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BANACH FRAMES FOR MULTIVARIATE α-MODULATION SPACES

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Abstract. The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $\alpha \in [0,1]$, form a family of spaces that include the Besov and modulation spaces as special cases. This paper is concerned with construction of Banach frames for α-modulation spaces in the multivariate setting. The frames constructed are unions of independent Riesz sequences based on tensor products of univariate brushlet functions, which simplifies the analysis of the full frame. We show that the multivariate α-modulation spaces can be completely characterized by the Banach frames constructed, and as an application we consider Jackson-type estimates for m-term nonlinear approximation.

1. Introduction

The α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$, $\alpha \in [0,1]$, are a parameterized family of spaces that include the Besov and modulation spaces as special cases corresponding to $\alpha = 1$ and $\alpha = 0$, respectively.

Besov spaces, see e.g. [22] for the definition, are based on coverings of frequency space $\mathbb{R}^d$ by balls $B(a_n, r_n)$ satisfying $|a_n| \asymp |B(a_n, r_n)|^{1/d}$, that is to say there exist constants $c, C \in (0,\infty)$ such that $c|a_n| \leq |B(a_n, r_n)|^{1/d} \leq C|a_n|$ for all the balls. On the other hand, the modulation spaces introduced by Feichtinger in [6] are based on uniform coverings of the frequency space, i.e., coverings satisfying $|a_n|^0 \asymp |B(a_n, r_n)|^{1/d}$, and it was pointed out by Feichtinger and Gröbner [8, 7] that Besov and modulation spaces are special cases of an abstract construction, the so-called decomposition type Banach spaces $D(Q, B, Y)$. Gröbner [12] used the methods in [8] to define the α-modulation spaces as a family of intermediate spaces. Gröbners idea was to define spaces corresponding to coverings based on the rule $|a_n|^\alpha \asymp |B(a_n, r_n)|^{1/d}$, $0 \leq \alpha \leq 1$. The precise definition of an α-modulation space will be given in Section 3. The coverings giving rise to α-modulation spaces have also been considered (independently) by Päivärinta and Somersalo in [20]. Päivärinta and Somersalo used the partitions to extend the Calderón-Vaillancourt boundedness result for pseudodifferential operators to the local Hardy spaces h_p.

The family of α-modulation spaces arise naturally in several applications. In [1], pseudodifferential operators on α-modulation spaces are studied in the univariate case. It was proved that certain pseudodifferential operators with "excotic" symbols of Hörmander type extends naturally to bounded operators on α-modulation spaces. The proof are based on the brushlet characterization of the α-modulation spaces given in [3]. The mapping properties of pseudodifferential operators on α-modulation spaces are also studied by Holschneider and Nazaret in [19]. These results can be seen as extensions of earlier classical results by Córdoba and Fefferman [5]. Pseudodifferential operators on modulation spaces have also been studied, see e.g. [21, 14].

One successful approach to study function spaces and operators on such spaces is to construct an unconditional basis for the space and use the corresponding norm characterization of the elements in the space to study various operators on the space. One striking example is the study of Calderón-Zygmund operators in smooth wavelet bases, see e.g. [17]. Another family of orthonormal bases for $L_2(\mathbb{R})$ is brushlet bases. Brushlets are the

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image of a local trigonometric basis under the Fourier transform, and such systems were introduced by Laeng [15]. Later Coifman and Meyer [16] used brushlets as a tool for image compression. The present authors proved in [2] that "nice" brushlets form unconditional bases for $L_p(\mathbb{R}^d)$, $1 < p < \infty$. In [3], the freedom to choose the frequency localization of a brushlet system was used to construct (orthonormal) unconditional brushlet bases for the univariate α-modulation spaces. The structure of the real line was used extensively in [3], and we cannot see any straightforward way of extending the bases to the multivariate setting. Fornasier has studied Gabor-type Banach frames for univariate α-modulation spaces in [9]. However, as Fornasier also states in [9], it also seems to be technically difficult to extend the construction to the multivariate case. The purpose of the present paper is to introduce an easy construction of Banach frames based on brushlet systems for multivariate α-modulation spaces. The frames constructed are not orthonormal bases, but each system is "locally" orthonormal (in a certain sense that will be explained in Section 4). One of the main tools we use to construct the brushlet frames is the theory of localized frames introduced by Gröchenig [13]. Localized frames are frames with a Gram matrix having fast decay (usually polynomial or exponential decay) of the entries away from the diagonal. In particular, we use an important recent result by Fornasier and Gröchenig [10] that a so-called self-localized frame has a self-localized dual frame. As an application of the frames, we consider Jackson-type estimates for m-term nonlinear approximation.

The structure of the paper is as follows. In Section 2 we recall the definition of a self-localized frame and we describe some recent results on such frames by Fornasier and Gröchenig [10]. The α-modulation spaces are defined in Section 3, and Section 4 contains the construction of the multivariate brushlet systems that will form Banach frames for the α-modulation spaces. In Section 5 we summarize the results and prove that suitable multivariate brushlet systems form Banach frames for the α-modulation spaces. In Section 6 we consider an application of the Banach frames to approximation theory and derive Jackson estimates for best m-term approximation. Finally, there is an appendix where we prove some technical results related to the partitions of unity used to define the α-modulation spaces.

2. Self-localized frames

In this section we recall the definition of a frame for $L_2(\mathbb{R}^d)$. We also discuss some recent results by Fornasier and Gröchenig [10] on so-called self-localized frames for $L_2(\mathbb{R}^d)$. The results will be used in Section 5 to construct brushlet-type systems that form Banach frames for the α-modulation spaces $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

A countable subset $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\} \subset L_2(\mathbb{R}^d)$ is a frame for $L_2(\mathbb{R}^d)$ if there exist constants $0 < A, B < \infty$ such that

$$A\|f\|^2_{L_2} \leq \sum_{n\in\mathbb{Z}^d} |\langle f, g_n\rangle|^2 \leq B\|f\|^2_{L_2}, \qquad \forall f \in L_2(\mathbb{R}^d).$$

Define the coefficient (analysis) operator $C = C_{\mathcal{G}}$ by $C_{\mathcal{G}} f = (\langle f, g_n\rangle)_{n\in\mathbb{Z}^d}$ and the synthesis operator $D = D_{\mathcal{G}} = C^*_{\mathcal{G}}$ by $Dc = \sum_n c_n g_n$, and let $S = S_{\mathcal{G}}$ be the frame operator $S = DC = C^*C$. It is well-known (see [4]) that S is positive and boundedly invertible, and the set $\widetilde{\mathcal{G}} := S^{-1}\mathcal{G}$ is again a frame for $L_2(\mathbb{R}^d)$, called the *canonical dual frame* to $\mathcal{G}$. We have the reconstruction formula

$$f = SS^{-1}f = \sum_n \langle f, S^{-1}g_n\rangle g_n = \sum_n \langle f, \tilde{g}_n\rangle g_n = S^{-1}Sf = \sum_n \langle f, g_n\rangle \tilde{g}_n.$$

Next we introduce the notion of a self-localized frame. To simplify the definition in [10], we assume that the frame is indexed by $\mathbb{Z}^d$. Let $\langle x\rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.

Definition 2.1. A frame $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$ for $L_2(\mathbb{R}^d)$ is called polynomially s-self-localized, $s > d$, if

$$|\langle g_n, g_m\rangle| \le C\langle n-m\rangle^{-s}, \qquad n, m \in \mathbb{Z}^d.$$

We now state a special case of a result by Fornasier and Gröchenig.

Theorem 2.2 ([10]). *Let $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$ be a frame for $L_2(\mathbb{R}^d)$. If $\mathcal{G}$ is polynomially s-self-localized, for some $s > d$, then $\widetilde{\mathcal{G}}$ is also polynomially s-self-localized.*

We need the notion of a Banach frame for a Banach space.

Definition 2.3. A Banach frame for a separable Banach space B is a sequence $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$ in the dual space B' with an associated sequence space B_d on $\mathbb{Z}^d$ such that the following properties hold.

(1) The coefficient operator $C_{\mathcal{G}}$ is bounded from B into B_d.
(2) Norm equivalence:

$$\|f\|_B \asymp \|(\langle f, g_n\rangle)_{n\in\mathbb{Z}^d}\|_{B_d}, \qquad f \in X.$$

(3) There exists a bounded operator R from B_d onto B, a so-called synthesis or reconstruction operator, such that

$$RC_{\mathcal{G}}f = R(\langle f, g_n\rangle)_{n\in\mathbb{Z}^d} = f.$$

When $B = L_2(\mathbb{R}^d)$ and $B_d = \ell^2(\mathbb{Z}^d)$, Definition 2.3 coincides with the usual definition of a frame for $L_2(\mathbb{R}^d)$.

We now define a class of Banach spaces associated with a Banach frame $\mathcal{G}$. A weight $m : \mathbb{R}^d \to [0,\infty)$ is called s-moderate if $m(x+y) \le \langle x\rangle^s m(y)$ for all $x, y \in \mathbb{R}^d$. Let $\ell^p_m(\mathbb{Z}^d)$ be the weighted ℓ^p space with weight m. Assume that m is s-moderate and $\ell^p_m(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$. We define

$$\mathcal{H}^p_m(\mathcal{G}, \widetilde{\mathcal{G}}) = \left\{ f \in L_2(\mathbb{R}^d) : f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde{g}_n\rangle g_n, \quad (\langle f, \tilde{g}_n\rangle)_{n\in\mathbb{Z}^d} \in \ell^p_m(\mathbb{Z}^d) \right\}.$$

Notice that $\mathcal{H}^p_m(\mathcal{G}, \widetilde{\mathcal{G}}) \subset L_2(\mathbb{R}^d)$ is well-defined since $\ell^p_m(\mathbb{Z}^d) \subseteq \ell^2(\mathbb{Z}^d)$. The following important result was proved in [10].

Theorem 2.4 ([10]). *Suppose $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$ is a polynomially r-self-localized frame for $L_2(\mathbb{R}^d)$, for some $r > d$. Let m be an s-moderate weight, $0 < s < r - d$. Then, for p with $d/(r-s) < p \le \infty$,*

$$\mathcal{H}^p_m(\mathcal{G}, \widetilde{\mathcal{G}}) = \left\{ f \in L_2(\mathbb{R}^d) : f = \sum_{n\in\mathbb{Z}^d} c_n g_n, \quad (c_n)_{n\in\mathbb{Z}^d} \in \ell^p_m(\mathbb{Z}^d) \right\},$$

and $\|f\|_{\mathcal{H}^p_m} \asymp \inf\{\|c\|_{\ell^p_m} : c \in \ell^p_m,\ f = D_{\mathcal{G}}c\}$. Moreover, both $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are Banach frames for $\mathcal{H}^p_m(\mathcal{G}, \widetilde{\mathcal{G}})$.

3. α-Modulation spaces

We now define the α-modulation spaces. The spaces are defined by a parameter α, belonging to the interval $[0,1]$. This parameter determines a segmentation of the frequency domain from which the spaces are built. Thus, we need to define "nice" partitions of the frequency space. Let $B(c, r) \subset \mathbb{R}^d$ denote the ball with center c and radius r.

Definition 3.1. A countable set $\mathcal{Q}$ of subsets $Q \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{Q\in\mathcal{Q}} Q$ and there exists $n_0 < \infty$ such that $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \le n_0$ for all $Q \in \mathcal{Q}$. Let

$$r_Q = \sup\{r \in \mathbb{R} \colon B(c_r, r) \subset Q \text{ for some } c_r \in \mathbb{R}^d\},$$
$$R_Q = \inf\{R \in \mathbb{R} \colon Q \subset B(c_R, R) \text{ for some } c_R \in \mathbb{R}^d\}$$

denote respectively the radius of the inscribed and circumscribed sphere of $Q \in \mathcal{Q}$. An admissible covering is called an α-covering, $0 \le \alpha \le 1$, of $\mathbb{R}^d$ if $|Q| \asymp \langle x\rangle^{\alpha d}$ (uniformly) for all $x \in Q$ and for all $Q \in \mathcal{Q}$, and there exists a constant $K \ge 1$ such that $R_Q/r_Q \le K$ for all $Q \in \mathcal{Q}$.

We also need partitions of unity compatible with the covers from Definition 3.1. We let $\mathcal{F}(f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $f \in L_2(\mathbb{R}^d)$, denote the Fourier transform.

Definition 3.2. Given $p \in (0, \infty]$ and an α-covering $\mathcal{Q}$ of $\mathbb{R}^d$. A corresponding bounded admissible partition of unity of order p (p-BAPU) $\{\psi_Q\}_{Q\in\mathcal{Q}}$ is a family of functions satisfying

- $\operatorname{supp}(\psi_Q) \subset Q$
- $\sum_{Q\in\mathcal{Q}} \psi_Q(\xi) = 1$
- $\sup_Q |Q|^{1/\tilde{p}-1}\|\mathcal{F}^{-1}\psi_Q\|_{L_{\tilde{p}}} < \infty$, $\tilde{p} := \min(1, p)$.

Remark 3.3. It is proved in Lemma A.1 that an α-covering with a corresponding p-BAPU actually exist for every $\alpha \in [0, 1]$ and $p > 0$.

We have the following definition of the α-modulation spaces.

Definition 3.4. Given $0 < p, q \le \infty$, $s \in \mathbb{R}$, and $0 \le \alpha \le 1$, let $\mathcal{Q}$ be an α-covering of $\mathbb{R}^d$ for which there exists a p-BAPU Ψ. Then we define the α-*modulation space*, $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ as the set of distributions $f \in S'(\mathbb{R}^d)$ satisfying

$$\|f\|_{M^{s,\alpha}_{p,q}} := \left(\sum_{Q\in\mathcal{Q}} \langle\xi_Q\rangle^{qs}\big\|\mathcal{F}^{-1}(\psi_I\mathcal{F}f)\big\|^q_{L_p}\right)^{1/q} < \infty, \tag{3.1}$$

with $\{\xi_Q\}_{Q\in\mathcal{Q}}$ a sequence satisfying $\xi_Q \in Q$. For $q = \infty$ we have the usual change of the sum to sup over $Q \in \mathcal{Q}$.

It is easy to see that (3.1) defines a quasi-norm (or a norm if $p, q \ge 1$) on $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ and that two different sequences $\{\xi_Q\}_{Q\in\mathcal{Q}}$ give equivalent norms. Furthermore, Theorem 4.1 below, shows that two different p-BAPU's give equivalent norms too. Thus, the α-modulation space is well defined. Notice that the definition is given for the full range og p and q, extending Gröbners original definition.

It can be proved that $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ is a quasi-Banach space, and that $\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^{s,\alpha}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)'$, see [3]. Moreover, if $p, q < \infty$, $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$.

3.1. **Admissible coverings.**

In this section we discuss a specific construction of an α-covering of $\mathbb{R}^d$. This type of covering was considered in [12] and in [20]. A Proof of Lemma 3.5 below can be found in [12], but since Gröbner's work has never been published, we have included a proof for the sake of completeness. Construction of α-coverings are also considered (from another perspective) in [20].

Notice that the set of balls $\{B(k, \sqrt{d})\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is an admissible 0-covering of $\mathbb{R}^d$. Define for some $\beta \in (-1, \infty)$, the bijection δ_β on $\mathbb{R}^d$ by $\delta_\beta(\xi) := \xi\,|\xi|^\beta$ (with inverse $\delta_{\beta'}$, $\beta' = -\beta/(1 + \beta)$). Since the set $\{B(k, R)\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is admissible for $R \ge \sqrt{d}$, so is $\{\delta_\beta(B(k, R))\}_{k\in\mathbb{Z}^d\setminus\{0\}}$. Moreover, we have the following result.

Lemma 3.5. *Suppose $\beta \ge 0$. Given $R > 0$, there exists an $r > 0$, such that*

$$\delta_\beta(B(z, R)) \subseteq B(\delta_\beta(z), r|z|^\beta), \qquad \textit{for all } z \in \mathbb{R}^d, \textit{ with } |z| \ge 1. \tag{3.2}$$

Likewise, given $r > 0$ there exists an $R > 0$, such that

$$B(\delta_\beta(z), r|z|^\beta) \subseteq \delta_\beta(B(z, R)), \qquad \textit{for all } z \in \mathbb{R}^d. \tag{3.3}$$

Proof. The proof is based on the following observation. For two points $x, z \in \mathbb{R}^d$ and $\beta \in (-1, \infty)$, we have

$$\begin{aligned}
|\delta_\beta(x) - \delta_\beta(z)| &= \Big|x|x|^\beta - z|z|^\beta\Big| \\
&\leq \Big|x|x|^\beta - x|z|^\beta\Big| + \Big|x|z|^\beta - z|z|^\beta\Big| \\
&= |x|\Big||x|^\beta - |z|^\beta\Big| + |z|^\beta|x - z| \\
&= \Big(|\beta||x||\tilde{x}|^{\beta-1} + |z|^\beta\Big)|x - z| \qquad (3.4)
\end{aligned}$$

for some $\tilde{x} \in L(x, z)$, by the mean value theorem.

Given $R > 0$, suppose $x \in B(z, R)$. Then since $|z| \geq 1$, (3.4) yields $|\delta_\beta(x) - \delta_\beta(z)| \leq r|z|^\beta$ for some $r > 0$ depending only on β and R. Now, take any $y \in \delta_\beta(B(z, R))$, i.e., $y = \delta_\beta(x)$ for some $x \in B(z, R)$. Then $|y - \delta_\beta(z)| \leq r|z|^\beta$, which proves (3.2).

We turn to (3.3). Suppose first that $|z| \leq K$ for some $K > r^{1+\beta}$. Then it is easy to verify that there exists a radius $P > 0$ such that $B(\delta_\beta(z), r|z|^\beta) \subset B(0, P)$ for all z. Likewise, there exists a radius R such that $B(0, P) \subset \delta_\beta(B(z, R))$ for all z. This proves (3.3) for $|z| \leq K$.

Suppose now that $|z| > r^{1+\beta}$. Recall that $\delta_\beta^{-1} = \delta_{\beta'}$, where $\beta' := -\beta/(\beta + 1)$. Thus, to show the inclusion (3.3) is equivalent to show that

$$\delta_{\beta'}(B(z, r|z|^{-\beta'})) \subseteq B(\delta_{\beta'}(z), R). \qquad (3.5)$$

Suppose $x \in B(z, r|z|^{-\beta'})$ for some $\beta' > -1$, then $(1 - r|z|^{-(1+\beta')})|z| \leq |x| \leq (1 + r|z|^{-(1+\beta')})|z|$. Since $1 + \beta = (1 + \beta')^{-1}$, (3.4) yields

$$|\delta_{\beta'}(x) - \delta_{\beta'}(z)| \leq R|z|^{\beta'}|z|^{-\beta'} = R$$

for some $R > 0$ depending only on r and β'. Now, take any $y \in \delta_{\beta'}(B(z, r|z|^{-\beta'}))$, i.e., $y = \delta_{\beta'}(x)$ for some $x \in B(z, r|z|^{-\beta'})$. Then, $|y - \delta_{\beta'}(z)| \leq R$, which proves (3.5). $\square$

We can now deduce the following result from Lemma 3.5.

Theorem 3.6. *Given $0 \leq \alpha < 1$, let $\beta = \alpha/(1 - \alpha)$. Then there exists a constant $r_1 > 0$ such that*

$$\{B(k|k|^\beta, r|k|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}} \qquad (3.6)$$

is an α-covering for any $r > r_1$.

Proof. By (3.2) there exists a radius r_1 such that $\mathbb{R}^d \subset \cup_{k \in \mathbb{Z}^d \setminus \{0\}} B(\delta_\beta(k), r|k|^\beta)$ for all $r \geq r_1$. Fix such an r and let $R := R(r)$ be given such that (3.3) holds. Then, since $\{\delta_\beta(B(k, R(r)))\}_{k \in \mathbb{Z}^d \setminus \{0\}}$ is an admissible covering of $\mathbb{R}^d$, so is $\{B(\delta_\beta(k), r|k|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}}$.

It is easy to see that $|B(\delta_\beta(k), r|k|^\beta)| \asymp \langle y \rangle^{d\alpha}$ for all $y \in B(\delta_\beta(k), r|k|^\beta)$ independent of $k \in \mathbb{Z}^d \setminus \{0\}$. Thus $\{B(\delta_\beta(k), r|k|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}}$ is an α-covering for any $r > r_1$. $\square$

Denote by $Q(c, r)$ the cube with center c and side lengths $2r$. We have the following corollary to Theorem 3.6.

Corollary 3.7. *Given $0 \leq \alpha < 1$, let $\beta = \alpha/(1 - \alpha)$. Then there exists a constant $r_1 > 0$ such that*

$$\{Q(k|k|^\beta, r|k|^\beta)\}_{k \in \mathbb{Z}^d \setminus \{0\}}$$

is an α-covering of $\mathbb{R}^d$ for any $r > r_1$.

3.2. **A specific α-covering.** Given $0 \le \alpha < 1$. In Section 4.2 below we will give an equivalent norm for the α-modulation spaces using a fixed α-covering $\mathbb{P}$ defined as follows.

Let r_1 be the constant from Corollary 3.7 and fix $r > r_1$. Then according to Corollary 3.7 the set

$$\mathbb{P} := \mathbb{P}_{\alpha,r} := \{Q(k|k|^{\alpha/(1-\alpha)}, r|k|^{\alpha/(1-\alpha)})\}_{k\in\mathbb{Z}^d\setminus\{0\}} \tag{3.7}$$

is an α-covering. Fix an $\varepsilon \in (0, 1/2)$ such that $r-\varepsilon > r_1$. For $Q = Q(k|k|^{\alpha/(1-\alpha)}, r|k|^{\alpha/(1-\alpha)}) \in \mathbb{P}$ define

$$Q_\varepsilon := Q(k|k|^{\alpha/(1-\alpha)}, (r-\varepsilon)|k|^{\alpha/(1-\alpha)}). \tag{3.8}$$

Then the set $\{Q_\varepsilon\}_{Q\in\mathbb{P}}$, is an α-covering too. In particular $\cup_{Q\in\mathbb{P}} Q_\varepsilon = \mathbb{R}^d$.

4. Localized multivariate brushlet systems

Univariate brushlet bases have proven succesfull in characterizing univariate α-modulation spaces. However, it is still an open question how to construct orthonormal multivariate brushlet bases. In this section we define separable multivariate brushlet systems and study their localization properties. The study concludes in Section 5 where we will see that nice separable brushlet systems in fact constitutes Banach Frames for multivariate α-modulation spaces.

4.1. **Brushlet systems.** Let us first recall the definition of a univariate brushlet. Take a non-negative ramp function $\rho \in C^r(\mathbb{R})$, for some $r \ge 1$, satisfying

$$\rho(\xi) = \begin{cases} 0 & \text{for } \xi \le 0, \\ 1 & \text{for } \xi \ge 1. \end{cases} \tag{4.1}$$

Given $\varepsilon \in (0, 1/2)$ as in Section 3.2, define g by

$$\hat{g}(\xi) := \rho\Big(\frac{\xi}{\varepsilon}\Big)\rho\Big(\frac{1-\xi}{\varepsilon}\Big), \tag{4.2}$$

where $\hat{g}$ denotes the Fourier transform of g.

For an interval $I = [a_I, a'_I)$, we define the *bell function*

$$b_I(\xi) := \hat{g}\big(|I|^{-1}(\xi - a_I)\big) = \rho\Big(\frac{\xi - a_I}{\varepsilon|I|}\Big)\rho\Big(\frac{a'_I - \xi}{\varepsilon|I|}\Big). \tag{4.3}$$

Notice that $\operatorname{supp}(b_I) \subseteq I$ and $b_I(\xi) = 1$ for $\xi \in [a_I + \varepsilon|I|, a'_I - \varepsilon|I|]$. Now for each $n \in \mathbb{N}_0$ we define the univariate brushlet $w_{n,I}$ by

$$\hat{w}_{n,I}(\xi) = \sqrt{\frac{2}{|I|}}\, b_I(\xi)\cos\Big(\pi\big(n+\tfrac{1}{2}\big)\frac{\xi - a_I}{|I|}\Big). \tag{4.4}$$

The brushlets also have an explicit representation in the time domain. Let for notational convenience

$$e_{n,I} := \frac{\pi\big(n+\frac{1}{2}\big)}{|I|}.$$

Then,

$$w_{n,I}(x) = \sqrt{\frac{|I|}{2}}\, e^{ia_I x}\big\{g\big(|I|(x + e_{n,I})\big) + g\big(|I|(x - e_{n,I})\big)\big\}. \tag{4.5}$$

By a straight forward calculation it can be verified that there exists a constant $C < \infty$, such that

$$|g(x)| \le C(1 + |\varepsilon x|)^{-r}, \tag{4.6}$$

with $r \geq 1$ given by the smoothness of the ramp function. Thus a univariate brushlet $w_{n,I}$ essentially consists of two humps at $\pm e_{n,I}$. We call r in (4.6) the decay rate of the brushlet.

Let $Q = \prod_{i=1}^{d} I_i$ be a cube in $\mathbb{R}^d$, and let

$$w_{n,Q} := \bigotimes_{i=1}^{d} w_{n_i,I_i}, \qquad n \in \mathbb{N}_0^d,$$

be the associated *multivariate brushlet*. Notice that $\{w_{n,Q}\}_{n\in\mathbb{N}_0^d}$ is an orthonormal system in $L_2(\mathbb{R}^d)$ for a fixed cube Q. We say that the brushlet $w_{n,Q}$ has decay rate $r > 0$ if each w_{n_i,I_i}, $i = 1, 2, \dots, d$, has decay rate r.

We associate a family of projection operators to the brushlets as follows. These operators will be used to obtain an equivalent norm for the α-modulation spaces in Section 4.2.

Given an interval $I \subset \mathbb{R}$, define the operator $P_I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by

$$\widehat{P_I f}(\xi) := b_I(\xi)\big[b_I(\xi)\hat{f}(\xi) + b_I(2a_I - \xi)\hat{f}(2a_I - \xi) - b_I(2a_I' - \xi)\hat{f}(2a_I' - \xi)\big].$$

It can be verified that P_I is an orthogonal projection, mapping $L_2(\mathbb{R})$ onto $\overline{\operatorname{Span}}\{w_{n,I} : n \in \mathbb{N}_0\}$.

For $Q = \prod_{i=1}^{d} I_i$ a cube in $\mathbb{R}^d$, we define the operator P_Q by the corresponding tensor product. Clearly, P_Q is a projection operator $P_Q : L_2(\mathbb{R}^d) \to \overline{\operatorname{Span}}\{w_{n,Q} : n \in \mathbb{N}_0^d\}$. Moreover, given $f \in L_2(\mathbb{R}^d)$ we have $\operatorname{supp}(\widehat{P_Q f}) \subseteq Q$, and $\widehat{P_Q f}(\xi) = \hat{f}(\xi)$ for all $Q \in \mathbb{P}$ and $\xi \in Q_\varepsilon$.

Finally, notice that,

$$P_Q = S_Q\Big[\bigotimes_{i=1}^{d}(\mathrm{Id_i} + R_{a_{I_i}} - R_{a'_{I_i}})\Big]S_Q, \tag{4.7}$$

where $\widehat{S_Q f} := b_Q \hat{f}$ and $R_a f(x) := e^{i2a} f(-x)$, $x, a \in \mathbb{R}$.

4.2. **Characterization of α-modulation spaces.** Now we show that it is possible to rewrite the $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$-norm using the projection operators P_Q associated with the α-covering $\mathbb{P}$. This leads to a characterization of the α-modulation space norm using the multivariate brushlet system.

The main result is the following.

Theorem 4.1. *Given $0 \leq \alpha < 1$, $0 < p, q \leq \infty$, and $s \in \mathbb{R}$. Let $\mathbb{P}$ be the disjoint α-covering defined in (3.7), and let P_Q, $Q \in \mathbb{P}$, be the associated projection operators generated from a brushlet system with decay rate $r > \max(1, 1/p)$. Then for any $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ we have*

$$\|f\|_{M^{s,\alpha}_{p,q}} \asymp \Big(\sum_{Q\in\mathbb{P}} \langle \xi_Q\rangle^{qs} \|P_Q f\|^q_{L_p}\Big)^{1/q}. \tag{4.8}$$

Proof. Let Ψ be a p-BAPU subordinate to an α-covering $\mathcal{Q}$. Take $f \in M^{s,\alpha}_{p,q}(\mathbb{R}^d)$. Then,

$$P_Q f = \sum_{Q'\in A_Q} P_Q(\mathcal{F}^{-1}(\psi_{Q'}\hat{f})), \qquad Q \in \mathbb{P},$$

in $\mathcal{S}'(\mathbb{R})$, where A_Q is the set of cubes $Q' \in \mathcal{Q}$ with $Q \cap Q' \neq \emptyset$. According to Lemma B.2 in Appendix B,

$$\sup_{Q\in\mathbb{P}} \#A_Q \leq d_A < \infty.$$

Using (4.3) we have that $\|\mathcal{F}^{-1}b_Q\|_{L_p} \leq C_p|Q|^{1-1/p}$, for $0 < p \leq \infty$. Now, by the identity (4.7) and since $\mathcal{F}^{-1}(\psi_{Q'}\hat{f}) \in L_p$, $0 < p \leq \infty$, for any $Q' \in \mathcal{Q}$, Proposition 1.5.1 in [22] implies

$$\begin{aligned}\|P_Q f\|_{L_p} &\leq C|Q|^{1/\tilde{p}-1}\|\mathcal{F}^{-1}b_Q\|_{L_{\tilde{p}}} \sum_{Q'\in A_Q} \|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p} \qquad (\tilde{p} := \min(1,p))\\ &\leq C' \sum_{Q'\in A_Q} \|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\end{aligned}$$

with C' independent of Q. Clearly, $\langle\xi_Q\rangle \asymp \langle\xi_{Q'}\rangle$ for any $\xi_Q \in Q$, $\xi_{Q'} \in Q'$, when $Q' \in A_Q$. Hence

$$\langle\xi_Q\rangle^s\|P_Q f\|_{L_p} \leq C' \sum_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p} \qquad \text{for } 0 < p \leq \infty, \tag{4.9}$$

with C' independent of Q. Suppose $0 < q \leq 1$, then

$$\begin{aligned}\sum_{Q\in\mathbb{P}}\Big(\sum_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\Big)^q &= \sum_{Q\in\mathbb{P}}\Big(\sum_{Q'\in\mathcal{Q}} \mathbf{1}_{A_Q}(Q')\langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\Big)^q\\ &\leq \sum_{Q'\in\mathcal{Q}}\sum_{Q\in\mathbb{P}}\big(\mathbf{1}_{A_Q}(Q')\langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\big)^q,\end{aligned}$$

where $\mathbf{1}_{A_Q}(Q') = 1$ for $Q' \in A_Q$ and 0 for $Q' \in \mathcal{Q}\setminus A_I$. Since $\mathbf{1}_{A_Q}(Q') = \mathbf{1}_{A_{Q'}}(Q)$, for any $Q \in \mathbb{P}$ and $Q' \in \mathcal{Q}$, this gives

$$\sum_{Q\in\mathbb{P}}\Big(\sum_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\Big)^q \leq d_A \sum_{Q'\in\mathbb{P}} \langle\xi_{Q'}\rangle^{qs}\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}^q.$$

Likewise, for $q = \infty$,

$$\begin{aligned}\sup_{Q\in\mathbb{P}}\sum_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p} &\leq d_A \sup_{Q\in\mathbb{P}}\sup_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\\ &= d_A \sup_{Q'\in\mathcal{Q}} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}.\end{aligned}$$

For $1 < q < \infty$, Hölder's inequality with $1 = 1/q + 1/q'$ implies

$$\begin{aligned}&\sum_{Q\in\mathbb{P}}\Big(\sum_{Q'\in A_Q} \langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\Big)^q\\ &\qquad\leq \sum_{Q\in\mathbb{P}}\Big(\sum_{Q'\in\mathcal{Q}}(\mathbf{1}_{A_Q}(Q'))^{q'}\Big)^{q/q'}\Big(\sum_{Q\in\mathbb{P}}\big(\mathbf{1}_{A_Q}(Q')\langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\big)^q\Big)\\ &\qquad\leq d_A^{q-1}\sum_{Q\in\mathbb{P}}\sum_{Q'\in\mathcal{Q}}\mathbf{1}_{A_Q}(Q')\big(\langle\xi_{Q'}\rangle^s\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}\big)^q\\ &\qquad\leq d_A\sum_{Q'\in\mathcal{Q}}\langle\xi_{Q'}\rangle^{qs}\|\mathcal{F}^{-1}(\psi_{Q'}\hat{f})\|_{L_p}^q.\end{aligned}$$

The lower bound in (4.8) now follows by combining the above estimates with the inequality (4.9). The upper bound can be proved in a similar fashion. $\square$

We now have a characterization of the $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$-norm using the Lebesgue norm of the projection operators P_Q. If the associated brushlets have a sufficiently high decay rate, this Lebesgue norm can be given by the size of the brushlet coefficients.

Corollary 4.2. *Suppose $f \in L_2(\mathbb{R}^d)$, and Q is a cube in $\mathbb{R}^d$. Given $p \in (0,\infty]$. If each brushlet $w_{n,Q}$, $n \in \mathbb{N}_0^d$, has decay rate $r > \max(1, 1/p)$, then $P_Q f \in L_p(\mathbb{R}^d)$ if and only if $\{\langle f, w_{n,Q}\rangle\}_{n\in\mathbb{N}_0^d} \in \ell_p$. In fact, if one of these conditions are satisfied we have*

$$\|P_Q f\|_{L_p} \asymp |Q|^{\frac{1}{2}-\frac{1}{p}} \Big(\sum_{n\in\mathbb{N}_0^d} |\langle f, w_{n,Q}\rangle|^p \Big)^{1/p}, \qquad 0 < p < \infty, \tag{4.10}$$

with equivalence independent of Q. When $p = \infty$ the sum in (4.10) *is changed to* sup *over $n \in \mathbb{N}_0^d$.*

Proof. From (4.6) we have that $g \in L_p(\mathbb{R})$. This, together with the representation (4.5), imply that

$$\sup_{x\in\mathbb{R}^d} \sum_{n\in\mathbb{N}_0^d} |w_{n,Q}(x)|^p \le C|Q|^{\frac{p}{2}} \quad \text{and} \quad \sup_{n\in\mathbb{N}_0^d} \|w_{n,Q}\|_{L_p}^p \le C'|Q|^{\frac{p}{2}-1}. \tag{4.11}$$

Suppose $p \le 1$. Since $\hat{w}_{n,Q}$ is compactly supported, we have (see e.g. [22, p. 18])

$$\begin{aligned} \sum_{n\in\mathbb{N}_0^d} |\langle f, w_{n,Q}\rangle|^p &= \sum_{n\in\mathbb{N}_0^d} |\langle P_Q f, w_{n,Q}\rangle|^p \\ &\le C|Q|^{1-p} \sum_{n\in\mathbb{N}_0^d} \int_{\mathbb{R}^d} |P_Q f(x)|^p |w_{n,Q}(x)|^p \, dx \le C'|Q|^{1-\frac{p}{2}} \|P_Q f\|_{L_p}^p. \end{aligned}$$

Likewise,

$$\|P_Q f\|_{L_p}^p \le \sum_{n\in\mathbb{N}_0^d} |\langle f, w_{n,Q}\rangle|^p \|w_{n,Q}\|_{L_p}^p \le C|Q|^{\frac{p}{2}-1} \sum_{n\in\mathbb{N}_0^d} |\langle f, w_{n,Q}\rangle|^p.$$

For $1 < p < \infty$ the lemma follows using the two estimates (4.11) for $p = 1$, together with Hölder's inequality (see e.g. [18, §2.5]). The case $p = \infty$ is left for the reader. $\square$

Let us introduce a new notation for the brushlets $w_{n,Q}$ associated with the α-covering $\mathbb{P}$. Recall that each cube $Q \in \mathbb{P}$ is of the form

$$Q = Q_k = Q(k|k|^{\frac{\alpha}{1-\alpha}}, r|k|^{\frac{\alpha}{1-\alpha}}), \quad k \in \mathbb{Z}^d \setminus \{0\}.$$

For $Q_k \in \mathbb{P}$ we use the short hand notation

$$w_{n,k} := w_{n,Q_k}, \qquad k \in \mathbb{Z}^d \setminus \{0\}.$$

Using Lemma 4.2 we can derive the following result from Theorem 4.1.

Proposition 4.3. *Given $0 < p, q \le \infty$, and $s \in \mathbb{R}$. Let $\mathcal{B} = \{w_{n,k}\}_{k\in\mathbb{Z}^d\setminus\{0\}, n\in\mathbb{N}_0^d}$ be a brushlet system with decay rate $r > \max(1, 1/p)$ associated with the α-covering $\mathbb{P}$, $0 \le \alpha < 1$. Then we have the characterization*

$$\|f\|_{M_{p,q}^{s,\alpha}} \asymp \Big(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \Big(\sum_{n\in\mathbb{N}_0^d} \big(|k|^{\frac{1}{1-\alpha}(s+\frac{\alpha d}{2}-\frac{\alpha d}{p})} |\langle f, w_{n,k}\rangle| \big)^p \Big)^{q/p} \Big)^{1/q}.$$

4.3. **Localized brushlets.** In this section we will see that brushlet systems with sufficiently high decay rate are self-localized.

Lemma 4.4. *Given $n, n' \in \mathbb{N}_0$, and two intervals $I, I' \subset \mathbb{R}$. Suppose the associated univariate brushlets $w_{n,I}$ and $w_{n',I'}$ have decay rate $r \ge 2N$, $N \in \mathbb{N}$. Then we have*

$$|\langle w_{n,I}, w_{n',I'}\rangle| \le \begin{cases} C_N \varepsilon^{-2N} \langle n - n'\rangle^{-2N} & \textit{if } I \cap I' \ne \emptyset, \\ 0 & \textit{otherwise}, \end{cases}$$

with ε given in (4.2).

Proof. By definition,

$$\langle w_{n,I}, w_{n',I'}\rangle = \frac{2}{\sqrt{|I||I'|}}\int_{\mathbb{R}} b_I(\xi) b_{I'}(\xi)\cos(e_{n,I}(\xi-a_I))\cos(e_{n',I'}(\xi-a_{I'}))\,d\xi.$$

Clearly, the inner product equals zero if I and I' are disjoint. Suppose that $I\cap I'\neq\emptyset$. Writing the cosines as complex exponentials, and substituting $\xi\to(\xi-a_I)/|I|$, the inner product is given by four terms of the form

$$2\sqrt{\frac{|I|}{|I'|}}e^{ih(n,n',I,I')}\int_{\mathbb{R}}\hat{g}(\xi)\hat{g}\Big(\frac{|I|}{|I'|}\xi+\frac{a_I-a_{I'}}{|I'|}\Big)e^{\pm i((n+1/2)\pm|I||I'|^{-1}(n'+1/2))\xi}\,d\xi, \tag{4.12}$$

with h a real valued function of the indices. Let us temporarily denote $\gamma:=\pm(n+1/2)\pm|I||I'|^{-1}(n'+1/2)$ for a particular choice of signs. Define $L_\xi:=1-\frac{d^2}{d\xi^2}$. Then by partial integration (4.12) can be rewritten

$$2\sqrt{\frac{|I|}{|I'|}}e^{ih(n,n',I,I')}\langle\gamma\rangle^{-2N}\int_{\mathbb{R}}L_\xi^N\Big[\hat{g}(\xi)\hat{g}\Big(\frac{|I|}{|I'|}\xi+\frac{a_I-a_{I'}}{|I'|}\Big)\Big]e^{ia\xi}\,d\xi,$$

Choose $A>0$ such that $A^{-1}\le|I||I'|^{-1}\le A$. Since $\rho\in C^{2N}(\mathbb{R})$, $\mathrm{supp}(\hat{g})\subseteq[0,1]$, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}}L_\xi^N\Big[\hat{g}(\xi)\hat{g}\Big(\frac{|I|}{|I'|}\xi+\frac{a_I-a_{I'}}{|I'|}\Big)\Big]e^{ia\xi}\,d\xi\Big| &\le\sup_{\xi\in\mathbb{R}}\Big|L_\xi^N\Big[\hat{g}(\xi)\hat{g}\Big(\frac{|I|}{|I'|}\xi+\frac{a_I-a_{I'}}{|I'|}\Big)\Big]\Big|\\ &\le C_N(A/\varepsilon)^{2N},\end{aligned}$$

and thus

$$\begin{aligned}|\langle w_{n,I},w_{n',I'}\rangle| &\le C_N(A/\varepsilon)^{2N}\langle(n+1/2)-|I||I'|^{-1}(n'+1/2)|\rangle^{-2N}\\ &\le C_N'A^{4N}\varepsilon^{-2N}\langle n-n'\rangle^{-2N}.\end{aligned}$$

□

For the multivariate brushlets we obtain the following corollary.

Corollary 4.5. *Given $n,n'\in\mathbb{N}_0^d$, and two cubes $Q,Q'\subset\mathbb{R}^d$ with $A^{-1}\le|Q||Q'|^{-1}\le A$. Suppose the associated brushlets $w_{n,Q}$ and $w_{n',Q'}$ have decay rate $r\ge 2N$, $N\in\mathbb{N}$. Then we have*

$$|\langle w_{n,Q},w_{n',Q'}\rangle|\le\begin{cases}C\langle n-n'\rangle^{-2N} & \text{if } Q\cap Q'\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

with a constant depending only on N, ε and A.

Proof. The result follows directly from Lemma 4.4 since $\prod_{i=1}^d\langle x_i\rangle\ge\langle x\rangle$ for any $x\in\mathbb{R}^d$. □

Remark 4.6. In the following section we will show that a collection of brushlet functions $\mathcal{B}=\{w_{n,k}\}_{n\in\mathbb{N}_0^d,k\in\mathbb{Z}^d\setminus\{0\}}$ with enough decay form a polynomially localized frame for $L_2(\mathbb{R}^d)$. Since $\#\{Q'\in\mathbb{P}:Q\cap Q'\neq\emptyset\}\le n_0$, $Q\in\mathbb{P}$, for some uniform finite constant n_0, we only have to prove localization w.r.t. the n-index as given in Corollary 4.5.

5. Banach frames for α-modulation spaces

We now have the tools needed to show that a nice brushlet system constitute a Banach frame for the α-modulation spaces. We have the following result which shows that the family of brushlet functions form a polynomially localized frame for $L_2(\mathbb{R}^d)$, see also Remark 4.6.

Theorem 5.1. *Let $\mathcal{B} = \{w_{n,k}\}_{n\in\mathbb{N}_0^d, k\in\mathbb{Z}^d\setminus\{0\}}$ be a collection of brushlet functions in $L_2(\mathbb{R}^d)$, with decay rate $2N$, $N \in \mathbb{N}$, and α-covering $\mathbb{P}$ given by (3.7), $0 \le \alpha < 1$. Then $\mathcal{B}$ is a polynomially $2N$-self-localized frame for $L_2(\mathbb{R}^d)$.*

Proof. The covering $\mathbb{P}$ has finite height, i.e., $\sum_{Q_k\in\mathbb{Z}^d\setminus\{0\}} \chi_{Q_k}(\xi) \le C < \infty$ for some fixed constant C. We can therefore write $\mathbb{P} = \cup_{j=1}^N \mathbb{P}_j$, with $\{w_{n,k}\}_{n\in\mathbb{N}_0, Q_k\in\mathbb{P}_j}$ an orthonormal system for $j = 1, 2, \ldots, N$. The brushlet system $\mathcal{B}$ can thus be written as a union of orthonormal sequences and $\mathcal{B}$ therefore has a finite upper frame bound. Next we turn to the lower frame bound. Let $f \in L_2(\mathbb{R}^d)$. Consider the orthogonal projection

$$P_{Q_k}\hat{f} = \sum_{n\in\mathbb{N}_0^d} \langle \hat{f}, \hat{w}_{n,k}\rangle \hat{w}_{n,k}, \qquad Q_k \in \mathbb{P},$$

which (by construction) satisfies $P_Q|_{Q_\varepsilon} = Id_{L_2(Q_\varepsilon)}$, where Q_ε is given by (3.8). Therefore, using $\cup_{Q\in\mathbb{P}} Q_\varepsilon = \mathbb{R}^d$,

$$\begin{aligned}\|f\|_{L_2}^2 = \|\hat{f}\|_{L_2}^2 &\le \int_{\mathbb{R}^d} \sum_{Q\in\mathbb{P}} \chi_{Q_\varepsilon}(\xi)|\hat{f}(\xi)|^2 d\xi = \sum_{Q\in\mathbb{P}} \int_{\mathbb{R}^d} \chi_{Q_\varepsilon}(\xi)|\hat{f}(\xi)|^2 d\xi \\ &\le \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \|P_{Q_k}\hat{f}\|_{L_2}^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \sum_{n\in\mathbb{N}_0^d} |\langle \hat{f}, \hat{w}_{n,k}\rangle|^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \sum_{n\in\mathbb{N}_0^d} |\langle f, w_{n,k}\rangle|^2,\end{aligned}$$

which proves the existence of a lower frame bound ≥ 1. So $\mathcal{B}$ is a frame for $L_2(\mathbb{R}^d)$. Corollary 4.5 shows that $\mathcal{B}$ is polynomially r-self-localized. □

Remark 5.2. We do not have a closed formula for the canonical dual frame to $\mathcal{B} = \{w_{n,k}\}_{n\in\mathbb{N}_0^d, k\in\mathbb{Z}^d\setminus\{0\}}$ since $\mathcal{B}$ is not tight. However, we can still expand an arbitrary L_2 function in the frame using the iterative frame algorithm, see e.g. [4, Lemma 1.2.3]. In fact, there is an efficient method to implement brushlet expansions using the FFT in the Fourier domain, see [16]. One can use this fast expansion algorithm to implement the frame algorithm.

Proposition 5.3. *Let $\mathcal{B} = \{w_{n,k}\}_{n\in\mathbb{N}_0^d, k\in\mathbb{Z}^d\setminus\{0\}}$ be a collection of brushlet functions in $L_2(\mathbb{R}^d)$, with decay rate $r > d$, and α-covering $\mathbb{P}$ given by (3.7), $0 \le \alpha < 1$. Let $\tilde{\mathcal{B}}$ be its canonical dual. Then $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are Banach frames for the α-modulation spaces $M_{p,p}^{s,\alpha}(\mathbb{R}^d)$, for $\frac{1}{r} < p \le 2$ and $s \ge \alpha(\frac{d}{p} - \frac{d}{2})$. Moreover, we have the identity*

$$\mathcal{H}_{m_{\tilde{s}}}^p(\mathcal{B}, \widetilde{\mathcal{B}}) = M_{p,p}^{s,\alpha}(\mathbb{R}^d), \qquad \frac{1}{r} < p \le 2, \quad s \ge \alpha\left(\frac{d}{p} - \frac{d}{2}\right),$$

where

$$m_{\tilde{s}}(n,k) = |k|^{\frac{1}{1-\alpha}\tilde{s}} \quad \textit{and} \quad \tilde{s} = s + \alpha\left(\frac{d}{2} - \frac{d}{p}\right).$$

Proof. Notice that $\ell_{m_{\tilde{s}}}^p(\mathbb{N}^d \times \mathbb{Z}^d \setminus \{0\}) \subseteq \ell^2(\mathbb{N}^d \times \mathbb{Z}^d \setminus \{0\})$, and by Theorem 2.4 and Proposition 4.3,

$$\mathcal{H}_{m_{\tilde{s}}}^p(\mathcal{B}, \widetilde{\mathcal{B}}) \supset M_{p,p}^{s,\alpha}(\mathbb{R}^d).$$

Conversely, suppose $f \in \mathcal{H}_{m_{\tilde{s}}}^p(\mathcal{B}, \widetilde{\mathcal{B}})$. Then by Corollary 4.5 and [10, Lemma 2.1] we have $\big(\langle f, w_{n,k}\rangle\big)_{n,k} \in \ell_{m_{\tilde{s}}}^p(\mathbb{N}^d \times \mathbb{Z}^d \setminus \{0\})$, and thus $f \in M_{p,p}^{s,\alpha}(\mathbb{R}^d)$ by Proposition 4.3. The result now follows from Theorem 2.4. □

6. Nonlinear approximation

In this section we consider an application of the Banach frames for the α-modulation spaces to nonlinear approximation. We are mainly interested in Jackson estimates for best m-term approximation, and using the stabillity of the frames, we can apply a general theory introduced in [11].

We let $\mathcal{B} = \{w_{n,k}\}_{k\in\mathbb{Z}^d\setminus\{0\},n\in\mathbb{N}_0^d}$ be a brushlet system with decay rate $r > 0$, and associated with an α-covering $\mathbb{P}$ for some $0 < \alpha \leq 1$. Consider the normalized functions

$$\tilde{w}_{n,k} = \frac{w_{n,k}}{\|w_{n,k}\|_{M^{s,\alpha}_{p,p}(\mathbb{R}^d)}}, \qquad k \in \mathbb{Z}^d \setminus \{0\}, n \in \mathbb{N}_0^d.$$

Notice that for any finite brushlet expansion $f = \sum_{n,k} c_{n,k}\tilde{w}_{n,k}$ we have,

$$\|f\|_{M^{s,\alpha}_{p,p}(\mathbb{R}^d)} \leq C\Big(\sum_{k\in\mathbb{Z}^d\setminus\{0\},n\in\mathbb{N}_0^d} |c_{n,k}|^p\Big)^{1/p}, \qquad \frac{1}{r} < p,$$

by Proposition 4.3, i.e., $\{\tilde{w}_{n,k}\}_{n,k}$ is ℓ_p-hilbertian (as defined in [11]).

Let us introduce some notation that will be needed to explore nonlinear approximation with brushlet bases. Let $\mathcal{D} = \{g_k\}_{k\in\mathbb{N}}$ be a dictionary in a Banach space X. We consider the collection of all possible m-term expansions with elements from $\mathcal{D}$:

$$\Sigma_m(\mathcal{D}) := \Big\{\sum_{i\in\Lambda} c_i g_i \;\Big|\; c_i \in \mathbb{C}, \#\Lambda \leq m\Big\}.$$

The error of the best m-term approximation to an element $f \in X$ is then

$$\sigma_m(f,\mathcal{D})_X := \inf_{f_m\in\Sigma_m(\mathcal{D})} \|f - f_m\|_X.$$

The corresponding approximation spaces are defined as follows.

Definition 6.1 (Approximation spaces)**.** The approximation space $\mathcal{A}^\gamma_q(X,\mathcal{D})$ is defined by

$$|f|_{\mathcal{A}^\gamma_q(X,\mathcal{D})} := \left(\sum_{m=1}^{\infty} \big(m^\gamma \sigma_m(f,\mathcal{D})_X\big)^q \frac{1}{m}\right)^{1/q} < \infty,$$

and (quasi)normed by $\|f\|_{\mathcal{A}^\gamma_q(X,\mathcal{D})} = \|f\|_X + |f|_{\mathcal{A}^\gamma_q(X,\mathcal{D})}$ for $0 < q, \gamma < \infty$, with the ℓ_q norm replaced by the sup-norm, when $q = \infty$.

We can now use the stability of the brushlet frames in the α-modulation spaces and apply [11, Theorem 6] to obtain a Jackson inequality for the brushlet frames.

Proposition 6.2. *Let $\{w_{n,k}\}_{k\in\mathbb{Z}^d\setminus\{0\},n\in\mathbb{N}_0^d}$ be a brushlet system with decay rate $r > d$, associated with a disjoint α-covering $\mathbb{P}$ for some $0 \leq \alpha \leq 1$, and let*

$$\mathcal{D} = \big\{w_{n,k}/\|w_{n,k}\|_{M^{s,\alpha}_{p,p}(\mathbb{R}^d)}\big\}_{k\in\mathbb{Z}^d\setminus\{0\},n\in\mathbb{N}_0^d}$$

for $\frac{1}{r} < p \leq 2$ and $s \geq \alpha(\frac{d}{p} - \frac{d}{2})$. Then

$$M^{\beta,\alpha}_{\tau,\tau}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^\gamma_\tau\big(M^{s,\alpha}_{p,p}(\mathbb{R}^d),\mathcal{D}\big) \qquad \textit{for}\ \gamma = \frac{1}{\tau} - \frac{1}{p} = \frac{\beta - s}{\alpha d} > 0,$$

with equivalent norms.

Proof. The Proposition follows from [11, Theorem 6] since $\{\tilde{w}_{n,k}\}_{n,k}$ is ℓ_p-hilbertian, and one can verify (using the notation of [11]) that $\mathcal{K}^\tau_\tau(M^{s,\alpha}_{p,p}(\mathbb{R}^d),\mathcal{D}) = M^{\beta,\alpha}_{\tau,\tau}(\mathbb{R}^d)$ with $\beta = s + \alpha d\big(\frac{1}{\tau} - \frac{1}{p}\big)$. □

Appendix A. Bounded admissible partitions of unity and their properties

The α-modulation spaces are defined using a bounded admissible partition of unity, but the spaces are actually independent of the specific choice. We have the following construction.

Proposition A.1. *For $\alpha \in [0,1)$, $0 < p \leq \infty$, there exists an α-covering of $\mathbb{R}^d$ with a corresponding p-BAPU $\{\psi_k\}_{k\in\mathbb{Z}^d\setminus\{0\}} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying*

$$|\partial^\beta \psi_k(\xi)| \leq C_\beta \langle\xi\rangle^{-|\beta|\alpha},$$

for every multi-index β and $k \in \mathbb{Z}^d \setminus \{0\}$.

Proof. For $r > 0$, and $k \in \mathbb{Z}^d \setminus \{0\}$ we define the ball

$$B_k^r := \{\xi \in \mathbb{R}^d : \big|\, \xi - |k|^{\frac{\alpha}{1-\alpha}} k \,\big| < r|k|^{\frac{\alpha}{1-\alpha}}\}.$$

By Lemma 3.5, there exists $r_1 > 0$ such that $\{B_k^{r_1}\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is an α-covering of $\mathbb{R}^d$. There also exists $0 < r_2 < r_1$, such that $\{B_k^{r_2}\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ is pairwise disjoint.

Fix $r > r_1$. We take $\Phi \in C^\infty(\mathbb{R}^d)$ satisfying $\inf_{\xi\in B(0,r_1)} |\Phi(\xi)| := c > 0$ and $\mathrm{supp}(\Phi) \subset B(0,r)$. Let

$$g_k(\xi) := \Phi(|c_k|^{-\alpha}(\xi - c_k)), \qquad k \in \mathbb{Z}^d \setminus \{0\},$$

where $c_k := |k|^{\frac{\alpha}{1-\alpha}} k$. Clearly, $g_k \in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(g_k) \subset B_k^r$. In fact, $\{\mathrm{supp}(g_k)\}_k$ is an α-covering of $\mathbb{R}^d$. The covering is admissible (see Lemma 3.5) since $\{B_k^{r_2}\}_{k\in\mathbb{Z}^d\setminus\{0\}}$, with $B_k^{r_2} \subset \mathrm{supp}(g_k)$, is pairwise disjoint. It is easy to see that the partition has "finite height", i.e., $\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \chi_{\mathrm{supp}(g_k)}(\xi) \leq n_1$ for some uniform constant n_1.

Notice that

$$|\partial^\beta g_k(\xi)| = |c_k|^{-\alpha|\beta|} |(\partial^\beta \Phi)(|c_k|^{-\alpha}(\xi - c_k))| \leq C_\beta |c_k|^{-\alpha|\beta|},$$

and since $|c_k| \geq 1$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, we have

$$|\partial^\beta g_k(\xi)| \leq C'_\beta \langle c_k\rangle^{-\alpha|\beta|} \asymp \langle\xi\rangle^{-\alpha|\beta|} \qquad \text{for all } \xi \in B_k^r.$$

Since we want a p-BAPU, we consider the sum $g(\xi) := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} g_k(\xi)$. Now, $\{\mathrm{supp}(g_k)\}_k$ has finite height, so g is well-defined, and the finite overlap ensures that $|\partial^\beta g(\xi)| \leq C'_\beta \langle\xi\rangle^{-|\beta|\alpha}$. Recall that $g_k(\xi) \geq c$ for all $\xi \in B_k^{r_1}$, and since $\{B_k^{r_1}\}_{k\in\mathbb{Z}^d\setminus\{0\}}$ covers $\mathbb{R}^d$, we have $g(\xi) \geq c$. Thus, we can define

$$\psi_n(\xi) := \frac{g_n(\xi)}{\sum_{k\in\mathbb{Z}^d\setminus\{0\}} g_k(\xi)}.$$

It is straightforward to show that $|\partial^\beta \psi_k(\xi)| \leq C_\beta \langle\xi\rangle^{-|\beta|\alpha}$. In order to conclude, we need to verify that $\sup_Q |Q|^{1/\tilde{p}-1} \|\mathcal{F}^{-1}\psi_k\|_{L_{\tilde{p}}} < \infty$, where $\tilde{p} = \min\{1,p\}$. Let

$$\widetilde{\psi}_k(\xi) = \psi_k(|c_k|^\alpha \xi + c_k) = \frac{\Phi(\xi)}{\sum_{k'} \Phi(|c_k|^\alpha |c_{k'}|^{-\alpha}(\xi - c_{k'}) + c_k)}.$$

Notice that for every $\beta \in \mathbb{N}^d$ there exists a constant C_β independent of $k \in \mathbb{Z}^d \setminus \{0\}$ such that

$$|\partial_\xi^\beta \widetilde{\psi}_k(\xi)| \leq C_\beta \chi_{B(0,r)}(\xi). \tag{A.1}$$

By a simple substitution in each of the following integrals, we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}\psi_k\|_{L_{\tilde{p}}}^{\tilde{p}} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\psi_k(\xi)e^{ix\cdot\xi}\,d\xi\right|^{\tilde{p}}dx\\ &= |c_k|^{\alpha d(\tilde{p}-1)}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\widetilde{\psi}_k(\xi)e^{ix\cdot\xi}\,d\xi\right|^{\tilde{p}}dx, \qquad \text{and since } |c_k|^{d\alpha}\asymp|Q|,\\ &\le |Q|^{\tilde{p}-1}C_d\Big(\sum_{|\beta|\le\lceil(d+1)/\tilde{p}\rceil}\|\partial^\beta\widetilde{\psi}_k\|_{L_1}\Big)^{\tilde{p}}\int_{\mathbb{R}^d}\langle x\rangle^{-d-1}dx \le C_d'|Q|^{\tilde{p}-1},\end{aligned}$$

where we have used integration by parts and (A.1) for the last estimate. We conclude that $\{\psi_k\}_k$ is a p-BAPU corresponding to the α-covering $\{\mathrm{supp}(g_k)\}_k$. □

Appendix B. Some properties of α-coverings

Let $s_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ be the volume of the unit ball in $\mathbb{R}^d$. Given an α-covering $\mathcal{Q}$, let r_Q and R_Q denote respectively the radius of the inscribed and circumscribed sphere of $Q \in \mathcal{Q}$. Let $K \ge 1$ be such that $R_Q/r_Q \le K$ for all $Q \in \mathcal{Q}$. Notice that for $Q \in \mathcal{Q}$ we have $s_d \cdot r_Q^d \le |Q| \le s_d \cdot R_Q^d$, so

$$s_d \le \frac{|Q|}{r_Q^d} = \frac{R_Q^d}{r_Q^d}\cdot\frac{|Q|}{R_Q^d} \le K^d \cdot s_d,$$

and consequently $|Q| \asymp R_Q^d \asymp r_Q^d$ independent of Q.

Given two α-coverings $\mathcal{Q}$ and $\mathcal{Q}'$, suppose $Q \in \mathcal{Q}$ and $Q' \in \mathcal{Q}'$ have nonempty intersection. Then from the observation above, we have $R_Q \asymp R_{Q'}$. Let d_Q and $d_{Q'}$ denote the center of the circumscribed sphere of Q and Q' respectively, and let c_Q be the center of the inscribed sphere of Q. Then there exists a constant $\kappa > 2K$ such that

$$Q' \subseteq \overline{B(d_{Q'}, R_{Q'})} \subset B(d_Q, \tfrac{\kappa}{2K}R_Q) \subset B(c_Q, \kappa r_Q). \tag{B.1}$$

Lemma B.1. *Given an α-covering $\mathcal{Q}$, there exist $n_0 < \infty$ subsets $\mathcal{Q}_i \subseteq \mathcal{Q}$, $i = 1, 2, \ldots, n_0$, such that $\mathcal{Q} = \cup_{i=1}^{n_0}\mathcal{Q}_i$ and the elements of $\mathcal{Q}_i$ are pairwise disjoint.*

The proof is given in Lemma c.1.1 and Lemma c.8.3 in [12], but will be given here for completeness.

Proof. Given $Q \in \mathcal{Q}$ define $Q^* := \{Q' \in \mathcal{Q} \colon Q \cap Q' \ne \emptyset\}$. By (B.1) we have $\cup_{Q'\in Q^*}Q' \subset B(c_Q, \kappa r_Q)$. Since $\mathcal{Q}$ has finite height, there exists a constant n_2 such that

$$n_2 C^d|Q| \ge n_2 s_d \kappa^d r_Q^d > \sum_{Q'\in Q^*}|Q'| \ge c(\#Q^*)|Q|,$$

i.e., $\#Q^*$ is bounded by a constant n_0 independent of Q.

Let $\mathcal{Q}_1 \subseteq \mathcal{Q}$ be a maximal set of pairwise disjoint elements, and let inductively $\mathcal{Q}_i \subset \mathcal{Q} \setminus \cup_{k=1}^{i-1}\mathcal{Q}_k$, $i = 2, 3, \ldots$, be a maximal set of pairwise elements. Suppose $Q \in \mathcal{Q}_{n_0+1}$. Then for each $k = 1, 2, \ldots, n_0$ there exists a $Q_k \in \mathcal{Q}_k$ such that $Q_k \cap Q \ne \emptyset$. But this is a contradiction to the fact that $\#Q^* \le n_0$. □

Lemma B.2. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two α-coverings. For each $Q \in \mathcal{Q}$ let*

$$A_Q = \{Q' \in \mathcal{Q}' \colon Q' \cap Q \ne \emptyset\}.$$

Then there exists a constant d_A such that $\#A_Q \le d_A$ independent of Q.

Proof. Recall that there exists a constant δ such that $|Q| \le \delta|Q'|$ for all $Q' \in A_Q$, independent of $Q \in \mathcal{Q}$. According to (B.1) and Lemma B.1 we have

$$|Q| \ge s_d r_Q^d \ge \frac{\delta s_d}{\kappa^d n_0}(\#A_Q)|Q|,$$

i.e., $\#A_Q \leq \frac{\kappa^d n_0}{\delta s_d}$. □

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