)\in\Lambda}c^{\ell,p}_I\psi^{\ell,p}_I\Big\|_p\le CMm^{-s},\qquad m\in\mathbb{N}.$$

Proof. The proof is an easy extension of the proof of Theorem 7.5 in [5]. Let

$$\Lambda_j=\{I: 2^{-j}<|c^{\ell,p}_I|\le 2^{-j+1}\}.$$

Since $(c^{\ell,p}_I)_{I,\ell}\in\ell_{\tau,\infty}$ we have that

$$\operatorname{card}\Lambda_j\le M^\tau 2^{j\tau}.$$

For $k\in\mathbb{N}$, define $T_{m_k}:=\sum_{j=-\infty}^k\sum_{(I,\ell)\in\Lambda_j}c^{\ell,p}_I\psi^{\ell,p}_I$. Notice that $T_{m_k}\in\Sigma_{m_k}$, where

$$m_k=\sum_{j=-\infty}^k\operatorname{card}\Lambda_j\le CM^\tau 2^{k\tau},$$

with C depending only on τ. Now, fix $m \in \mathbb{N}$ such that $m_k \leq m \leq m_{k+1}$. Suppose $T_m = \sum_{(I,\ell)\in\Lambda} c_I^{\ell,p}\psi_I^{\ell,p}$ is an m-term approximation consisting of the m largest coefficients. Consider the inequality

$$\|f - T_m\|_p \leq \|f - T_{m_{k+1}}\|_p + \|T_{m_{k+1}} - T_m\|_p. \tag{4.7}$$

The estimates in (4.6) gives

$$\begin{aligned}
\|f - T_{m_{k+1}}\|_p &\leq \sum_{j=m_{k+1}+1}^{\infty} \Big\| \sum_{(I,\ell)\in\Lambda_j} c_I^{\ell,p}\psi_I^{\ell,p} \Big\|_p \\
&\leq C \sum_{j=m_{k+1}+1}^{\infty} 2^{-j}(\operatorname{card}\Lambda_j)^{1/p} \\
&\leq C' \sum_{j=m_{k+1}+1}^{\infty} M^{\tau/p}2^{j(\tau/p-1)} \\
&\leq CM(m_{k+1})^{-s} \leq CMm^{-s}.
\end{aligned} \tag{4.8}$$

Denote $\tilde{\Lambda} := \Lambda \setminus \cup_{j=-\infty}^{k}\Lambda_j$ and notice that $\tilde{\Lambda} \subset \Lambda_{k+1}$. Now, according to (4.6), $\|T_{m_{k+1}} - T_m\|_p \leq C2^{-k}(\operatorname{card}\tilde{\Lambda})^{1/p}$, and thus

$$\begin{aligned}
\|T_{m_{k+1}} - T_m\|_p &\leq C2^{-k}(\operatorname{card}\Lambda_{k+1})^{1/p} \\
&\leq C'M^{\tau/p}2^{(k+1)(\tau/p-1)} \leq C''M(m_{k+1})^{-s} \leq C''Mm^{-s}.
\end{aligned} \tag{4.9}$$

Finally, using (4.8), and (4.9) in (4.7) the result follows. □

4.2. **Tight wavelet frames with vanishing moments.** It is possible to prove that $\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$ is a (quasi) Banach space for any system $X(\Psi)$. However, when we have a "nice" system, we can actually identify $\mathcal{K}_{\tau,q}(L_p(\mathbb{R}^d), X(\Psi))$ with the space given by interpolation between $L_p(\mathbb{R}^d)$ and a Besov space. This will lead to a Jackson inequality for a nice TWF, for functions that are smooth measured on the classical Besov scale. Let us give the details.

For $1 < p < \infty$, $1 < q \leq \infty$ and $s \geq p$ we recall the homogeneous discrete Besov space $\dot{b}_{p,q}^s$ as the space of sequences $\{c_I\}_{I\in D}$ satisfying

$$\|\{c_I\}\|_{\dot{b}_{p,q}^s} := \Big(\sum_{j\in\mathbb{Z}} 2^{jdq(1/p-1/2-s/d)}\Big(\sum_{|I|=2^{jd}} |c_I|^p\Big)^{q/p}\Big)^{1/q} < \infty.$$

Let $\Psi = \{\psi^\ell\}_{\ell=1}^L$ be the generators of a TWF for $L_2(\mathbb{R}^d)$. For $1 < p < \infty$, $1 < q \leq \infty$ and $s > 0$ we define

$$\mathcal{B}_q^s(L_p, \Psi) := \Big\{ f \in L_p(\mathbb{R}^d) : \|f\|_{\mathcal{B}_q^s(L_p,\Psi)} := \|f\|_p + \sum_{\ell=1}^{L} \|\{\langle f, \psi_I^\ell\rangle\}\|_{\dot{b}_{p,q}^s} < \infty \Big\}.$$

Theorem 4.8. *Given $r \in \mathbb{N}$, let $\Psi = \{\psi^\ell\}_{\ell=1}^L$ be the generators of a TWF for $L_2(\mathbb{R}^d)$ with $\psi^\ell \in S^r(\mathbb{R}^d)$ for all $\ell = 1, 2, \ldots, L$ (cf.* (3.4)*). Then, for $1 < p < \infty$, $1 < q \leq \infty$ and $s \leq r$, the following identity holds, with equivalent norms,*

$$B_q^s(L_p(\mathbb{R}^d)) = \mathcal{B}_q^s(L_p, \Psi).$$

Proof. The embedding $\mathcal{B}_q^s(L_p,\Psi) \hookrightarrow B_q^s(L_p(\mathbb{R}^d))$ follows from the theory of atomic decomposition of $B_q^s(L_p(\mathbb{R}^d))$ (see e.g. [9]). To get the other inclusion, let $\{\psi^{m,k}\}_{k=1}^{2^d-1}$ be the Meyer wavelets defined on $\mathbb{R}^d$. Then for any $f \in B_q^s(L_p(\mathbb{R}^d))$ we have an expansion $f = \sum_{I\in D}\sum_{k=1}^{2^d-1} d_{I,k}\psi_I^{m,k}$, with $\{d_I\}_{I\in D} \in \dot{b}_{p,q}^s$, where $d_I := (\sum_{k=1}^{2^d-1}|d_{I,k}|^2)^{1/2}$. Now, the framelet coefficient $\langle f,\psi_I^\ell\rangle$ is given by

$$\langle f,\psi_I^\ell\rangle = \sum_{I'\in D}\sum_{k=1}^{2^d-1}\langle \psi_{I'}^{m,k},\psi_I^\ell\rangle d_{I',k}.$$

Since $\psi^{m,s}$ are Meyer wavelets and ψ^ℓ satisfies (3.4), the matrix $M^{s,\ell}$ having $\langle\psi_{I'}^{m,s},\psi_I^\ell\rangle$ as coefficients, is a sparse matrix and thus bounded on $\dot{b}_{p,q}^s$ provided that $r \geq s$ (see e.g. [9, Lemma 3.3]). In particular, this implies that

$$\|\{\langle f,\psi_I^\ell\rangle\}\|_{\dot{b}_{p,q}^s} \leq C\|\{d_I\}\|_{\dot{b}_{p,q}^s} \leq C'\|f\|_{B_{p,q}^s}.$$

□

With this characterization in hand, we read off the following result.

Corollary 4.9. *Let $r\in\mathbb{N}$, and let $\Psi = \{\psi^\ell\}_{\ell=1}^L$ be the generators of a TWF for $L_2(\mathbb{R}^d)$ with $\psi^\ell \in S^r(\mathbb{R}^d)$ for all $\ell = 1,2,\ldots,L$. We have, for $1<p<\infty$, $\tau<p$, and $\alpha = 1/\tau-1/p$,*

$$\mathcal{K}_{\tau,\tau}(L_p(\mathbb{R}^d),X(\Psi)) = B_\tau^{\alpha d}(L_\tau(\mathbb{R}^d)).$$

Moreover, we have the Jackson inequality

$$\sigma_m(f,X(\Psi)) \leq Cm^{-\alpha}\|f\|_{B_\tau^{\alpha d}(L_\tau(\mathbb{R}^d))}, \qquad \forall m\in\mathbb{N}.$$

4.3. **On complete characterizations of the approximation space.** The ultimate goal is to completely characterize the approximation space $\mathcal{A}_q^\alpha(L_p(\mathbb{R}^d),X(\Psi))$ in terms of a smoothness space. The most difficult step to get such a characterization is to derive a Bernstein inequality for the TWF. In general, this is an open (and likely very hard) problem but we conclude this paper by mentioning one important case where a Bernstein inequality can be proved. The proof relies heavily on the rather deep result [6, Th. 5.6]. We have the following Proposition.

Proposition 4.10. *Let $\Psi = \{\psi^\ell\}_{\ell=1}^L$ be the generators of a framelet system in $L_2(\mathbb{R}^d)$ with ψ^ℓ compactly supported for $\ell = 1,2,\ldots,L$. Suppose the associated refinable scaling function ϕ has compact support, it has nonnegative two-scale coefficients, and there is $s>0$ such that $\phi\in W^s(L_\infty(\mathbb{R}^d))$. Suppose, furthermore, that $\hat{\phi}(2\pi j) = \delta_{j,0}$, $j\in\mathbb{Z}^d$ and $\partial^\nu\hat{\phi}(2\pi j) = 0$ for $j\neq 0$, $\nu\in\mathbb{N}^d$, $|\nu|<s$. Then there exists a constant $C<\infty$ depending only on ϕ, s, d, and p such that*

$$|g|_{B_\tau^\alpha(L_\tau)} \leq Cm^{\alpha/d}\|g\|_{L_p}, \qquad \tfrac{1}{\tau} = \tfrac{\alpha}{d}+\tfrac{1}{p},\ 0<\alpha<s, \tag{4.10}$$

for $g\in\Sigma_m(X(\Psi))$.

Proof. The result is based on Theorem 5.6 in [6]. According to this result we have (4.10) for any $g \in \Sigma_m(X(\phi))$, with a constant depending only on ϕ and p. Now, since ϕ has compact support, there exists a constant K depending only on ϕ such that for $\ell = 1,2,\dots,L$, $\psi^\ell(x) = \sum_{I\in\Gamma_\ell} d_{\ell,I}\phi_I(x)$, with $\operatorname{card}\Gamma_\ell \le K$. Thus if $g \in \Sigma_m(X(\Psi))$, then $g \in \Sigma_{Km}(X(\phi))$. $\square$

Finally we can combine Proposition 4.10 and Corollary 4.9 to get

Corollary 4.11. *Let $r \in \mathbb{N}$, and let $\Psi = \{\psi^\ell\}_{\ell=1}^L$ be the generators of a framelet system for $L_2(\mathbb{R}^d)$ with $\psi^\ell \in S^r(\mathbb{R}^d)$ and ψ^ℓ is compactly supported, for $\ell = 1,2,\dots,L$. Suppose the associated scaling function ϕ satisfies the hypothesis of Proposition 4.10. Then*

$$\mathcal{A}_q^{\gamma/d}(L_p(\mathbb{R}^d), X(\Psi)) = \Big(L_p(\mathbb{R}^d), B_\tau^\alpha(L_\tau(\mathbb{R}^d))\Big)_{\gamma/\alpha,q},$$

for $1 < p < \infty$, $0 < \alpha < \min(r,s)$, and $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$.

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