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QUASI-GREEDY SYSTEMS OF INTEGER TRANSLATES

MORTEN NIELSEN† AND HRVOJE ŠIKIĆ*

ABSTRACT. We consider quasi-greedy systems of integer translates in a finitely generated shift invariant subspace of $L_2(\mathbb{R}^d)$, that is systems for which the thresholding approximation procedure is well behaved. We prove that every quasi-greedy system of integer translates is also a Riesz basis for its closed linear span. The result shows that there are no *conditional* quasi-greedy basis of integer translates in a finitely generated shift invariant space.

1. INTRODUCTION

We are interested in finitely generated shift-invariant (FSI) subspaces of $L_2(\mathbb{R}^d)$. By this we mean a subspace $S \subset L_2(\mathbb{R}^d)$ for which there exists a finite family $\Phi = \{\varphi_j\}_{j\in E}$ of $L_2(\mathbb{R}^d)$-functions such that

$$S = \overline{\text{span}\{\varphi(\cdot - k) : \varphi \in \Phi, k \in \mathbb{Z}^d\}}.$$

For many applications it is useful to have a stable generating set for S. Here the word "stable" covers a broad spectrum from weak notions such as (local) linear indepence all the way to an unconditional basis. Given the structure of S, it is natural to consider stable generating sets of integer translates. That is, a system of the form

$$\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}, \tag{1.1}$$

where $\Psi \subset L_2(\mathbb{R}^d)$ is a finite subset. Often we take $\Psi = \Phi$, but Ψ may be different from Φ, and the two sets need not have the same cardinality.

In this paper, we study the special case where S has a Schauder basis (1.1) having the additional property that decreasing rearrangements are norm convergent. Put another way, we assume that the approximants, obtained by thresholding an expansion in the system (1.1), converge. Thresholding is a very natural way to build approximants, and for Riesz bases it is known to be (up to a constant) the best possible way to build m-terms approximants.

Systems for which decreasing rearrangements are norm convergent are known as quasi-greedy bases. Every unconditional basis is also a quasi-greedy basis, but it is well-known that conditional quasi-greedy bases exist, even in an infinite dimensional Hilbert space, see [13] and [18]. However, our main result shows that there are no conditional quasi-greedy bases of integer translates. Put another way, the structure of

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the system (1.1) implies any such quasi-greedy basis is automatically an *unconditional* Riesz basis for S. We thereby obtain a characterization of all the possible quasi-greedy basis of the form (1.1). The result also shows that for every quasi-greedy basis of integer translates, thresholding is the optimal way to build m-term approximants.

Problems on translates of functions have a long history, let us mention classical results by Kolmogoroff [12] and Helson [10]. More recently, FSI subspaces have been used in several applications. Wavelets and other multiscale methods are based on FSI subspaces [3, 4, 11], and FSI subspaces play an important role in multivariate approximation theory such as spline approximation [7] and approximation with radial basis functions [9, 15]. The fundamental structure of FSI spaces has been studied in a number of papers, see for example [1, 2, 5, 6, 16]. The problem of characterizing when (1.1) forms a Schauder basis for S has been considered recently by the present authors. In the univariate case with one generator, a complete characterization is given in [14].

2. Notation and main result

Let us begin by introducing some notation and recalling some necessary results. First we consider the precise definition of a quasi-greedy system in a Hilbert space $\mathcal{H}$. A biorthogonal system is a family $(x_n, x_n^*)_{n\in\mathbb{N}} \subset \mathcal{H}\times\mathcal{H}$ such that $\langle x_n, x_m^*\rangle = \delta_{n,m}$. We fix a biorthogonal system $(x_n, x_n^*)_{n\in\mathbb{N}}$ with $\operatorname{span}_n(x_n)$ dense in $\mathcal{H}$. We assume that the system is quasi-normalized, i.e., $\inf_n \|x_n\|_{\mathcal{H}} > 0$ and $\sup_n \|x_n^*\|_{\mathcal{H}^*} < \infty$. For each $x \in \mathcal{H}$ and $m \in \mathbb{N}$, we define

$$\mathcal{G}_m(x) = \sum_{n\in A} x_n^*(x) x_n,$$

where A is a set of cardinality m satisfying $|x_n^*(x)| \geq |x_k^*(x)|$ whenever $n \in A$ and $k \notin A$. Whenever A is not uniquely defined, we arbitrarily pick any such set. The definition of $\mathcal{G}_m$ leads directly to the definition of a quasi-greedy system, see [13].

Definition 2.1. A quasi-normalized biorthogonal system $(x_n, x_n^*)_{n\in\mathbb{N}} \subset \mathcal{H}\times\mathcal{H}$, with $\operatorname{span}_n(x_n)$ dense in $\mathcal{H}$, is called a quasi-greedy system if there exists a constant Q such that

$$\|\mathcal{G}_m(x)\|_{\mathcal{H}} \leq Q\|x\|_{\mathcal{H}}, \qquad \forall x \in \mathcal{H}. \tag{2.1}$$

If the system is also a Schauder basis for $\mathcal{H}$, we will use the term quasi-greedy basis.

Remark 2.2. In was proved by Wojtaszczyk in [18] that a system is quasi-greedy if and only if for each $x \in \mathcal{H}$, the sequence $\mathcal{G}_m(x)$ converges to x in norm.

It turns out that a quasi-greedy basis in a Hilbert space is "very close" to a Riesz basis. Let us now state the Theorem that justifies this claim in its general form, in order to compare it to our main result on integer translates stated below in Theorem 2.4. Let use introduce some needed additional notation.

For a sequence $\{a_n\}_{n=1}^\infty$ we denote by $\{a_n^*\}$ a non-increasing rearrangement of the sequence $\{|a_n|\}$. Then we define the Lorentz norms

$$\|\{a_n\}\|_{2,\infty} := \sup_n n^{1/2} a_n^* \quad \text{and} \quad \|\{a_n\}\|_{2,1} := \sum_{n=1}^\infty n^{-1/2} a_n^*.$$

The following general result is proved in [18].

Theorem 2.3 ([18]). *Let $\mathcal{B} = \{b_n\}_{n\in\mathbb{N}}$ be a quasi-greedy basis in a Hilbert space $\mathcal{H}$. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for any coefficients $\{a_n\}$*

$$c_1\|\{a_n\}\|_{2,\infty} \le \Big\| \sum_{n\in\mathbb{N}} a_n b_n \Big\|_{\mathcal{H}} \le c_2\|\{a_n\}\|_{2,1}.$$

It is known that *conditional* quasi-greedy bases exist in an infinite dimensional Hilbert space, see [13] and [18]. Therefore, in the general case, we cannot expect an improvement of Theorem 2.3 that would give a complete characterization of the Hilbert space. However, for systems of integer translates Theorem 2.3 can be strengthened. Let us state our main result that there are no conditional quasi-greedy systems of integer translates. A quasi-greedy system of integer translates is also a Riesz-basis for its closed linear span.

Theorem 2.4. *Suppose that $\mathcal{Q} = \{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}$ is a quasi-greedy system in a FSI space S. Then $\mathcal{Q}$ is a Riesz-basis for S, i.e., there exist constants $0 < c \le C < \infty$ such that*

$$c_1\|\{c_\psi(k)\}\|_{\ell_2} \le \Bigg\| \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}^d} c_\psi(k)\psi(\cdot - k) \Bigg\|_{L_2(\mathbb{R}^d)} \le C\|\{c_\psi(k)\}\|_{\ell_2}, \tag{2.2}$$

for every finite sequence $\{c_\psi(k)\}$.

It may not be evident at first, but Theorem 2.4 is essentially a result on properties of trigonometric polynomials in a certain vector-valued space. The reason for this is that the Fourier transform translates problems on FSI spaces to problems on trigonometric polynomials. The proof of Theorem 2.4 is given in Section 3.

3. Quasi-greedy shift invariant systems

This section contains the proof of Theorem 2.4. To prove the result, we use the Fourier transform to study expansions in a FSI space with a generating set Ψ with cardinality $\#\Psi := N < \infty$. For notational convenience, we assume that some ordering has been imposed on Ψ. We define the Fourier transform by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{ix\cdot\xi}\,dx, \qquad f \in L_2(\mathbb{R}^d), \tag{3.1}$$

and we let $\mathbb{T}^d = [-\pi,\pi)^d$ denote the fundamental domain for Fourier series. Following [6], we introduce the so-called bracket product given by

$$[f,g] : \mathbb{T}^d \to \mathbb{C} : x \to \sum_{k\in\mathbb{Z}^d} f(x+2\pi k)\overline{g(x+2\pi k)},$$

for $f,g \in L_2(\mathbb{R}^d)$. With this setup, we have the fundamental identity

$$(2\pi)^d\langle f,g\rangle_{L_2(\mathbb{R}^d)} = \langle \hat{f},\hat{g}\rangle_{L_2(\mathbb{R}^d)} = \int_{\mathbb{T}^d}[\hat{f},\hat{g}]\,d\xi, \qquad f,g, \in L_2(\mathbb{R}^d). \tag{3.2}$$

Let us now consider an expansion

$$f = \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}^d} c_\psi(k)\psi(x-k),$$

relative to the system generated by Ψ. An application of the Fourier transform yields

$$\hat{f}(\xi) = \sum_{\psi\in\Psi}\Bigg(\sum_{k\in\mathbb{Z}^d} c_\psi(k)e^{ik\cdot\xi}\Bigg)\hat{\psi} := \sum_{\psi\in\Psi}\tau_\psi(\xi)\hat{\psi}.$$

We can now calculate the norm of $\hat{f}$ using the bracket product and (3.2). We form the vector $\tau = [\tau_\psi]_{\psi\in\Psi}$, and we let τ^H denote the Hermitian transpose of the vector τ. We obtain

$$\|\hat{f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{T}^d}[\hat{f},\hat{f}] = \int_{\mathbb{T}^d}\tau(\xi)^H G(\xi)\tau(\xi)\,d\xi, \tag{3.3}$$

where $G := G(\Psi)$ is the Hermitian positive semi-definite $N\times N$-matrix given by

$$G(\Psi) = \big([\hat{\eta},\hat{\psi}]\big)_{\eta,\psi\in\Psi}. \tag{3.4}$$

G is known as the Grammian matrix associated with Ψ. Notice that the Cauchy-Schwarz inequality shows that each entry in G is contained in $L_1(\mathbb{T}^d)$ since $\Psi \subset L_2(\mathbb{R}^d)$.

The analysis so far shows that studying metric properties of the shift invariant system (1.1) is equivalent to studying the same properties of the trigonometric system

$$\{e^{ik\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d, j=1,\dots,N},$$

where $\mathbf{e}_j$, $j = 1,\dots,N$, is the standard basis for $\mathbb{C}^N$, in the vector-valued weighted space

$$L_2(\mathbb{T}^d,\mathbb{C}^N;G) := \Bigg\{f:\mathbb{T}^d\to\mathbb{C}^N : \|f\|^2_{L_2(\mathbb{T}^d,\mathbb{C}^N;G)} := \int_{\mathbb{T}^d} f(\xi)^H G(\xi) f(\xi)\,d\xi < \infty\Bigg\}.$$

At this point, we can outline the most important idea in the proof of Theorem 2.4. The idea is to "probe" the Grammian G in (3.4) using translates of the Dirichlet kernel in equation (3.3). The Dirichlet kernel has two important features. The square of it (properly normalized) forms an approximation to the identity, and moreover, it has "flat" coefficients.

It is possible to estimate the norm of expansions with flat coefficients relative to a quasi-greedy system. We need the following result due to Wojtaszczyk [18], see also [8]. It shows that quasi-greedy bases are unconditional for constant coefficients.

Lemma 3.1. *Suppose $\{e_k\}_{k\in\mathbb{N}}$ is a quasi-greedy system in a Hilbert space $\mathcal{H}$. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for every choice of signs $\varepsilon_k = \pm 1$ and any finite subset $A \subset \mathbb{N}$ we have*

$$c_1\Big\|\sum_{k\in A} e_k\Big\|_{\mathcal{H}} \le \Big\|\sum_{k\in A}\varepsilon_k e_k\Big\|_{\mathcal{H}} \le c_2\Big\|\sum_{k\in A} e_k\Big\|_{\mathcal{H}}, \tag{3.5}$$

where c_1 and c_2 depend only on the quasi-greedy constant for the system.

For our purpose, Lemma 3.1 is not quite enough. When we consider translates of the Dirichlet kernel, we need to be able to handle arbitrary unimodular complex coefficients and not only ± 1 as covered by Lemma 3.1. The following proposition, which will be essential for the proof of Theorem 2.4, takes care of that.

Proposition 3.2. *Suppose that $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}$ is a quasi-greedy system in S. Then there exist constants c_1, c_2 such that for every finite unimodular sequence $\{\alpha_k^\psi\}_{(\psi,k)\in F} \subset \mathbb{C}$, $F \subset \Psi \times \mathbb{Z}^d$, and every scalar sequence $\{v_\psi\}_{\psi\in\Psi}$, we have*

$$c_1(\max_{\psi\in\Psi}|v_\psi|)L^{1/2} \le \Big\| \sum_{(\psi,k)\in F} v_\psi \alpha_k^\psi \psi(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)} \le c_2(\max_{\psi\in\Psi}|v_\psi|)(\#F)^{1/2}. \tag{3.6}$$

where $L := \min_{\psi\in\Psi}\#\{k\in\mathbb{Z}^d : \alpha_k^\psi \neq 0\}$.

Proof. We begin by proving the upper estimate. An easy application of the triangle and Hölder inequalities yield

$$\Big\| \sum_{(\psi,k)\in F} v_\psi \alpha_k^\psi \psi(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)} \le (\max_{\psi\in\Psi}|v_\psi|) \sum_{\psi\in\Psi} \Big\| \sum_{k:(\psi,k)\in F} \alpha_k^\psi \psi(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)}. \tag{3.7}$$

For technical reasons we define a new scalar sequence $\{\beta_k^\psi\}$ by

$$\beta_k^\psi = \begin{cases} \alpha_k^\psi, & (\psi,k)\in F,\\ 1, & (\psi,k)\in(\Psi\times\mathbb{Z}^d)\backslash F.\end{cases}$$

Now observe that $\{\beta_k^\psi\psi(\cdot - k)\}_{\psi\in\Psi,k\in\mathbb{Z}^d}$ is also a quasi-greedy system. In fact, the greedy approximation operator $\tilde{\mathcal{G}}_m$ for $\{\beta_k^\psi\psi(\cdot - k)\}_{\psi\in\Psi,k\in\mathbb{Z}^d}$ is idential to the approximation operator $\mathcal{G}_m$ for $\{\psi(\cdot - k)\}_{\psi\in\Psi,k\in\mathbb{Z}^d}$. This follows from the trivial observation that if f_k^ψ is the dual element to $\psi(\cdot - k)$, then $\beta_k^\psi f_k^\psi$ is the dual element to $\beta_k^\psi\psi(\cdot - k)$, since $|\beta_k^\psi| = 1$. We use this fact together with Lemma 3.1 to obtain (for fixed $\psi \in \Psi$)

$$\begin{aligned}\Big\| \sum_{k:(\psi,k)\in F} \alpha_k^\psi \psi(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)} &\asymp \mathrm{Avg}_{\varepsilon_k^\psi=\pm1} \Big\| \sum_{k:(\psi,k)\in F} \varepsilon_k^\psi[\beta_k^\psi\psi(\cdot - k)]\Big\|_{L_2(\mathbb{R}^d)}\\ &\asymp \Big(\sum_{k:(\psi,k)\in F} \|\beta_k^\psi\psi(\cdot - k)\|_{L_2(\mathbb{R}^d)}\Big)^{1/2}\\ &\le C(\#F)^{1/2}, \end{aligned}\tag{3.8}$$

where we used that $L_2(\mathbb{R}^d)$ has Rademacher type and cotype 2, see [17, §III.A]. This together with (3.7) proves the upper estimate in (3.6). We turn to the lower estimate in (3.6). Pick an index $\psi' \in \Psi$ such that $|v_{\psi'}| = \max_{\psi\in\Psi}|v_\psi|$. Now we use that the system $\{\beta_k^\psi\psi(\cdot - k)\}_{\psi\in\Psi,k\in\mathbb{Z}^d}$ is quasi-greedy to conclude that for every $\varepsilon > 0$,

$$\begin{aligned}(1+\varepsilon)|v_{\psi'}|\Big\| \sum_{(\psi',k)\in F} \alpha_k^{\psi'}\psi'(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)}\\ \le Q\Big\|(1+\varepsilon)v_{\psi'}\sum_{(\psi',k)\in F}\alpha_k^{\psi'}\psi'(\cdot - k) + \sum_{\psi\in\Psi,\psi\neq\psi'} v_\psi \sum_{k\in\mathbb{Z}} \alpha_k^\psi\psi(\cdot - k)\Big\|_{L_2(\mathbb{R}^d)},\end{aligned}\tag{3.9}$$

where Q is the quasi-greedy constant for $\{\beta_k^\psi \psi(\cdot - k)\}_{\psi\in\Psi,k\in\mathbb{Z}^d}$. We let $\varepsilon \to 0^+$ to conclude that

$$|v_{\psi'}| \Big\| \sum_{(\psi',k)\in F} \alpha_k^{\psi'} \psi'(\cdot - k) \Big\|_{L_2(\mathbb{R}^d)} \leq Q \Big\| \sum_{(\psi,k)\in F} v_\psi \alpha_k^\psi \psi(\cdot - k) \Big\|_{L_2(\mathbb{R}^d)}.$$

Using the same argument as in (3.8), we see that there exists a constant c such that

$$L := \min_{\psi\in\Psi} \#\{k \in \mathbb{Z}^d : \alpha_k^\psi \neq 0\} \leq c \Big\| \sum_{(\psi',k)\in F} \alpha_k^{\psi'} \psi'(\cdot - k) \Big\|_{L_2(\mathbb{R}^d)}^2,$$

and the lower estimate in (3.6) follows. □

Remark 3.3. Notice that the *only* properties of the system $\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}$ that are used in the proof of Proposition 3.2 are quasi-greedyness and the fact that the system is quasi-normalized in $L_2(\mathbb{R}^d)$. Hence, the estimate (3.6) holds for an abstract quasi-normalized and quasi-greedy system $\{e_k^\psi : \psi \in \Psi, k \in \mathbb{Z}^d\}$ in a Hilbert space $\mathcal{H}$.

Let G be the Gram matrix (3.4) associated with the finite collection Ψ of functions in $L_2(\mathbb{R}^d)$. The entries in G are $L_1(\mathbb{T}^d)$-functions and consequently finite a.e. We let $\lambda(\xi)$ and $\Lambda(\xi)$ denote the smallest, respectively largest, eigenvalue of the Gram matrix $G(\xi)$. On the null-set of $\mathbb{T}^d$ where G may not exist, we assign any convenient value to $\lambda(\xi)$ and $\Lambda(\xi)$. We can now give a proof of Theorem 2.4.

Proof of Theorem 2.4. Let $\mathcal{L} \subseteq \mathbb{T}^d$ denote the common set of Lebesgue points for the entries in G. We notice that $\mathcal{L}$ has full measure. Pick $u \in \mathcal{L}$, and let $\mathbf{v} \in \mathbb{C}^N$ be an ℓ_2-normalized eigenvector corresponding to the smallest eigenvalue of $G(u)$. We use the Dirichlet kernel

$$D_K(\xi) := \sum_{k\in\mathbb{Z}^d:|k_i|\leq K} e^{ik\cdot\xi}, \qquad K \geq 1,$$

to create the vector functions

$$\tau_K(\xi) := D_K(u - \xi)\mathbf{v},$$

$K \in \mathbb{N}$. Notice that $D_K(u - \xi)$ is a trigonometric polynomial with exactly $\mathcal{O}(K^d)$ non-zero *unimodular* coefficients, so $\|D_K\|_{L_2(\mathbb{T}^d)} \asymp K^{d/2}$. We let

$$f_K = \sum_{(\psi,k)\in F_K} \alpha_k^{\psi,K} \psi(\cdot - k)$$

be the "pre-image" expansion in (1.1) that gives the Fourier transform

$$\widehat{f_K} = \sum_{\psi\in\Psi} [\tau_K]_\psi \hat{\psi}.$$

Since $\mathbf{v}$ is normalized in ℓ_2, we have $\max_i |v_i| \geq 1/\sqrt{N}$. We use this fact together with Proposition 3.2 to obtain the estimate,

$$\|f_K\|_{L_2(\mathbb{R}^d)} \asymp K^{d/2},$$

uniformly in K and $u \in \mathbb{T}^d$. The crucial step is to observe that

$$\frac{|D_K(u-\xi)|^2}{\|D_K\|^2_{L_2(\mathbb{T}^d)}}$$

is an approximation of the identity at u. We have

$$\frac{1}{\|D_K\|^2_{L_2(\mathbb{T}^d)}}\tau_K(\xi)^H G(\xi)_{i,j}\tau_K(\xi) = \sum_{i=1}^N\sum_{j=1}^N G(\xi)_{i,j}\overline{v_i}v_j\frac{|D_K(u-\xi)|^2}{\|D_K\|^2_{L_2(\mathbb{T}^d)}}$$

We thus obtain the estimate

$$\lim_{K\to\infty}\int_{\mathbb{T}^d}\frac{1}{\|D_K\|^2_{L_2(\mathbb{T}^d)}}\tau_K(\xi)^H G(\xi)\tau_K(\xi)\,d\xi = \sum_{i=1}^N\sum_{j=1}^N G(u)_{i,j}\overline{v_i}v_j = \lambda(u).$$

At the same time,

$$\int_{\mathbb{T}^d}\frac{1}{\|D_K\|^2_{L_2(\mathbb{T}^d)}}\tau_K(\xi)^H G(\xi)\tau_K(\xi)\,d\xi = \frac{\|f_K\|^2_{L_2(\mathbb{R}^d)}}{\|D_K\|^2_{L_2(\mathbb{T}^d)}} \asymp 1,$$

uniformly in K and $u \in \mathbb{T}^d$. Hence there exists $c > 0$ such that $c \le \lambda(u)$ for a.e. $u \in \mathbb{T}^d$. To get the estimate for $\Lambda(u)$, we repeat the argument with $\mathbf{w} \in \mathbb{C}^N$, a normalized eigenvector corresponding to the largest eigenvalue of $G(u)$.

To conclude the proof, we take an arbitrary finite expansion

$$f = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}^d} c_\psi(k)\psi(\cdot - k),$$

with $\hat{f} = \sum_{\psi\in\Psi}\tau_\psi\hat{\psi}$, where $\tau_\psi = \sum_k c_\psi(k)e^{ik\cdot\xi}$. We have, using the standard Rayleigh-Ritz estimate and Plancherel's Theorem,

$$\begin{aligned}
(2\pi)^d c\|\{c_\psi(k)\}\|^2_{\ell_2} &\le \operatorname*{ess\,inf}_{u\in\mathbb{T}^d}\lambda(u)\sum_{\psi\in\Psi}\int_{\mathbb{T}^d}|\tau_\psi(\xi)|^2\,d\xi \\
&= \operatorname*{ess\,inf}_{u\in\mathbb{T}^d}\lambda(u)\cdot\int_{\mathbb{T}^d}\tau(\xi)^H\tau(\xi)\,d\xi \\
&\le \int_{\mathbb{T}^d}\tau(\xi)^H G(\xi)\tau(\xi)\,d\xi \\
&= (2\pi)^d\|f\|^2_{L_2(\mathbb{R}^d)} \\
&\le \operatorname*{ess\,sup}_{u\in\mathbb{T}^d}\Lambda(u)\cdot\int\tau(\xi)^H\tau(\xi)\,d\xi \\
&= \operatorname*{ess\,sup}_{u\in\mathbb{T}^d}\Lambda(u)\sum_{\psi\in\Psi}\int_{\mathbb{T}^d}|\tau_\psi(\xi)|^2\,d\xi \\
&\le (2\pi)^d C\|\{c_\psi(k)\}\|^2_{\ell_2}.
\end{aligned}$$

It follows that (2.2) holds, so the quasi-greedy system (1.1) is indeed a Riesz basis for S. □

An application to vector valued spaces. The proof of Theorem 2.4 relies heavily on properties of the trigonometric system in a certain weighted vector-valued space. Here we spell out the equivalent version of Theorem 2.4 in this vector-valued setting since the result may be of some interest in its own right.

For $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ a positive definite matrix-valued function, we can consider the $\mathbb{C}^N$-valued L_2 space defined by

$$L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi)) := \left\{ f : \mathbb{T}^d \to \mathbb{C}^N : \|f\|^2_{L_2(\mathbb{T}^d,\mathbb{C}^d;W(\xi))} := \int_{\mathbb{T}^d} f(\xi)^H W(\xi) f(\xi)\, d\xi < \infty \right\}.$$

For such a space, we have the following Corollary to the proof of Theorem 2.4. We let $\mathbf{e}_j$, $j = 1,\dots,N$, denote the standard basis for $\mathbb{C}^N$, and we let $\lambda(\xi)$ and $\Lambda(\xi)$ denote the smallest, resp. largest, eigenvalue for $W(\xi)$. It is proved in [6, Lemma 3.25] that λ and Λ are measurable functions whenever the entries of $W(\xi)$ are in $L_1(\mathbb{T}^d)$.

Corollary 3.4. *Let $W : \mathbb{T}^d \to \mathbb{C}^{N\times N}$ be a positive definite matrix-valued function with entries in $L_1(\mathbb{T}^d)$. The trigonometric system $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d,j=1,\dots,N}$ forms a quasi-greedy basis for the vector-valued space $L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))$ if and only if there exists positive constants c, C such that the spectrum $\sigma(W(\xi))$ of $W(\xi)$ satisfies*

$$c \le \min \sigma(W(\xi)) \le \max \sigma(W(\xi)) \le C, \tag{3.10}$$

for a.e. $\xi \in \mathbb{T}^d$. Moreover, whenever $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d,j=1,\dots,N}$ forms a quasi-greedy basis for $L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))$, it is also a Riesz basis for the same space.

Proof. Suppose $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d,j=1,\dots,N}$ is a quasi-greedy system in $L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))$. To derive the lower estimate in (3.10) we follow the proof of Theorem 2.4. For $u \in \mathbb{T}^d$, a common Lebesgue point of the entries of $W(\xi)$, we let let $\mathbf{v} \in \mathbb{C}^N$ be an ℓ_2-normalized eigenvector corresponding to the smallest eigenvalue of $W(u)$. Then we form the trigonometric polynomials $\tau_K(\xi) := D_K(u-\xi)\mathbf{v}$, $K \in \mathbb{N}$. We notice that the quasi-greedy system $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d,j=1,\dots,N}$ is quasi-normalized in $L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))$, so following Remark 3.3 we conclude that $\|\tau_K\|_{L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))} \asymp K^{d/2}$. The lower estimate in (3.10) now follows along the lines of the proof of Theorem 2.4. The upper estimate in (3.10) is proved the same way.

Next, we let $f(\xi) := \sum_{j=1}^N \sum_{k\in\mathbb{Z}^d} c_{j,k} e^{ik\cdot\xi}\mathbf{e}_j$ be a finite expansion. We have, using the standard Rayleigh-Ritz estimate,

$$\begin{aligned}
c\|\{c_{j,k}\}\|^2_{\ell_2} &\le \operatorname*{ess\,inf}_{\xi\in\mathbb{T}^d} \lambda(\xi)\cdot \int_{\mathbb{T}^d} f(\xi)^H f(\xi)\, d\xi \\
&\le \int_{\mathbb{T}^d} f(\xi)^H W(\xi) f(\xi)\, d\xi \\
&\le \operatorname*{ess\,sup}_{\xi\in\mathbb{T}^d} \Lambda(\xi)\cdot \int f(\xi)^H f(\xi)\, d\xi \\
&\le C\|\{c_{j,k}\}\|^2_{\ell_2}.
\end{aligned}$$

It follows immediately that $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d,j=1,\dots,N}$ is a Riesz basis for $L_2(\mathbb{T}^d,\mathbb{C}^N;W(\xi))$.

The converse result is straightforward. Suppose (3.10) holds. Then $L_2(\mathbb{T}^d, \mathbb{C}^N; W(\xi))$ is isomorphic to $L_2(\mathbb{T}^d, \mathbb{C}^N; \mathrm{Id})$ and it follows that $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d, j=1,\dots,N}$ is a Riesz-basis for $L_2(\mathbb{T}^d, \mathbb{C}^N; W(\xi))$ since it is an orthonormal basis for $L_2(\mathbb{T}^d, \mathbb{C}^N; \mathrm{Id})$. In particular, $\{e^{ik\cdot\xi}\mathbf{e}_j\}_{k\in\mathbb{Z}^d, j=1,\dots,N}$ is a quasi-greedy basis for $L_2(\mathbb{T}^d, \mathbb{C}^N; W(\xi))$. □

Remark 3.5. A natural question is whether Corollary 3.4 holds true for more general systems than the trigonometric basis. An analysis of the proof of Corollary 3.4 shows that we only use two properties of the system:

- The system is a character group for a compact group. This means that the translated Dirichlet kernel $D_K(u+\xi)$ (here "+" is the addition on the compact group) is exactly $D_K(\xi)$ modified by unimodular coefficients.
- The kernel $|D_K(u+\xi)|^2 / \|D_K\|_{L_2}$ is an approximate identity at u as $K \to \infty$.

So for a given system we just have to check the two conditions. We mention, as an example, that the corollary holds true for the Walsh system.

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