SCHAUDER BASES OF INTEGER TRANSLATES

MORTEN NIELSEN AND HRVOJE ŠIKIĆ

Abstract. For a function $\Psi \in L^2(\mathbf{R})$, we give necessary and sufficient conditions for the family $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ to be a Schauder basis for the space $\overline{\text{Span}}\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$.

1. Introduction

In this note we address the following question. Given a function $\Psi \in L^2(\mathbf{R})$, what are necessary and sufficient conditions for the family $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ to be a Schauder basis for the space

$$W_0 := \overline{\text{Span}}\{\Psi(\cdot - k) : k \in \mathbf{Z}\}?$$

Since in a Schauder basis the order of its vectors matters, we shall assume here that $\mathbf{Z}$ is ordered as $0, 1, -1, 2, -2, \ldots$. We believe that the question is of interest to the wavelet community since several important properties of wavelet expansions in $L^2(\mathbf{R})$ are contained in the basic resolution level W_0 (see, e.g., [2], [4], and [5]). As it turned out the answer to this question has existed in the literature for more than thirty years (see Theorem 8 in [3]), although this fact is not necessarily obvious, since in [3] Schauder bases have not been considered specifically. However, we were inspired and motivated by a recent article of Heil and Powell [1] in which they explore many interesting consequences of [3] in a related context. In [1] we actually have all the ingredients to answer the question above (see, in particular, Corollary 5.4 in [1]), but again it is easy to overlook this fact. Therefore, we decided to write this note.

Let us begin by introducing some notation and recalling some necessary results. Roughly, we follow the notation from [2], i.e., the Fourier transform of a function $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ is $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\xi} dx$, while the inverse Fourier transform is given with a factor $(2\pi)^{-1}$. Hence, $(\hat{f})^{\vee} = f$, for $f \in L^2(\mathbf{R})$, while $\langle f, g \rangle = (2\pi)^{-1} \langle \hat{f}, \hat{g} \rangle$ for $f, g \in L^2(\mathbf{R})$. We denote by $L^p(\mathbf{T})$, $p = 1$ or 2, the L^p space on the torus $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z})$ with respect to the measure $\frac{d\xi}{2\pi}$. Functions on $\mathbf{T}$ are considered as 2π-periodic functions on $\mathbf{R}$. When a 2π-periodic measurable density (or weight) w is taken into account, we denote by $L^2(\mathbf{T}; w)$ the L^2 space on $\mathbf{T}$ with respect to the measure $w(\xi)\frac{d\xi}{2\pi}$.

Observe that W_0 is the smallest shift invariant space in $L^2(\mathbf{R})$ generated by Ψ. Hence, it is known that

$$t \to (t \cdot \hat{\Psi})^{\vee} \tag{1}$$

is an isometry between $L^2(\mathbf{T};P_\Psi)$ and W_0, where P_Ψ is the periodization of $|\hat{\Psi}|^2$, given by

$$P_\Psi(\xi) := \sum_{k\in\mathbf{Z}} |\hat{\Psi}(\xi+2\pi k)|^2, \qquad \xi \in \mathbf{R}. \tag{2}$$

Since $\Psi \in L^2(\mathbf{R})$, P_Ψ is a 2π-periodic function in $L^1(\mathbf{T})$ and

$$\int_{\mathbf{T}} P_\Psi(\xi)\frac{d\xi}{2\pi} = \frac{1}{2\pi}\|\hat{\Psi}\|_2^2 = \|\Psi\|_2^2. \tag{3}$$

The isometry (1) maps the exponential $e_k(\xi) := e^{-ik\xi}$, $k \in \mathbf{Z}$, into $\Psi(\cdot - k)$. As a consequence, our question is equivalent to the following one. What are the necessary and sufficient conditions on P_Ψ for the family $\{e_k : k \in \mathbf{Z}\}$ (ordered properly) to be a Schauder basis for $L^2(\mathbf{T};P_\Psi)$?

A family $\mathcal{B} = \{x_n : n \in \mathbf{N}\}$ is a *basis* of vectors in a Hilbert space $\mathbf{H}$ is a basis for $\mathbf{H}$ if for every $x \in \mathbf{H}$ there exists a unique sequence $\{\alpha_n := \alpha_n(x) : n \in \mathbf{N}\}$ of scalars such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n = x$$

in the norm topology of $\mathbf{H}$. The unique choice of scalars implies that $x \to \alpha_n(x)$ is a linear functional, for every $n \in \mathbf{N}$. If a basis has the property that all these functionals are continuous, the we call $\mathcal{B}$ a *Schauder basis*. In a Hilbert space, every basis is a Schauder basis (see [6] for this and other basic results on bases). Furthermore, for every $n \in \mathbf{N}$, there exists a unique vector y_n such that $\alpha_n(x) = \langle x, y_n\rangle$. It follows that

$$\langle x_m, y_n\rangle = \delta_{m,n}, \quad m,n \in \mathbf{N} \tag{4}$$

and that there exists a smallest constant $C = C(\mathcal{B}) \geq 1$ such that, for every $n \in \mathbf{N}$,

$$1 \leq \|x_n\| \cdot \|y_n\| \leq C. \tag{5}$$

A pair of sequences $(\{u_n\}_{n\in\mathbf{N}}, \{v_n\}_{n\in\mathbf{N}})$ in $\mathbf{H}$ is a *bi-orthogonal system* if $\langle u_m, v_n\rangle = \delta_{m,n}$, $m,n \in \mathbf{N}$. We say that $\{v_n\}_{n\in\mathbf{N}}$ is the *dual sequence* to $\{u_n\}_{n\in\mathbf{N}}$, and vice versa. The dual sequence is not necessarily unique. In fact, it is unique if and only if the original sequence is complete in $\mathbf{H}$ (i.e., if the span of the original sequence is dense in $\mathbf{H}$). A complete sequence $\{x_n : n \in \mathbf{N}\}$ with dual sequence $\{y_n\}$ is a Schauder basis for $\mathbf{H}$ if and only if the partial sum operators $S_N(x) = \sum_{n=1}^{N}\langle x, y_n\rangle x_n$ are uniformly bounded on $\mathbf{H}$. Obviously, (4) shows that every (Schauder) basis $\{x_n : n \in \mathbf{N}\}$ for $\mathbf{H}$ belongs to a bi-orthogonal system $(\{x_n\}, \{y_n\})$ with a uniquely determined dual sequence. Furthermore, the dual sequence $\{y_n\}$ is also a Schauder basis for $\mathbf{H}$. Hence, we consider bi-orthogonal systems first.

Proposition 1.1. *Let $\Psi \in L^2(\mathbf{R})$. The sequence $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ belongs to a bi-orthogonal system in W_0 if and only if*

$$\frac{1}{P_\Psi} \in L^1(\mathbf{T}) \tag{6}$$

(in particular, $P_\Psi > 0$ a.e.). If this is the case, the the dual system is unique, given by

$$\left(\frac{e^{ik\xi}}{P_\Psi(\xi)} \cdot \hat{\Psi}\right)^\vee, \qquad k \in \mathbf{Z}. \tag{7}$$

Proof. As we have seen, it is equivalent to consider $\{e_k : k \in \mathbf{Z}\}$ in $L^2(\mathbf{T}; P_\Psi)$. Suppose first that $\{y_k : k \in \mathbf{Z}\}$ is the dual sequence of $\{e_k : k \in \mathbf{Z}\}$ in $L^2(\mathbf{T}; P_\Psi)$. Observe that y_k are then functions on $\mathbf{T}$ and they satisfy, for $m, n \in \mathbf{Z}$,

$$\delta_{m,n} = \int_{\mathbf{T}} e_m \cdot \overline{y_n} P_\Psi \frac{d\xi}{2\pi} = \int_{\mathbf{T}} e_m \cdot \overline{(y_n P_\Psi)} \frac{d\xi}{2\pi}.$$

It follows that $y_n \cdot P_\Psi$ are $L^1(\mathbf{T})$ functions whose Fourier coefficients are all zero, expect the coefficient n, which is 1. Hence, for every $k \in \mathbf{Z}$,

$$e_k = y_k \cdot P_\Psi \in L^2(\mathbf{T}).$$

Since $e_k(\xi) \neq 0$, we conclude that

$$P_\Psi(\xi) > 0, \quad \text{for a.e. } \xi \in \mathbf{T}. \tag{8}$$

Hence, we can divide by $P_\Psi(\xi)$ for a.e. ξ. Since $y_k \in L^2(\mathbf{T}; P_\Psi)$ we get

$$+\infty > \|y_k\|_{L^2(\mathbf{T};P_\Psi)} = \int_{\mathbf{T}} \left|\frac{e_k(\xi)}{P_\Psi(\xi)}\right|^2 \cdot P_\Psi(\xi) \frac{d\xi}{2\pi} = \frac{1}{2\pi} \int_{\mathbf{T}} \left|\frac{1}{P_\Psi(\xi)}\right|^2 d\xi.$$

It shows that $1/P_\Psi \in L^1(\mathbf{T})$. Observe that (7) now follows via the isometry (1). Uniqueness of the dual sequence (7) is the consequence of the completeness of $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ in W_0.

Suppose now that $1/P_\Psi \in L^1(\mathbf{T})$. It follows that $e_k/P_\Psi \in L^2(\mathbf{T}; P_\Psi)$ and that they form the dual sequence to $\{e_k\}$ in $L^2(\mathbf{T}; P_\Psi)$. □

Observe also that, if (6) is satisfied, we have

$$1 \leq \|e_k\|_{L^2(\mathbf{T};P_\Psi)} \cdot \|y_k\|_{L^2(\mathbf{T};P_\Psi)} = \frac{1}{2\pi} \sqrt{\int_{\mathbf{T}} P_\Psi(\xi)\, d\xi} \cdot \sqrt{\int_{\mathbf{T}} \frac{1}{P_\Psi(\xi)}\, d\xi}, \tag{9}$$

for every $k \in \mathbf{Z}$. In particular, the linear operators $x \mapsto \langle x, y_k \rangle e_k$, with $\langle \cdot, \cdot \rangle$ the standard inner product on $L^2(\mathbf{T})$, are uniformly bounded on $L^2(\mathbf{T}; P_\Psi)$ and their operator norms are given by the right hand side of (9).

Turning our attention to the main question of Schauder bases, we shall slightly modify the weight condition from [3]. In this way we avoid some trivial sub-cases which are irrelevant in our set up (see (8)).

Definition 1.2. *A measurable, 2π-periodic function $w : \mathbf{R} \to (0, \infty)$ is an $\mathcal{A}_2(\mathbf{T})$-weight if there exists a constant $M > 0$ such that for every interval $I \subseteq \mathbf{T}$,*

$$\left(\frac{1}{|I|} \int_I w(\xi)\, d\xi\right)\left(\frac{1}{|I|} \int_I \frac{1}{w(\xi)}\, d\xi\right) \leq M.$$

Notice that for any $\mathcal{A}_2(\mathbf{T})$-weight w, we have $w, 1/w \in L^1(\mathbf{T})$. We can now state the main result of this note.

Proposition 1.3. *Let $\Psi \in L^2(\mathbf{R})$. The family $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ is a Schauder basis for W_0 if and only if P_Ψ is an $\mathcal{A}_2(\mathbf{T})$-weight.*

Remark 1.4. *The family above is considered as an $\mathbf{N}$-indexed sequence, where $\mathbf{Z}$ is ordered as $0, 1, -1, 2, -2, \ldots$, as mentioned at the beginning. In the usual Banach space terminology (see [6], for example), Proposition 1.3 provides necessary and sufficient conditions for our sequence to be a* basis sequence.

Proof. Suppose first that $\{e_k\}$ is a Schauder basis for $L^2(\mathbf{T}, P_\Psi)$. In particular, $\{e_k\}$ belongs to a bi-orthogonal system. By Proposition 1.1, $P_\Psi > 0$ a.e. and $1/P_\Psi \in L^1(\mathbf{T})$. Furthermore, we have an explicit formulae for the dual sequence $\{y_k\}$. We define

$$S_N f = \sum_{n=1}^{N} \langle f, y_n \rangle_{L^2(\mathbf{T};P_\Psi)} e_n,$$

taken with the proper reordering of the bi-orthogonal system $(\{e_n\}, \{y_n\})$. Notice that for every $f \in L^2(\mathbf{T}; P_\Psi)$,

$$\langle f, y_k \rangle_{L^2(\mathbf{T};P_\Psi)} = \langle f, e_k \rangle.$$

Hence, the Fourier partial sum operators

$$T_N f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$$

on $L^2(\mathbf{T}; P_\Psi)$ satisfy $S_{2N+1} f = T_N f$. Since, by assumption, the system $(\{e_n\}, \{y_n\})$ is complete and regular (i.e., forms a Schauder basis), it follows that

$$\sup_{N \in \mathbf{N}} \|S_N\|_{L^2(\mathbf{T};P_\Psi) \to L^2(\mathbf{T};P_\Psi)} < +\infty. \tag{10}$$

Hence, the Fourier partial sum operators $\{T_N\}$ are uniformly bounded on $L^2(\mathbf{T}; P_\Psi)$, which, by Theorem 8 in [3], implies that P_Ψ is an $\mathcal{A}_2(\mathbf{T})$-weight.

If P_Ψ is an $\mathcal{A}_2(\mathbf{T})$-weight, then $P_\Psi > 0$ a.e., and $P_\Psi, 1/P_\Psi \in L^1(\mathbf{T})$. Hence $\{e_n\}$ belongs to a bi-orthogonal system which is also complete. In order to prove that $\{e_n\}$ is a Schauder basis it remains to prove (10). By Theorem 8 in [3], the operators S_{2N+1} are uniformly bounded on $L^2(\mathbf{T}; P_\Psi)$, while (9) and

$$S_{2N+2} = S_{2N+1} + \langle f, y_{2N+2} \rangle_{L^2(\mathbf{T};P_\Psi)} e_{2N+2},$$

imply (10). □

Let us recall standard results on orthonormal and Riesz bases (see [4], for example) in order to complete this note with the following list of results.

Corollary 1.5. *Let $\Psi \in L^2(\mathbf{R})$ (hence $P_\Psi \in L^1(\mathbf{T})$). Then the family $\{\Psi(\cdot - k) : k \in \mathbf{Z}\}$ in W_0*

a) *belongs to a bi-orthogonal system if and only if $1/P_\Psi \in L^1(\mathbf{T})$;*
b) *is a Schauder basis for W_0 if and only if P_Ψ is an $\mathcal{A}_2(\mathbf{T})$-weight.;*
c) *is a Riesz basis for W_0 if and only if P_Ψ and $1/P_\Psi$ are bounded a.e.;*
d) *is an orthonormal basis for W_0 if and only if $P_\Psi \equiv 1$ a.e. (hence, $1/P_\Psi \equiv 1$ a.e.).*

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Department of Mathematics, Washington University, Campus Box 1146, St. Louis, MO 63130, USA

E-mail address: `mnielsen@math.wustl.edu`

Department of Mathematics, University of Zagreb, Bijenička 30, HR-10000, Zagreb, Croatia and Department of Mathematics, Washington University, Campus Box 1146, St. Louis, MO 63130, USA

E-mail address: `sikic@math.wustl.edu`