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ON THE QUASI-GREEDY PROPERTY AND UNIFORMLY BOUNDED ORTHONORMAL SYSTEMS

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ABSTRACT. We derive a necessary condition for a uniformly bounded orthonormal basis for $L^2(\Omega)$, Ω a probability space, to be quasi-greedy in $L^p(\Omega)$, $p \neq 2$, and then use this condition to prove that many classical systems, such as the trigonometric system and Walsh system, fail to be quasi-greedy in L^p, $p \neq 2$, i.e., thresholding is not well-behaved in L^p, $p \neq 2$, for such systems.

1. INTRODUCTION

Consider a quasi-normalized Schauder basis $\{e_k\}_{k\in\mathbf{N}}$ for a Banach space X with associated coefficient functionals $\{e_k^*\}_{k\in\mathbf{N}}$. For each element $x \in X$ we define the greedy ordering of the coefficients as the map $\rho : \mathbf{N} \to \mathbf{N}$ with $\rho(\mathbf{N}) \supseteq \{j : e_j^*(x) \neq 0\}$ such that for $j < k$ we have either $|\, e_{\rho(k)}^*(x)| < |\, e_{\rho(j)}^*(x)|$ or $|\, e_{\rho(k)}^*(x)| = |\, e_{\rho(j)}^*(x)|$ and $\rho(k) > \rho(j)$. Then the greedy m-term approximant to x is given by $\mathcal{G}_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x) e_{\rho(j)}$. Thus, $\mathcal{G}_m(x)$ is basically obtained by thresholding the expansion coefficients. The question is whether $\mathcal{G}_m(x) \to x$. Whenever $\{e_k\}_{k\in\mathbf{N}}$ is an *unconditional basis* this is clearly the case, but it may also happen for bases that are not unconditional. This leads to the definition of a quasi-greedy basis.

Definition 1.1. *A (quasi-normalized) Schauder basis for a Banach space X is* quasi-greedy *if* $\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\|_X = 0$ *for every element $x \in X$.*

Wojtaszczyk proved in [9] that a basis is quasi-greedy if and only if there exists a constant $C < \infty$ such that for every $x \in X$, $\sup_m \|\mathcal{G}_m(x)\|_X \leq C\|x\|_X$.

Whenever a basis fails to be quasi-greedy we may conclude that there exists an element $x_0 \in X$ such that $\mathcal{G}_m(x_0) \not\to x_0$ in the norm of X. It was proved in [2] that the trigonometric system in $L^p[0,1)$, $1 \leq p < 2$, fails to be quasi-greedy, and another proof of this fact can be found in [7]. Much deeper results concerning the pointwise divergence of such partial sums almost everywhere were derived by Körner [5].

The purpose of this short note is to give yet another proof of the fact that the trigonometric system fails to be quasi-greedy in $L^p[0,1)$, $p \neq 2$, but this time with a more general proof that applies to other classical uniformly bounded orthonormal systems too, such as the Walsh system. We should also note that uniformly

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bounded orthonormal systems cannot form an unconditional basis for $L^p(\Omega)$, $p \neq 2$, due to a classical result by Orlicz so such systems are never trivially quasi-greedy.

2. A NECESSARY CONDITION TO BE QUASI-GREEDY

The crucial observation we need about quasi-greedy bases is the following lemma due to Wojtaszczyk [9], see also [3]. We give the simple proof for the sake of completeness.

Lemma 2.1. *Suppose $\{e_k\}_{k\in\mathbf{N}}$ is a quasi-greedy basis in X. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for every choice of signs $\epsilon_k = \pm 1$ and any finite subset $A \subset \mathbf{N}$ we have*

$$c_1 \Big\| \sum_{k\in A} e_k \Big\|_X \le \Big\| \sum_{k\in A} \epsilon_k e_k \Big\|_X \le c_2 \Big\| \sum_{k\in A} e_k \Big\|_X. \tag{1}$$

Proof. First, note that for any subset $B \subset A$ and any $\epsilon > 0$, we have by the quasi-greedy property $\big\| \sum_{k\in B}(1+\epsilon)e_k \big\|_X \le C \big\| \sum_{k\in B}(1+\epsilon)e_k + \sum_{k\in A\setminus B} e_k \big\|_X$. Letting $\epsilon \to 0$ we obtain $\big\| \sum_{k\in B} e_k \big\|_X \le C \big\| \sum_{k\in A} e_k \big\|_X$. This clearly implies $\big\| \sum_{k\in A} \epsilon_k e_k \big\|_X \le 2C \big\| \sum_{k\in A} e_k \big\|_X$. The other inequality is obtained the same way. □

We now specialize to uniformly bounded orthonormal bases $\mathcal{B} = \{e_k\}_{k\in\mathbf{N}}$ in $L^2(\Omega)$, with Ω a probability space. Notice that using the Hölder inequality it is easy to see that the uniform boundedness of the system, together with $\|e_k\|_{L^2(\Omega)} = 1$, implies that $\mathcal{B}$ is quasi-normalized in $L^p(\Omega)$, $1 \le p \le \infty$. We have the following result that gives a necessary condition for a uniformly bounded orthonormal basis to be quasi-greedy in some $L^p(\Omega)$, $p \neq 2$.

Proposition 2.2. *Let $\mathcal{B} = \{e_k\}_{k\in\mathbf{N}}$ be a uniformly bounded orthonormal basis for $L^2(\Omega)$. Suppose that $\mathcal{B}$ is quasi-greedy in $L^p(\Omega)$ for some $1 \le p < \infty$, $p \neq 2$. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for any $\epsilon = \{\epsilon_k\}_{k\in\mathbf{N}} \in \{-1,1\}^{\mathbf{N}}$ and any finite subset $A \subset \mathbf{N}$,*

$$c_1 |A|^{1/2} \le \Big\| \sum_{k\in A} \epsilon_k e_k \Big\|_{L^1(\Omega)} \le c_2 |A|^{1/2}.$$

Proof. First we consider the case $1 \le p < 2$. Let $r_1, r_2, \dots$ be the Rademacher functions on $[0,1]$, and take any finite subset of integers $A = \{k_1, k_2, \dots, k_N\} \subset \mathbf{N}$. Using Lemma 2.1, and the fact that $L^p(\Omega)$ has cotype 2 (see e.g. [8, Chap. 3]), we obtain

$$\Big\| \sum_{n=1}^N \epsilon_{k_n} e_{k_n} \Big\|_{L^p(\Omega)} \asymp \int_0^1 \Big\| \sum_{n=1}^N r_n(t)\, e_{k_n} \Big\|_{L^p(\Omega)} dt \ge C \Big(\sum_{n=1}^N \|e_{k_n}\|_{L^p(\Omega)}^2 \Big)^{1/2} \asymp N^{1/2}.$$

Put $D_N = \sum_{n=1}^N \epsilon_{k_n} e_{k_n}$. The system $\mathcal{B}$ is orthonormal so we can apply Hölder's inequality to get

$$N^{1/2} \lesssim \|D_N\|_{L^p(\Omega)} \le \|D_N\|_{L^1(\Omega)}^{\alpha} \|D_N\|_{L_2(\Omega)}^{1-\alpha} = \|D_N\|_{L^1(\Omega)}^{\alpha} N^{(1-\alpha)/2},$$

$1/p = \alpha + (1-\alpha)/2$, so $N^{1/2} \lesssim \|D_N\|_{L^1(\Omega)} \leq \|D_N\|_{L^2(\Omega)} = N^{1/2}$ follows. Now suppose $2 < p < \infty$. Then $L^p(\Omega)$ has type 2 ([8, Chap. 3]), and using Lemma 2.1, we get the estimate

$$\Big\| \sum_{n=1}^{N} \epsilon_{k_n} e_{k_n} \Big\|_{L^p(\Omega)} \asymp \int_0^1 \Big\| \sum_{n=1}^{N} r_n(t)\, e_{k_n} \Big\|_{L^p(\Omega)} dt \leq C \Big(\sum_{n=1}^{N} \|e_{k_n}\|^2_{L^p(\Omega)} \Big)^{1/2} \asymp N^{1/2}.$$

We apply Hölder's inequality again

$$N^{1/2} = \|D_N\|_{L_2(\Omega)} \leq \|D_N\|^{\alpha}_{L^1(\Omega)} \|D_N\|^{1-\alpha}_{L^p(\Omega)} \lesssim \|D_N\|^{\alpha}_{L^1(\Omega)} N^{(1-\alpha)/2},$$

$1/2 = \alpha + (1-\alpha)/p$, and $\|D_N\|_{L^1(\Omega)} \asymp N^{1/2}$ follows as in the previous case. □

We have the following immediate corollary.

Corollary 2.3. *Let $\mathcal{B} = \{e_k\}_{k\in\mathbf{N}}$ be a uniformly bounded orthonormal basis for $L^2(\Omega)$. Suppose there exists a family of finite subsets $A_n \subset \mathbf{N}$ for which*

$$|A_n|^{-1/2} \| \sum_{k\in A_n} e_k \|_{L^1(\Omega)} = o(1),$$

then $\mathcal{B}$ is not quasi-greedy in $L^p(\Omega)$, $p \neq 2$.

Proof. If $\mathcal{B}$ is not a Schauder basis for $L^p(\Omega)$ there is nothing to prove; if it is a Schauder basis then the result follows directly from Proposition 2.2. □

The necessary conditions for a uniformly bounded orthonormal system to be quasi-greedy are quite restrictive and most "classical" system will fail to be quasi-greedy. Let us give some explicit examples.

Example 2.4. *Let $\mathcal{B}$ be the trigonometric system on $\mathbf{T}^n$, $n \geq 1$ (with any ordering). We claim that $\mathcal{B}$ is not quasi-greedy in $L^p(\mathbf{T}^n)$, $p \neq 2$. To verify the claim, we consider the spherical Dirichlet kernel, $D_R(x) = \sum_{|k|\leq R} e^{ik\cdot x}$. For $n = 1$ it is well-known that $\|D_R\|_{L^1(\mathbf{T}^n)} = O(\log(R))$ and for $n \geq 2$ we have $\|D_R\|_{L^1(\mathbf{T}^n)} = O(R^{(n-1)/2})$ which was proved by Shapiro* [6]*. However, in both cases, $\|D_R\|_{L^2(\mathbf{T}^n)} \asymp R^{n/2}$ so Corollary 2.3 applies and we get the result.*

The next example shows that the same result holds for the Walsh system on $[0,1)$.

Example 2.5. *Let D_n be the Dirichlet kernel for the Walsh system. It is well-known that for this system, $\|D_n\|_{L^1[0,1)} = O(\log(n))$ so Corollary 2.3 applies and consequently the system is not quasi-greedy in $L^p[0,1)$, $p \neq 2$, even though the system forms a Schauder basis for $L^p[0,1)$, $1 < p < \infty$.*

Remark 2.6. *The condition given by Proposition 2.2 is restrictive, and the authors know of no orthonormal basis that satisfies the condition. The Rademacher functions do satisfy the condition, but the system is not complete in L_2 and thus not a basis. In fact, we conjecture the following: There are no uniformly bounded quasi-greedy Schauder bases in $L_p(\Omega)$, $p \neq 2$.*

3. Hausdorff-Young inequalities for uniformly bounded orthonormal quasi-greedy bases

In this section we show that a quasi-greedy uniformly bounded orthonormal system satisfies an improved Hausdorff-Young inequality. For the sake of completeness, we first prove a lemma considered by the authors in [4]. We will use the notation ℓ^p_q and $L^p_q(\Omega)$, respectively, to denote the standard Lorentz spaces with main index p and secondary index q.

Lemma 3.1. *Let $\mathcal{B} = \{e_k\}_{k\in\mathbf{N}}$ be a quasi-greedy basis in the Banach space X. Suppose for some $\beta \geq 1$, for every finite set of indices $A \subset \mathbf{N}$ and any choice of signs*

$$\Big\| \sum_{k\in A} \pm e_k \Big\|_X \asymp |A|^{1/\beta}.$$

Then there exist constants $0 < c_1 \leq c_2 < \infty$ such that for any coefficients $\{a_k\}$

$$c_1 \|\{a_k\}\|_{\ell^\beta_\infty} \leq \Big\| \sum_{k\in\mathbf{N}} a_k e_k \Big\|_X \leq c_2 \|\{a_k\}\|_{\ell^\beta_1}.$$

Proof. Let $f = \sum_{k\in\mathbf{N}} c_k g_k \in X$. Since $\mathcal{B}$ is quasi-greedy there is a constant C depending only on $\mathcal{B}$ [9] such that

$$\sup_N \| \sum_{k=1}^N c_k^\star g_{\pi(k)} \|_X \leq C \|f\|_X,$$

where $\{c_k^\star\}$ corresponds to the decreasing rearrangement of $\{|c_k|\}$. Using the Abel transform we get for any increasing sequence $\{\alpha_k\}$ of positive numbers that $\sup_N \| \sum_{k=1}^N \alpha_k c_k^\star g_{\pi(k)} \|_X \leq C(\sup_k \alpha_k) \|f\|_X$. Thus, for every $N \geq 1$ for which $c_N^\star \neq 0$ and $\alpha_k = |c_N^\star| |c_k^\star|^{-1}$, $k = 1, 2, \ldots, N$, we have

$$|c_N^\star| N^{1/\beta} \leq c^{-1} \Big\| \sum_{k=1}^N \frac{c_k^\star |c_N^\star|}{|c_k^\star|} g_{\pi(k)} \Big\|_X \leq c^{-1} C \|f\|_X.$$

It follows at once that $\|\{c_k\}\|_{\ell_{\beta,\infty}(\mathbf{N})} \leq c^{-1} C \|f\|_X$. Conversely, an extremal point argument shows that for all finite set $\Lambda \subset \mathbf{N}$,

$$\| \sum_{k\in\Lambda} c_k g_k \| \leq \sup_{k\in\Lambda} |c_k| \cdot \sup_{\varepsilon_k \in \{-1,1\}} \| \sum_{k\in\Lambda} \varepsilon_k g_k \| \leq C \sup_{k\in\Lambda} |c_k| |\Lambda|^{1/p}.$$

Let $f = \sum_{k\in\mathbf{N}} c_k g_k \in X$ and denote by $\Lambda_j = \{k : |c_k^\star| \geq 2^{-j}\}$. As $\mathcal{B}$ is quasi-greedy we can write

$$\begin{aligned} \|f\|_X &= \Big\| \sum_{j=-\infty}^{\infty} \sum_{k\in\Lambda_j\setminus\Lambda_{j-1}} c_k^\star g_{\pi(k)} \Big\|_X \leq \sum_{j=-\infty}^{\infty} \Big\| \sum_{k\in\Lambda_j\setminus\Lambda_{j-1}} c_k^\star g_{\pi(k)} \Big\|_X \\ &\leq \sum_{j\in\mathbf{Z}} C 2^{-(j-1)} |\Lambda_j \setminus \Lambda_{j-1}|^{1/\beta} \leq \tilde{C} \sum_j 2^{-j} |\Lambda_j|^{1/\beta} \leq C' \|\{c_k(f)\}\|_{\ell_{\beta,1}}. \end{aligned}$$

□

From the lemma and Proposition 2.2 we immediately get the following result for quasi-greedy systems. Notice that the lower estimate is an improvement of the generic Hausdorff-Young estimate.

Proposition 3.2. *Let $\mathcal{B} = \{e_k\}_{k\in\mathbf{N}}$ be a uniformly bounded orthonormal basis for $L^2(\Omega)$. Suppose $\mathcal{B}$ is quasi greedy in $L^p(\Omega)$ for some $1 \leq p < \infty$. Then there exist constants $0 < c_1 \leq c_2 < \infty$ such that for any coefficients $\{a_k\}$*

$$c_1\|\{a_k\}\|_{\ell^2_\infty} \leq \Big\| \sum_{k\in\mathbf{N}} a_k e_k \Big\|_{L^p(\Omega)} \leq c_2\|\{a_k\}\|_{\ell^2_1}.$$

4. Some other orthonormal systems that fail to be quasi-greedy

We conclude this note by considering the local sine and cosine bases of Coifman and Meyer [1] to show that in some cases the results of the previous section also work for uniformly bounded orthonormal bases in $L^2(\mathbf{R})$ as long as the basis functions have a local nature. For simplicity let us only consider sine systems. On the interval $I = [\alpha, \beta]$, such a basis will look like

$$\left\{ \psi_{I,k}(t) := \sqrt{\frac{2}{|I|}} b_I(t) \sin \frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha) \right\}_{k=0}^{\infty},$$

with b_I some nice smooth real-valued "bump function". The full system is obtained by taking the union of such systems corresponding to a partition of the real line with intervals I, see e.g. [1]. We have the following result:

Corollary 4.1. *Any local sine or cosine basis will fail to be quasi-greedy in $L^p(\mathbf{R})$ for $p \neq 2$.*

Proof. It suffices to show this on one of the intervals I. We notice that

$$D_n(t) := \sum_{k=0}^{n} \psi_{I,k}(t) = \sqrt{\frac{2}{|I|}} b_I(t) \sum_{k=0}^{n} \sin \frac{2k+1}{2}\frac{\pi}{|I|}(x-\alpha).$$

By a change of variable we may assume that $\alpha = 0$, and $|I| = 1$, so from the identity $\sum_{k=0}^{n-1} \sin[(2k+1)x] = \sin^2(nx)/\sin(x)$, we deduce the estimate $\|D_n\|_{L^1(\mathbf{R})} \leq C(I)\log(n)$. The result then follows from Corollary 2.3. □

References

[1] Ronald R. Coifman and Yves Meyer. Remarques sur l'analyse de Fourier à fenêtre. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(3):259–261, 1991.

[2] Antonio Córdoba and Pablo Fernández. Convergence and divergence of decreasing rearranged Fourier series. *SIAM J. Math. Anal.*, 29(5):1129–1139 (electronic), 1998.

[3] S. J. Dilworth, N. J. Kalton, Denka Kutzarova, and V. N. Temlyakov. The thresholding greedy algorithm, greedy bases, and duality. *Technical report*, 2001. IMI, Dept. of Mathematics, University of South Carolina.

[4] R. Gribonval and M. Nielsen. Some remarks on nonlinear approximation with Schauder bases. *East J. Approx.*, 7(3):267–285, 2001.

[5] T. W. Körner. Divergence of decreasing rearranged Fourier series. *Ann. of Math. (2)*, 144(1):167–180, 1996.

[6] H. S. Shapiro. Lebesgue constants for spherical partial sums. *J. Approx. Theory*,13:40–44, 1975.
[7] V. N. Temlyakov. Greedy algorithm and m-term trigonometric approximation. *Constr. Approx.*, 14(4):569–587, 1998.
[8] P. Wojtaszczyk. Banach spaces for analysts Cambridge University Press, Cambridge, 1991.
[9] P. Wojtaszczyk. Greedy algorithm for general biorthogonal systems. *J. Approx. Theory*, 107(2):293–314, 2000.

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