AALBORG UNIVERSITY

On Polynomial Symbols for Subdivision Schemes

by

Morten Nielsen

September 2004 R-2004-24

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: www.math.aau.dk/research/reports/reports.htm

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

ON POLYNOMIAL SYMBOLS FOR SUBDIVISION SCHEMES

MORTEN NIELSEN

ABSTRACT. Given a dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, and G a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$, we consider polynomial solutions M to the equation $\sum_{g\in G} M(\xi+g) = 1$ with the constraints that $M \geq 0$ and $M(0) = 1$. We prove that the full class of such functions can be generated using polynomial convolution kernels. Trigonometric polynomials of this type play an important role as symbols for interpolatory subdivision schemes. For isotropic dilation matrices, we use the method introduced to construct symbols for interpolatory subdivision schemes satisfying Strang-Fix conditions of arbitrary order.

1. INTRODUCTION

In this paper we present a general method to construct polynomial symbols associated with compactly supported refinable functions. Let A be a dilation matrix defined on $\mathbb{R}^d$, i.e., A is a $d \times d$-matrix with integer entries and all its eigenvalues have modulus larger than 1. A compatible compactly supported *refinable* function is a compactly supported function $\varphi \in L_2(\mathbb{R}^d)$ satisfying

$$\varphi(x) = |\det(A)| \sum_{k\in\mathbb{Z}^d} a_k \varphi(Ax-k) \tag{1.1}$$

for some finite sequence $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}^d} \in \ell_0(\mathbb{Z}^d)$ called the *mask* of φ. A compactly refinable function may not exist for every dilation matrix A. In case φ exists and satisfies (1.1), the symbol $M(\xi)$ associated with φ is given by $M(\xi) = \sum_{k\in\mathbb{Z}^d} a_k e^{i\langle k,\xi\rangle}$. A refinable function φ is called *interpolating* if it satisfies the additional constraint $\varphi(k) = \delta_{0,k}$ for all $k \in \mathbb{Z}^d$. A refinable function is called *nonseparable* if it cannot be written as a product of univariate refinable functions.

Interpolating refinable functions are used frequently in computer aided design and play an important role as generators of biorthogonal wavelet bases. In the one dimensional case, with $A = [2]$, the construction of an interpolating function is a crucial intermediate step in the well known method to construct compactly supported orthonormal wavelets [11], see also [21]. There is no straightforward generalization of method to construct wavelets in the univariate case to the multivariate case due to the unfortunate lack of a good substitute for the Fejér-Riesz factorization in $\mathbb{R}^d$ for $d \geq 2$. This has made the construction of compactly supported

Date: September 7, 2004.

Key words and phrases. Interpolating refinable function, polynomial symbols, subdivision scheme, multiwavelets.

Research partially supported by the Danish Technical Science Foundation, Grant no. 9701481, and by the Danish SNF-PDE network.

orthonormal multiwavelets a rather tedious and difficult task, and the contributions to date have dealt with somewhat limited contexts. Cohen and Daubechies [7] used univariate techniques to construct nonseparable wavelets in $\mathbb{R}^2$ for the class of dilations with $\det(A) = \pm 2$. The generators have an arbitrary number of vanishing moments, but are not even continuous. However, arbitrarily smooth bi-orthogonal wavelets were constructed in [7] for the dilation $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$. Belogay and Wang [2] constructed a family of arbitrarily smooth nonseparable wavelets for the dilation matrices $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right]$. Ayache [1] constructed arbitrarily smooth nonseparable wavelets for the isotropic dilations $\pm 2I$. Other specific examples of nonseparable wavelets can be found in, e.g., [5, 16, 14, 10, 20]. Haar-type wavelet bases for very general dilation matrices are considered in [17]. Smooth r-regular multivariate wavelets for arbitrary dilations with infinite masks are constructed in [4, 3]. The wavelets in [4, 3] do not have compact support.

Subdivision schemes and refinable functions in general are considered in the monograph [6]. Jia considered multidimensional interpolatory refinable functions induced by box splines [15]. The multidimensional case is also considered in [23]. Interpolatory functions resembling the one dimensional Daubechies construction is considered by Derado in [12, 13]. The smoothness of refinable functions is most often estimated by considering the decay of the Fourier transform of the function. A technique based on spectral methods to calculate the smoothness of refinable functions is considered in [9]. There are also estimates on the smoothness of refinable functions in [2, 7, 8, 13].

The author considered a method to construct finite filters for orthonormal wavelets in [22]. The method in [22] is based on an integral representation of the symbol for the filter, and the purpose was to construct finite filters providing the best possible approximation to the Shannon filter. Integral representations of univariate wavelet filters were also considered by Lemarie-Rieusset [18] and by Lemarie-Rieusset and Zahrouni [19]. In the present paper we adapt the method in [22] to the multidimentional case, and discuss the problem of creating symbols satisfying certain Strang-Fix conditions. The main advantage of the method in the present paper is that it is explicit, flexible, and easy to implement. In fact, we show that any mask for an interpolatory refinable function can (in principle) be constructed using the method.

The structure of the paper is as follows. Section 2 contains some basic and well-known material on refinable functions and their masks. In Section 3 we introduce a general method to construct masks for interpolatory subdivision schemes. Proposition 3.1 in Section 3.1 shows that one can generate non-negative polynomial symbols using polynomial convolution kernels, and the method works for arbitrary dilation matrices A. The polynomials obtained by this method unfortunately do not take on the value 1 at the origin, so there is no chance that they can be symbols of any convergent subdivision scheme. However, Proposition 3.5 in Section 3.2 will give a rather general method to "repair" the functions from Proposition 3.1 by an affine transformation so they take the value 1 at the origin. In Section 3.3 we show that the method from Sections 3.2 and 3.1 is *completely general.* Every finite mask associated with a refinable function (1.1) can be generated by the method. In the final section of the paper, Section 4, we show how to construct polynomial symbols satisfying Strang-Fix conditions of arbitrary order for a certain class of dilation matrices.

2. General Setting

For a given dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, we consider the refinement equation (1.1). The symbol $M(\xi)$ associated with the mask $\{a_k\}$ from (1.1) is defined by

$$M(\xi) = \sum_{k\in\mathbb{Z}^d} a_k e^{i\langle k,\xi\rangle}.$$

In the Fourier domain (1.1) becomes

$$\hat{\varphi}(\xi) = M(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi), \tag{2.1}$$

with $B := A^\top$ (this notation will be used throughout the paper). The most successful approach to actually construct refinable functions φ satisfying (1.1) is inspired by (2.1) and the idea is to first design the mask $M(\xi)$ and then simply define φ by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} M(B^{-j}\xi).$$

Obviously, some constraints have to be put on the mask M for this to work. Let us recall here the well known sufficient conditions a symbol M has to satisfy to generate an interpolatory refinable function, see [13, 9]. For the given dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, we let G be a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$. The trigonometric polynomial M satisfies *Cohen's condition* (see [7]) if there exists a compact set $T \subset \mathbb{R}^d$ satisfying

(A) T tiles $\mathbb{R}^d$, i.e., $\cup_{k\in\mathbb{Z}^d}(T + 2\pi k) = \mathbb{R}^d$ a.e.,
(B) $M(B^{-j}\xi) > 0$, for $\xi \in T$ and $j = 1, 2, \ldots$,
(C) $(-\varepsilon, \varepsilon)^d \subset T$ for some $\varepsilon > 0$.

Suppose that the symbol M satisfies

$$M \geq 0, \quad M(0) = 1, \tag{2.2}$$

$$\sum_{g\in G} M(\xi + g) = 1, \tag{2.3}$$

$$\text{Cohen's condition}, \tag{2.4}$$

then the product

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} M(B^{-1}\xi),$$

defines a interpolatory refinable functions satisfying (1.1), see [13].

3. Symbols for interpolatory subdivision schemes

This section contains the basic machinery we use to construct symbols for interpolatory subdivision schemes.

3.1. **Filters generated by convolution kernels.** The first result will show that whenever we have *any* (perhaps very "rough") solution to (2.3), then we can easily generate polynomial solutions to (2.3).

Proposition 3.1. *Consider a dilation matrix* $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$*, and* G *a complete set of coset representatives of* $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$*. Let* $K(\xi)$ *be a* $2\pi\mathbb{Z}^d$*-periodic function for which* $\sum_{g\in G} K(\xi+g) = 1$ *a.e. Then for any non-negative trigonometric polynomial* $P(\xi)$ *with* $\int_{[-\pi,\pi]^d} P(\xi)\,d\xi = 1$*, the function* M *defined by*

$$M(\xi) := \int_{[-\pi,\pi]^d} P(u)K(\xi-u)\,du$$

is a non-negative trigonometric polynomial, with degree at most $\deg(P)$*, that satisfies*

$$\sum_{g\in G} M(\xi+g) = 1.$$

Proof. The fact that M is a non-negative trigonometric polynomial of degree at most $\deg(P)$ follows from elementary properties of convolution operators. We have,

$$\begin{aligned}\sum_{g\in G} M(\xi+g) &= \sum_{g\in G}\int_{[-\pi,\pi]^d} P(u)K(\xi+g-u)\,du \\ &= \int_{[-\pi,\pi]^d} P(u)\Big\{\sum_{g\in G} K(\xi+g-u)\Big\}\,du \\ &= \int_{[-\pi,\pi]^d} P(u)\,du \\ &= 1.\end{aligned}$$

□

We will often use a "Shannon-type" filter as kernel K, and then an approximation to the identity as the polynomial P. This will ensure that the resulting filter M is a smooth approximation to the ideal Shannon filter with good frequency resolution. This imitates the construction of the Daubechies filters in the univariate case. The square of the Daubechies filters are know to approach the univariate Shannon filter $\chi_{[-\pi/2,\pi/2]}$.

The following elementary lemma shows that we can always find a function K needed for Proposition 3.1.

Lemma 3.2. *Let* $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$ *be a dilation matrix, and* G *a complete set of coset representatives of* $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$*. Suppose the set* $T \subset \mathbb{R}^d$ *is such that* $\sum_{k\in\mathbb{Z}^d} \chi_T(\xi + 2\pi k) = 1$ *a.e. Then the function*

$$K(\xi) := \sum_{k\in\mathbb{Z}^d} \chi_T(A^\top(\xi+2\pi k))$$

is $2\pi\mathbb{Z}^d$*-periodic and satisfies*

$$\sum_{g\in G} K(\xi+g) = 1 \qquad a.e.$$

Proof. A trivial computation shows that

$$\sum_{g\in G} K(\xi+g) = \sum_{g\in G}\sum_{k\in\mathbb{Z}^d} \chi_T(A^\top(\xi+g+2\pi k)) = \sum_{k\in\mathbb{Z}^d} \chi_T(A^\top\xi+2\pi k) = 1, \quad a.e.$$

□

Remark 3.3. The most elementary candidate for the function χ_T is $\chi_{[-\pi,\pi)^d}$. We obviously have $\sum_{k\in\mathbb{Z}^d}\chi_{[-\pi,\pi)^d}(\xi-2\pi k)=1$. Also, if we have $B^{-1}T\subset T\subset[-\pi,\pi]^d$ then it follows that $K|_{[-\pi,\pi]^d} = \chi_{B^{-1}T}$. Also notice that such K automatically satisfies Cohen's condition.

Let us now consider a first example of the approach. We consider the well-know quincunx dilation matrix, and use Proposition 3.1 with a product Fejér kernel.

Example 3.4. Consider the quincunx dilation

$$A := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.1}$$

It is clear that we can take $K|_{[-\pi,\pi]^2} = \chi_{B^{-1}[-\pi,\pi]^2}$. Let

$$F_n(\xi) = \frac{1}{n\pi}\frac{\sin(n\xi\pi/2)}{\sin(\xi/2)}$$

be the Fejér kernel and let $P_n(x,y)=F_n(x)F_n(y)$ be the associated product kernel. A direct computation shows that $\chi_{B^{-1}[-\pi,\pi]^2} = \frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}^2} a_k e^{i\langle k,\xi\rangle}$, with

$$a_{(k_1,k_2)} = \begin{cases} 1/2, & k_1=k_2=0, \\ 0, & |k_1|=|k_2|\neq 0, \\ 4\dfrac{(-1)^{k_2}-(-1)^{k_1}}{k_1^2-k_2^2}, & \text{otherwise.} \end{cases} \tag{3.2}$$

Thus we get the symbol $S(\xi)=\sum_k b_k e^{i\langle k,\xi\rangle}$ associated to P_n with coefficients:

$$b_{(k_1,k_2)} = \left(\frac{n-|k_1|}{n}\right)\left(\frac{n-|k_2|}{n}\right) a_{(k_1,k_2)}, \qquad |k_1|,|k_2|<n.$$

As an explicit example, let us consider $n=4$. Then we get the 7×7 quadratic mask $[b_{k_1,k_2}]_{k_1,k_2=-3}^{3}$ given by (the constant term of the symbol is boxed):

$$\begin{bmatrix} 0 & \frac{1}{20\pi^2} & 0 & \frac{1}{18\pi^2} & 0 & \frac{1}{20\pi^2} & 0 \\ \frac{1}{20\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 & \frac{1}{20\pi^2} \\ 0 & -\frac{1}{4\pi^2} & 0 & \frac{3}{2\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 \\ \frac{1}{18\pi^2} & 0 & \frac{3}{2\pi^2} & \boxed{\frac{1}{2}} & \frac{3}{2\pi^2} & 0 & \frac{1}{18\pi^2} \\ 0 & -\frac{1}{4\pi^2} & 0 & \frac{3}{2\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 \\ \frac{1}{20\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 & -\frac{1}{4\pi^2} & 0 & \frac{1}{20\pi^2} \\ 0 & \frac{1}{20\pi^2} & 0 & \frac{1}{18\pi^2} & 0 & \frac{1}{20\pi^2} & 0 \end{bmatrix} \tag{3.3}$$

□

One can check that the symbol for the filter has value ≈ 0.968 at 0, which makes the mask rather "useless", but we will show in the next section how to "repair" such symbols.

3.2. **Affine transform of symbols.** The major deficiency of the symbols obtained from Proposition 3.1 is that they never take on the value 1 at 0 (due to the fact that P is a polynomial and thus never identically zero on any region of non-zero measure) so they are not symbols of any convergent interpolatory subdivision scheme. The good news is that the deficiency can be "corrected" by an affine transform for a large number of such functions. We have the following result.

Proposition 3.5. *Given a dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, and G a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$. Suppose the non-negative $2\pi\mathbb{Z}^d$-periodic function $M(\xi)$ satisfies*

$$\sum_{g\in G} M(\xi+g)=1, \quad M(0)>|G|^{-1}, \quad \text{and} \quad M(\xi)\geq \frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}} M(g).$$

Then we can define

$$\tilde{M}(\xi)=1-\frac{M(0)-M(\xi)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}} M(g)}. \tag{3.4}$$

The function $\tilde{M}$ satisfies

$$\tilde{M}\geq 0, \quad \tilde{M}(0)=1, \quad \text{and} \quad \sum_{g\in G}\tilde{M}(\xi+g)=1.$$

Proof. Notice that the inequality $M(0)|G|>1$ implies that

$$M(0)>\frac{1-M(0)}{|G|-1} \Rightarrow M(0)>\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}} M(g),$$

so $\tilde{M}$ is well-defined. Obviously $\tilde{M}(0)=1$. The estimate $\tilde{M}\geq 0$ follows from the fact that $\tilde{M}$ is given by

$$\tilde{M}(\xi)=\frac{M(\xi)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}} M(g)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}} M(g)}.$$

We also have

$$
\begin{aligned}
\sum_{g\in G}\tilde{M}(\xi+g) &= |G| - \sum_{g\in G}\frac{M(0)-M(\xi+g)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}}M(g)}\\
&= |G| + \frac{1-|G|M(0)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}}M(g)}\\
&= |G| + \frac{(1-M(0))-(|G|-1)M(0)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}}M(g)}\\
&= |G| + \frac{\sum_{g\in G\setminus\{0\}}M(g)-(|G|-1)M(0)}{M(0)-\frac{1}{|G|-1}\sum_{g\in G\setminus\{0\}}M(g)}\\
&= |G| - (|G|-1)\\
&= 1.
\end{aligned}
$$

□

Remark 3.6. We notice that the conditions of the proposition are satisfied whenever the function M satisfies $M(0) > |G|^{-1}$ and $\min_{\xi\in[-\pi,\pi]^d} M(\xi) = \min_{g\in G} M(g)$.

Example 3.7. We apply Proposition 3.5 to the filter given by (3.3), and get the filter

$$
\begin{bmatrix}
0 & \frac{9}{1664} & 0 & \frac{5}{832} & 0 & \frac{9}{1664} & 0\\
\frac{9}{1664} & 0 & -\frac{45}{1664} & 0 & -\frac{45}{1664} & 0 & \frac{9}{1664}\\
0 & -\frac{45}{1664} & 0 & \frac{135}{832} & 0 & -\frac{45}{1664} & 0\\
\frac{5}{832} & 0 & \frac{135}{832} & \boxed{\frac{1}{2}} & \frac{135}{832} & 0 & \frac{5}{832}\\
0 & -\frac{45}{1664} & 0 & \frac{135}{832} & 0 & -\frac{45}{1664} & 0\\
\frac{9}{1664} & 0 & -\frac{45}{1664} & 0 & -\frac{45}{1664} & 0 & \frac{9}{1664}\\
0 & \frac{9}{1664} & 0 & \frac{5}{832} & 0 & \frac{9}{1664} & 0
\end{bmatrix}.
$$

which corresponds to a symbol $\tilde{S}$ that has value one at the origin. In terms of the symbols, Proposition 3.5 gives

$$
\tilde{S}(\xi_1,\xi_2) = \frac{S(\xi_1,\xi_2)-S(\pi,\pi)}{S(0,0)-S(\pi,\pi)} = \frac{1}{2} - \frac{45\pi^2}{832} + \frac{45\pi^2}{416}S(\xi_1,\xi_2).
$$

□

3.3. **On the generality of the method.** The method presented so far boils down to the following two simple steps.

- Create an intermediary symbol S using Proposition 3.1.
- (If possible) Correct the symbol S using Proposition 3.5.

The question is how general the method presented is. We now want to demonstrate that the method can generate every possible finite mask satisfying (2.2) and (2.3). Let us define the following full collection of symbols for (1.1). Given a dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, and G a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$, we let

$$\mathcal{M}(A,G) := \{M : M \text{ is a trig. polynomial on } \mathbb{R}^d \text{ satisfying (2.2) and (2.3)}\}.$$

First we prove the following simple but useful lemma.

Lemma 3.8. *Given a dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, and G a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$. Suppose $\tilde{M} \in \mathcal{M}(A,G)$. Define $M = \alpha+\beta\tilde{M}$ for $\alpha, \beta > 0$, $\alpha + \beta \leq 1$. Then substituting M in the right hand side of (3.4) recovers $\tilde{M}$.*

Proof. Notice that

$$\sum_{g\in G\setminus\{0\}} M(g) = (|G|-1)\alpha + \beta \sum_{g\in G\setminus\{0\}} \tilde{M}(g) = (|G|-1)\alpha.$$

Hence, (3.4) yields

$$1 - \frac{(\alpha+\beta) - (\alpha+\beta\tilde{M})}{\alpha+\beta-\alpha} = 1-(1-\tilde{M}) = \tilde{M}.$$

□

Proposition 3.9. *Consider a dilation matrix $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, and G a complete set of coset representatives of $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$. Given any symbol $\tilde{M} \in \mathcal{M}(A,G)$, there exist a positive kernel P and a symbol K such that Proposition 3.1 and Proposition 3.5 used successively recover $\tilde{M}$.*

Proof. We define a positive kernel by

$$P(u) = \frac{1}{(2\pi)^d}\frac{D_n(u)+L}{1+L},$$

where D_n is the Dirichlet (product) kernel given by

$$D_n(u) = \sum_{|k|\leq n} e^{ik\cdot u},$$

and L is a large positive number ensuring that $P \geq 0$. Notice that

$$\begin{aligned}\int_{[-\pi,\pi]^d} P(u)du &= \frac{1}{(2\pi)^d+(2\pi)^d L}\left(\int_{[-\pi,\pi]^d} \big(D_n(u)+L\big)du\right)\\ &= \frac{(2\pi)^d+(2\pi)^d L}{(2\pi)^d+(2\pi)^d L}\\ &= 1.\end{aligned}$$

Define M by Proposition 3.1 using the kernel P and $K = \tilde{M}$. Clearly,

$$M = \alpha + \beta\tilde{M},$$

and Lemma 3.8 shows that Proposition 3.5 recovers $\tilde{M}$. □

4. Symbols satisfying Strang-Fix conditions

The smoothness of a refinable function generated by the symbol M depends to some extend on the behavior of M at the origin, see [9, 13] for a detailed discussion. To have smooth interpolatory function, it is thus necessary to be able to design symbols that satisfy Strang-Fix conditions of high order.

We want to use the method from Proposition 3.5 to construct the symbol $\tilde{M}$ such that $\tilde{M}(\xi) = 1 - O(\|\xi\|^r)$ at the origin, which will imply that for each $g' \in G\backslash\{0\}$,

$$\tilde{M}(\xi + g') \leq \sum_{g \in G\backslash\{0\}} \tilde{M}(\xi + g) = O(\|\xi\|^r).$$

Let us assume we have a symbol M with $M(0) \approx 1$, satisfying the hypothesis of Proposition 3.5. From Proposition 3.5 we see that to have $\tilde{M}(\xi) = 1 - O(\|\xi\|^r)$ at the origin, it suffices to ensure that $M(0) - M(\xi) = O(\|\xi\|^r)$ at the origin.

4.1. **A "rotation" method for nonseparable kernels.** Let us first consider a variation on Proposition 3.1 that is sometimes useful to construct filters that satisfy Strang-Fix conditions.

Proposition 4.1. *Consider a dilation matrix* $A\colon \mathbb{Z}^d \to \mathbb{Z}^d$, *and* G *a complete set of coset representatives of* $2\pi(A^{-\top}\mathbb{Z}^d/\mathbb{Z}^d)$. *Let* $K(\xi)$ *be a* $2\pi\mathbb{Z}^d$*-periodic function for which* $\sum_{g\in G} K(\xi+g) = 1$ *a.e. Then for any non-negative trigonometric polynomial* $P(\xi)$ *with* $\int_{[-\pi,\pi]^d} P(\xi)d\xi = 1$, *the function* M *defined by*

$$M(\xi) := \int_{[-\pi,\pi]^d} P(u)K(A^\top u + \xi)du$$

is a non-negative trigonometric polynomial, with degree at most $\deg(P)$, *that satisfies*

$$\sum_{g\in G} M(\xi + g) = 1.$$

Proof. We have,

$$\begin{aligned}
\sum_{g\in G} M(\xi + g) &= \sum_{g\in G} \int_{[-\pi,\pi]^d} P(u)K(A^\top u + \xi + g)du \\
&= \int_{[-\pi,\pi]^d} P(u)\Big\{\sum_{g\in G} K(A^\top u + \xi + g)\Big\}du \\
&= \int_{[-\pi,\pi]^d} P(u)\cdot 1du \\
&= 1.
\end{aligned}$$

Let $\tilde{M}(\xi) := M(A^\top\xi)$, and let $\tilde{K}(u) = K(-A^\top u)$. We notice that $\tilde{M}(\xi) = P * \tilde{K}(-\xi)$, so $\tilde{M}(\xi)$ is a nonnegative trigonometric polynomial. Now, suppose

$K(u) = \sum_{k\in\mathbb{Z}^d} a_k e^{i\langle k,u\rangle}$, and we have $\tilde{K}(u) = \sum_{k\in\mathbb{Z}^d} a_k e^{-i\langle Ak,u\rangle}$. Hence, for $P(\xi) = \sum_k \beta_k e^{-i\langle k,\xi\rangle}$,

$$\tilde{M}(\xi) = P * \tilde{K}(-\xi) = \sum_{k\in\mathbb{Z}^d} \beta_{Ak} a_k e^{-i\langle Ak,\xi\rangle},$$

and

$$M(\xi) = \tilde{M}(A^{-\top}\xi) = \sum_{k\in\mathbb{Z}^d} \beta_{Ak} a_k e^{-i\langle (AA^{-1})k,\xi\rangle},$$

is a nonnegative trigonometric polynomial. □

4.2. **Daubechies-type Symbols for Isotropic dilations.** We now want to illustrate how Proposition 4.1 can be used to construct symbols that satisfies the Strang-Fix conditions of high order. As an example, we now construct Daubechies-type wavelets adapted to dilation matrices A satisfying $A^2 = 2I$. For simplicity we assume $d = 2$.

Let

$$U_N(\xi) := 1 - \frac{\int_0^\xi \sin^{2N-1}(u)\, du}{\int_0^\pi \sin^{2N-1}(u)\, du}$$

be the square of the univariate Daubechies filter of length $2N$. It was proved in [22] that $U_N(\xi) = \int_{-\pi/2}^{\pi/2} L_N(\xi - u)\, du$ with $L_N(u)$ some appropriate trigonometric polynomial given explicitely in [22, Lemma 2.1]. An associated nonnegative kernel is given by

$$\tilde{L}_N(u) = \frac{L_N(u) + \beta}{\beta + 1},$$

with $\beta = \max\{0, -\min_{\xi\in[-\pi,\pi]} L_N(\xi)\}$. We now let

$$P_N(u_1, u_2) = \frac{1}{4\pi^2}\tilde{L}_N(u_1)\tilde{L}_N(u_2) := \sum_{k_1,k_2=-2L+1}^{2L-1} b^N_{k_1,k_2} e^{i(k_1u_1+k_2u_2)},$$

and define

$$S_N(\xi) := \int_{[-\pi,\pi]^2} P_N(u) K(A^\top u + \xi)\, du, \qquad (A^\top)^2 := 2I,$$

with $K(u) = \chi_{B^{-1}[-\pi,\pi]^2}$. Notice that $S_N(\xi) = \sum_k c^N_k e^{i\langle k,\xi\rangle}$, with $c^N_{k_1,k_2} = b^N_{A\mathbf{k}} \cdot a_{k_1,k_2}$ and a_{k_1,k_2} given by (3.2). Then $S_N(\xi)$ is normalized using Proposition 3.5 to get the symbol

$$\tilde{S}_N(\xi) = \frac{S_N(\xi) - S_N(\pi,\pi)}{S_N(0,0) - S_N(\pi,\pi)}.$$

Example 4.2. Let us demonstrate the approach for $N = 2$, and A the quincunx dilation given by (3.1). We have $U_3(\xi) := 1/2 + 9/16\cos(\xi) - 1/16\cos(3\xi)$, and

$$U_3(\xi) = \int_{-\pi/2}^{\pi/2} L_3(\xi - u)\, du \quad \text{with} \quad L_3(u) = 1 + \frac{9\pi}{16}\cos(u) + \frac{3\pi}{16}\cos(3u).$$

An associated nonnegative kernel is given by $\tilde{L}_3(u) = \frac{L_3(u)+3}{4}$. We now let $P_3(u_1, u_2) = \frac{1}{4\pi^2}\tilde{L}_3(u_1)\tilde{L}_3(u_2) := \sum_{k_1,k_2=-3}^{3} b_{k_1,k_2} e^{i(k_1u_1+k_2u_2)}$, and define

$$S_3(\xi) := \int_{[-\pi,\pi]^2} P(u)K(A^\top u + \xi)\, du,$$

with $K(u) = \chi_{B^{-1}[-\pi,\pi]^2}$. Then $S_3(\xi)$ is normalized using Proposition 3.5 to get

$$\tilde{S}_3(\xi) = \frac{S_3(\xi) - S_3(\pi,\pi)}{S_3(0,0) - S_3(\pi,\pi)}.$$

We obtain the 7×7-mask corresponding to $\tilde{S}$:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{512} & 0 & 0 & 0 \\ 0 & 0 & -\frac{9}{512} & 0 & -\frac{9}{512} & 0 & 0 \\ 0 & -\frac{9}{512} & 0 & \frac{81}{512} & 0 & -\frac{9}{512} & 0 \\ \frac{1}{512} & 0 & \frac{81}{512} & \boxed{\frac{1}{2}} & \frac{81}{512} & 0 & \frac{1}{512} \\ 0 & -\frac{9}{512} & 0 & \frac{81}{512} & 0 & -\frac{9}{512} & 0 \\ 0 & 0 & -\frac{9}{512} & 0 & -\frac{9}{512} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{512} & 0 & 0 & 0 \end{bmatrix}$$

Notice the "diamond shape" of the mask which closely resembles the shape of $\chi_{B^{-1}[-\pi,\pi]^2}$. □

Example 4.3. The same approach for $N = 3$ yields the 11×11-mask

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{9}{131072} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{75}{131072} & 0 & -\frac{75}{131072} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{225}{65536} & 0 & \frac{625}{131072} & 0 & \frac{225}{65536} & 0 & 0 & 0 \\ 0 & 0 & \frac{225}{65536} & 0 & -\frac{1875}{65536} & 0 & -\frac{1875}{65536} & 0 & \frac{225}{65536} & 0 & 0 \\ 0 & -\frac{75}{131072} & 0 & -\frac{1875}{65536} & 0 & \frac{5625}{32768} & 0 & -\frac{1875}{65536} & 0 & -\frac{75}{131072} & 0 \\ \frac{9}{131072} & 0 & \frac{625}{131072} & 0 & \frac{5625}{32768} & \boxed{\frac{1}{2}} & \frac{5625}{32768} & 0 & \frac{625}{131072} & 0 & \frac{9}{131072} \\ 0 & -\frac{75}{131072} & 0 & -\frac{1875}{65536} & 0 & \frac{5625}{32768} & 0 & -\frac{1875}{65536} & 0 & -\frac{75}{131072} & 0 \\ 0 & 0 & \frac{225}{65536} & 0 & -\frac{1875}{65536} & 0 & -\frac{1875}{65536} & 0 & \frac{225}{65536} & 0 & 0 \\ 0 & 0 & 0 & \frac{225}{65536} & 0 & \frac{625}{131072} & 0 & \frac{225}{65536} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{75}{131072} & 0 & -\frac{75}{131072} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{131072} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

□

Proposition 4.4. *The symbols $\tilde{S}_N$, $N \geq 2$, associated with the quincunx dilation satisfy Cohen's condition, and $\tilde{S}_N$ satisfies the Strang-Fix condition of order N.*

Proof. It is not hard to see that the only zeros of $\tilde{S}_N$ are at $\pm(\pi,\pi)$. Therefore $\tilde{S}_N$ satisfies the Cohen condition for the tile $T = B^{-1}[-\pi,\pi]^2$. As explained in the introduction to this section, it suffices to prove that S_N satisfies the Strang-Fix condition of order N. However, if $s_N(\xi) := S_N(A^\top\xi) = P_N * \tilde{K}(-\xi)$ satisfies the condition, then S_N also satisfies the condition.

We notice that

$$\tilde{K}(\xi)\big|_{[-\pi,\pi]^2} = \chi_{[-\pi/2,\pi/2}(\xi_1)\chi_{[-\pi/2,\pi/2]}(\xi_2) + \chi_{[-\pi,-\pi/2]\cup[\pi/2,\pi]}(\xi_1)\chi_{[-\pi,-\pi/2]\cup[\pi/2,\pi]}(\xi_2).$$

Hence,

$$\frac{\partial^{r+s} s_N}{\partial^r \xi_1 \partial^s \xi_2}\bigg|_{(0,0)} = \int_{-\pi}^{\pi} \tilde{L}_N^{(r)}(u_1)\chi_{[-\pi/2,\pi/2]}(u_1) \int_{-\pi}^{\pi} \tilde{L}_N^{(s)}(u_2)\chi_{[-\pi/2,\pi/2]}(u_2) du_2\, du_1 + \int_{-\pi}^{\pi} \tilde{L}_N^{(r)}(u_1)\chi_{[-\pi,-\pi/2]\cup[\pi/2,\pi]}(u_1) \int_{-\pi}^{\pi} \tilde{L}_N^{(s)}(u_2)\chi_{[-\pi,-\pi/2]\cup[\pi/2,\pi]}(u_2) du_2\, du_1.$$

However,

$$\int_{-\pi}^{\pi} \tilde{L}_N^{(r)}(u_1)\chi_{[-\pi/2,\pi/2]}(u_1) du_1 = 0 = \int_{-\pi}^{\pi} \tilde{L}_N^{(s)}(u_2)\chi_{[-\pi,-\pi/2]\cup[\pi/2,\pi]}(u_2) du_2,$$

for $0 \le r, s \le 2N$ since the integrals equals (up to a multiplicative constant) $U_N^{(r)}(0)$ and $U_N^{(s)}(\pi)$, respectively.

□

Remark 4.5. One can check, modifying the argument above slightly, that the conclusion of Proposition 4.4 holds for matrices A satisfying $A^2 = F$, with F one of the following matrices

$$\pm \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \pm \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

References

[1] A. Ayache. Construction of non-separable dyadic compactly supported orthonormal wavelet bases for $L^2(\mathbf{R}^2)$ of arbitrarily high regularity. *Rev. Mat. Iberoamericana*, 15(1):37–58, 1999.

[2] E. Belogay and Y. Wang. Arbitrarily smooth orthogonal nonseparable wavelets in $\mathbf{R}^2$. *SIAM J. Math. Anal.*, 30(3):678–697 (electronic), 1999.

[3] M. Bownik. The construction of r-regular wavelets for arbitrary dilations. *J. Fourier Anal. Appl.*, 7(5):489–506, 2001.

[4] M. Bownik and D. Speegle. Meyer type wavelet bases in $\mathbb{R}^2$. *J. Approx. Theory*, 116(1):49–75, 2002.

[5] C. A. Cabrelli, C. Heil, and U. M. Molter. Multiwavelets in $\mathbb{R}^n$ with an arbitrary dilation matrix. In *Wavelets and signal processing*, Appl. Numer. Harmon. Anal., pages 23–39. Birkhäuser Boston, Boston, MA, 2003.

[6] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli. Stationary subdivision. *Mem. Amer. Math. Soc.*, 93(453):vi+186, 1991.

[7] A. Cohen and I. Daubechies. Nonseparable bidimensional wavelet bases. *Rev. Mat. Iberoamericana*, 9(1):51–137, 1993.

[8] A. Cohen, I. Daubechies, and G. Plonka. Regularity of refinable function vectors. *J. Fourier Anal. Appl.*, 3(3):295–324, 1997.

[9] A. Cohen, K. Gröchenig, and L. F. Villemoes. Regularity of multivariate refinable functions. *Constr. Approx.*, 15(2):241–255, 1999.

[10] W. Dahmen and C. A. Micchelli. Biorthogonal wavelet expansions. *Constr. Approx.*, 13(3):293–328, 1997.

[11] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
[12] J. Derado. Multivariate refinable interpolating functions. *Appl. Comput. Harmon. Anal.*, 7(2):165–183, 1999.
[13] J. Derado. Nonseparable, compactly supported interpolating refinable functions with arbitrary smoothness. *Appl. Comput. Harmon. Anal.*, 10(2):113–138, 2001.
[14] W. He and M.-J. Lai. Construction of bivariate compactly supported biorthogonal box spline wavelets with arbitrarily high regularities. *Appl. Comput. Harmon. Anal.*, 6(1):53–74, 1999.
[15] R.-Q. Jia. Interpolatory subdivision schemes induced by box splines. *Appl. Comput. Harmon. Anal.*, 8(3):286–292, 2000.
[16] J. Kovačević and M. Vetterli. Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$. *IEEE Trans. Inform. Theory*, 38(2, part 2):533–555, 1992.
[17] J. C. Lagarias and Y. Wang. Haar bases for $L^2(\mathbf{R}^n)$ and algebraic number theory. *J. Number Theory*, 57(1):181–197, 1996.
[18] P. G. Lemarie-Rieusset. Some remarks on orthogonal and bi-orthogonal wavelets. *Mat. Apl. Comput.*, 15(2):125–137, 1996.
[19] P. G. Lemarie-Rieusset and E. Zahrouni. More regular wavelets. *Appl. Comput. Harmon. Anal.*, 5(1):92–105, 1998.
[20] P. Maass. Families of orthogonal two-dimensional wavelets. *SIAM J. Math. Anal.*, 27(5):1454–1481, 1996.
[21] C. A. Micchelli. Interpolatory subdivision schemes and wavelets. *J. Approx. Theory*, 86(1):41–71, 1996.
[22] M. Nielsen. On the construction and frequency localization of finite orthogonal quadrature filters. *J. Approx. Theory*, 108(1):36–52, 2001.
[23] S. D. Riemenschneider and Z. Shen. Multidimensional interpolatory subdivision schemes. *SIAM J. Numer. Anal.*, 34(6):2357–2381, 1997.

Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, DK-9220 Aalborg East, Denmark

E-mail address: mnielsen@math.auc.dk