On convergence of wavelet packet expansions

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Abstract. It is well known that the Walsh-Fourier expansion of a function from the block space $\mathcal{B}_q([0,1))$, $1 < q \leq \infty$, converges pointwise a.e. We prove that the same result is true for the expansion of a function from $\mathcal{B}_q$ in certain periodized smooth periodic non-stationary wavelet packets bases based on the Haar filters. We also consider wavelet packets based on the Shannon filters and show that the expansion of L^p-functions, $1 < p < \infty$, converges in norm and pointwise almost everywhere.

Keywords: Wavelet packets, non-stationary wavelet packets, pointwise convergence, block spaces, Walsh functions.

Introduction

In a recent paper [6] a family of non-stationary wavelet packets called Walsh-type wavelet packets was introduced. The Walsh-type wavelet packets can be considered smooth generalizations of the Walsh functions and they have some of the same nice convergence properties for expansion of L^p-functions, $1 < p < \infty$, as the Walsh-Fourier series have. However, it was also demonstrated in [5, 6] that Walsh-type wavelet packet expansions fail when it comes to L^1-functions. There is thus a gap between the negative result in L^1 and the positive results for L^p, $1 < p < \infty$. The same gap exists for the Walsh system, and several families of spaces containing L^p, $1 < p < \infty$, have been introduced where the positive results about convergence almost everywhere can be shown to still hold true, see e.g. [7].

The purpose of this paper is to show that for Walsh-type wavelet packet expansion, we also obtain pointwise convergence almost everywhere for the expansion of functions from the block space $\mathcal{B}_q$, $1 < q \leq \infty$. This will be done by a careful analysis of the partial sum operator for the Walsh-type wavelet packet expansions. The block space $\mathcal{B}_q$ was introduced by Taibleson and Weiss [8, 9] to study convergence properties for trigonometric series, and we recall their definition in Section 1.

In Section 1 we will also give the definition of the Walsh-type wavelet packets that will be considered and give a precise statement and proof of the main result, Theorem 2.4.

The second part of this paper is devoted to a study of wavelet packets based on the Shannon filters. In a sense the Shannon filter is the opposite extreme to the Haar filter associated with the Walsh functions. It is the only filter with "perfect" frequency localization. In Section 3 we will consider wavelet packets based on the Shannon filters, and prove L^p-convergence for certain nonstationary wavelet packets based on the Shannon filter.

Finally, there is an appendix dealing with some elementary properties of the Walsh functions that is used in the proofs in Section 1.

1. Walsh-Type Wavelet Packets

The Walsh-type wavelet packets are one example of a family of so-called non-stationary wavelet packets; to define the non-stationary wavelet packets we need a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ (for the definition and properties, see e.g. [1, 4]). To every multiresolution analysis we have an associated scaling function ϕ and a wavelet ψ with the properties that

$$V_j = \overline{\operatorname{span}}\{2^{j/2}\phi(2^j\cdot -k)|k\in\mathbb{Z}\},$$

and

$$\{\psi_{j,k} \equiv 2^{j/2}\psi(2^j\cdot -k)|j,k\in\mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. We denote $W_j = \overline{\operatorname{span}}\{2^{j/2}\psi(2^j\cdot -k)|k\in\mathbb{Z}\}$.

We let $(F_0^{(p)}, F_1^{(p)})$, $p\in\mathbb{N}$, be a family of bounded operators on $\ell^2(\mathbb{Z})$ of the form

$$(F_\epsilon^{(p)}a)_k = \sum_{n\in\mathbb{Z}} a_n h_\epsilon^{(p)}(n-2k), \qquad \epsilon = 0,1$$

with $h_1^{(p)}(n) = (-1)^n h_0^{(p)}(1-n)$ a real-valued sequence in $\ell^1(\mathbb{Z})$ such that

$$\begin{aligned} F_0^{(p)*}F_0^{(p)} + F_1^{(p)*}F_1^{(p)} &= 1 \\ F_0^{(p)}F_1^{(p)*} &= 0. \end{aligned}$$

We define the family of functions $\{w_n\}_{n=0}^\infty$ recursively by letting $w_0 = \phi$, $w_1 = \psi$ and then for $n\in\mathbb{N}$

$$w_{2n}(x) = 2\sum_{\ell\in\mathbb{Z}} h_0^{(p)} w_n(2x-\ell) \tag{1}$$

$$w_{2n+1}(x) = 2\sum_{\ell\in\mathbb{Z}} h_1^{(p)}(q) w_n(2x-\ell), \tag{2}$$

where $2^p \leq n < 2^{p+1}$. The family $\{w_n\}_{n=0}^{\infty}$ is our basic non-stationary wavelet packets. It is easy to verify that

$$\{w_n(\cdot - k) | n \geq 0, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Moreover,

$$\{w_n(\cdot - k) | 2^j \leq n < 2^{j+1}, k \in \mathbb{Z}\}$$

is an orthonormal basis for $W_j = \overline{\text{span}}\{2^{j/2} w_1(2^j \cdot -k) | k \in \mathbb{Z}\}$.

Each pair $(F_0^{(p)}, F_1^{(p)})$ can be chosen as a pair of quadrature mirror filters associated with a multiresolution analysis, but this is not necessary. The trigonometric polynomials given by

$$m_0^{(p)}(\xi) = \frac{1}{\sqrt{2}} \sum_k h_0^{(p)}(k) e^{-ik\xi} \quad \text{and} \quad m_1^{(p)}(\xi) = \frac{1}{\sqrt{2}} \sum_k h_1^{(p)}(k) e^{-ik\xi}$$

are called the symbols of the filters. The Fourier transform of (1) is given by

$$\hat{w}_{2n}(\xi) = m_0^{(p)}\left(\frac{\xi}{2}\right) \hat{w}_n\left(\frac{\xi}{2}\right), \tag{3}$$

and (2) becomes

$$\hat{w}_{2n+1}(\xi) = m_1^{(p)}\left(\frac{\xi}{2}\right) \hat{w}_n\left(\frac{\xi}{2}\right). \tag{4}$$

The Haar low-pass quadrature mirror filter $\{h_0(k)\}_k$ is given by $h_0(0) = h_0(1) = 1/\sqrt{2}$, $h_0(k) = 0$ otherwise, and the associated high-pass filter $\{h_1(k)\}_k$ is given by $h_1(k) = (-1)^k h_0(1-k)$. We now give the definition of the family of non-stationary wavelet packets we will consider

DEFINITION 1.1. *Let $\{w_n\}_{n \geq 0, k \in \mathbb{Z}}$ be a family of non-stationary wavelet packets constructed by using a family $\{h_0^{(p)}(n)\}_{p=1}^{\infty}$ of finite filters for which there is a constant $K \in \mathbb{N}$ such that $h_0^{(p)}(n)$ is the Haar filter for every $p \geq K$. If $w_1 \in C^1(\mathbb{R})$ it is compactly supported then we call $\{w_n\}_{n \geq 0}$ a family of Walsh-type wavelet packets.*

We call the functions (basic) Walsh-type wavelet packets since it turns out that they share a number of metric properties with the Walsh system and they can therefore be considered a smooth generalization of the Walsh system (see [6]).

For technical reasons we would like the functions to have support on $[0, 1)$. This can be obtained by periodizing the Walsh-type wavelet packets.

DEFINITION 1.2. *Let $\{w_n\}_{n=0}^{\infty}$ be a family of Walsh-type basic wavelet packets. For $n \in \mathbb{N}_0$ we define the corresponding periodic Walsh-type wavelet packets $\widetilde{w_n}$ by*

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} w_n(x-k).$$

It follows easily from Fubini's theorem that $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[0,1)$.

1.1. Block spaces

We now briefly recall the definition of the block spaces, see also [8, 9].

DEFINITION 1.3. *A dyadic q-block is a function $\beta \in L^q[0,1)$ which is supported on some dyadic interval I such that $\|\beta\|_q \le |I|^{1/q-1}$. We let $\mathcal{B}_q$ denote the space of measurable functions f on $[0,1)$ which has an expansion*

$$f = \sum_{k=1}^{\infty} c_k\beta_k,$$

where each β_k is a q-block and the coefficients $\{c_k\}$ satisfy

$$|||\{c_k\}||| := \sum_{k:c_k\neq 0} |c_k|\left(1+\log\frac{\sum_{j=1}^{\infty}|c_j|}{|c_k|}\right) < \infty. \tag{5}$$

The quasi-norm of $f \in \mathcal{B}_q$ is given as the infimum of $|||\cdot|||$ over all possible decompositions of f into blocks:

$$\|f\|_{\mathcal{B}_q} := \inf_{f=\sum c_k\beta_k} |||\{c_k\}|||.$$

REMARK 1.4. *Notice that for $f \in \mathcal{B}_q$,*

$$\|f\|_1 \le \sum_{k=1}^{\infty}|c_k|\|\beta_k\|_1 \le \sum_{k=1}^{\infty}|c_k| < \infty$$

using (5) and the fact that for each k, $\|\beta_k\|_q \le |I|^{1/q-1}$ which implies that $\|\beta_k\|_1 \le 1$, i.e. $\mathcal{B}_q \subset L^1[0,1)$. Moreover, for $f \in L^q[0,1)$, $1<q<\infty$, $\beta = \|f\|_q^{-1}f$ is a q-block supported on $I=[0,1)$ so $L^q[0,1) \subset \mathcal{B}_q$.

The classical example to show that for each $q>1$ there exists $f \in \mathcal{B}_q$ which belong to none of the $L^p[0,1)$-spaces is the following: put

$$\beta_k(x) = \begin{cases} 2^k & 2^{-k} \le x < 2^{-k-1} \\ 0 & \textit{otherwise.} \end{cases}$$

Then $f = \sum_{k=1}^{\infty} k^{-2}\beta_k \in \mathcal{B}_q$ but $\|f\|_p^p = \sum_{k=1}^{\infty} k^{-2p}2^{k(p-1)} = \infty$ for every $p>1$.

2. Main Results

We are now in a position to state the main result on Walsh-type wavelet packets. We need the following definition.

DEFINITION 2.1. *Let $\{\widetilde{w_n}\}$ be a periodic Walsh-type wavelet packet basis. For any function $f \in L^1[0,1)$ we define the Carleson operator by*

$$\mathcal{G}f(x) = \sup_{N \geq 0} \left| \sum_{n=0}^{N} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x) \right|,$$

the weak Carleson operator G is defined by

$$Gf(x) = \limsup_{N \geq 0} \left| \sum_{n=0}^{N} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x) \right|,$$

and the dyadic Carleson operator $\mathcal{G}^d$ is defined by

$$\mathcal{G}^d f(x) = \sup_{\ell \geq 0} \left| \sum_{n=0}^{2^\ell - 1} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x) \right|.$$

REMARK 2.2. *It is obvious that $\mathcal{G}$, G, and $\mathcal{G}^d$ are sub-linear operators. The reason we use the term* weak *for G is of course that it is defined with "$\limsup$" in place of "$\sup$".*

The main result that will be proved below is.

THEOREM 2.3. *For $1 < q \leq \infty$ there exists a finite constant C_q such that*

$$|\{Gf > \alpha\}| \leq \frac{C_q \|f\|_{\mathcal{B}_q}}{\alpha}, \qquad \alpha > 0,$$

for every $f \in \mathcal{B}_p$.

Before we get to the details of the proof of Theorem 2.3 let us verify that the result implies convergence a.e. for the Walsh-type wavelet packet expansion of block functions. We have the following corollary.

COROLLARY 2.4. *Let $\{\widetilde{w_n}\}$ be a periodic Walsh-type wavelet packet basis. Then the Fourier expansion of any function $f \in \mathcal{B}_q$, $1 < q < \infty$, in $\{\widetilde{w_n}\}$ converges pointwise a.e.*

Proof. Define $\Sigma_N f(x) = \sum_{n=0}^{N} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x)$. With $f = \sum_{k=1}^{\infty} c_k \beta_k \in \mathcal{B}_q$ we let $g_K = \sum_{k=1}^{K} c_k \beta_k$ and notice that $\|f - g_K\|_{\mathcal{B}_q} \to 0$. For each

$x \in [0,1)$ we write $f - \Sigma_n f = (f - g_K) + (g_K - \Sigma_n g_K) + (\Sigma_n g_K - \Sigma_n f)$. Thus

$$
\begin{aligned}
&|\{x : \limsup_{n\to\infty} |f(x) - \Sigma_n f(x)| > \alpha\}| \\
&\qquad \le |\{x : \limsup_{n\to\infty} |f(x) - g_K(x)| > \alpha/3\}| + \\
&\qquad\qquad + |\{x : \limsup_{n\to\infty} |g_K(x) - \Sigma_n g_K(x)| > \alpha/3\}| \\
&\qquad\qquad\qquad + |\{x : \limsup_{n\to\infty} |\Sigma_n g_K(x) - \Sigma_n f(x)| > \alpha/3\}| \\
&\qquad \le \frac{3\|f - g_K\|_{\mathcal{B}_q}}{\alpha} + 0 + \frac{3C_q\|f - g_K\|_{\mathcal{B}_q}}{\alpha}.
\end{aligned}
$$

From this it follows that

$$|\{x : \limsup_{n\to\infty} |f(x) - \Sigma_n f(x)| > \alpha\}| = 0,$$

which proves the claim.

The proof of Theorem 2.3 will be based on a number of Lemmas, and the following result proved by the author in [6].

THEOREM 2.5. *Let $\{\widetilde{w_n}\}$ be a periodic Walsh-type wavelet packet basis, and let $\mathcal{G}$ be the associated Carleson operator. Then $\mathcal{G}$ is of strong type (p,p), $1 < p < \infty$, i.e. $\mathcal{G}$ is sub linear and there exists a constant K_p such that $\|\mathcal{G}f\|_p \le K_p\|f\|_p$.*

REMARK 2.6. *One cannot deduce from Theorem 2.5 that the Carleson operator for a periodic Walsh-type system maps $L^1[0,1)$ into weak-L^1. In fact, this type of result will most likely not be true similar to the situation for the Walsh system and the trigonometric system. The much weaker result that the Carleson operator cannot be bounded on $L^1[0,1)$ follows from a counterexample in [6].*

The first lemma deals with the dyadic partial sums of the wavelet packet expansion of a single q-block. Before we state and prove the lemma we want to recall two basic facts about the periodic multiresolution analysis $\{\widetilde{V_j}\}$ in which the wavelet packets live. We let $P_{\widetilde{V_j}}$ denote the orthogonal projection onto $\widetilde{V_j}$, and let $K_j(x,y)$ denote the operator kernel associated with $P_{\widetilde{V_j}}$. Then there exists a finite constant C (independent of j) such that

$$|K_j(x,y)| \le C|x-y|^{-1}, \tag{6}$$

and the operator $f \to \sup_j P_{\widetilde{V_j}} f(x)$ is of strong type (q,q), $1 < q < \infty$, see e.g. [10] for details.

LEMMA 2.7. *Let $\{\widetilde{w_n}\}$ be a periodic Walsh-type wavelet packet system. Then there exists a constant C_q such that for every q-block β, $1 < q$,*

$$|\mathcal{G}^d\beta > \alpha\}| \leq \frac{C_q}{\alpha}.$$

Proof. The dyadic partial sums for the expansion of a measuable (integrable) function f in the periodic Walsh-type wavelet packets,

$$S_N f(x) = \sum_{n=0}^{2^N-1} \langle f, \widetilde{w_n}\rangle \widetilde{w_n},$$

agrees everywhere with the projection onto the (periodized) scaling space $\widetilde{V_N}$ associated with the underlying multiresolution analysis, see [3]. It therefore suffices to consider the projection operators $P_{\widetilde{V_N}}$ onto the spaces $\widetilde{V_N}$. Suppose that the q-block β is associated with the dyadic interval $I \subset [0,1]$, and let $\alpha > 0$. If $1 < \alpha|I|$ then $|I|^{1-q}/\alpha^q \leq 1/\alpha$, and using the fact that the operator $f \to \sup_N P_{\widetilde{V_N}} f(x)$ (and thus $f \to \mathcal{G}^d f(x)$) is of strong type (q,q), we obtain

$$|\{\mathcal{G}^d f(x) > \alpha\}| \leq C_q \left(\frac{\|\beta\|_q}{\alpha}\right)^q \leq C_q \frac{|I|^{1-q}}{\alpha^q} \leq \frac{C_q}{\alpha}.$$

Next we suppose that $1 \geq \alpha|I|$ with $I = [a,b)$. Put $\tilde{I} = [\frac{3a-b}{2}, \frac{3b-a}{2}) \cap [0,1)$, and define $\bar{I} = [0,1)\backslash\tilde{I}$. We have

$$\begin{aligned} |\{\mathcal{G}^d\beta > \alpha\}| &\leq 2|I| + |\bar{I} \cap \{\mathcal{G}^d\beta > \alpha\}| \\ &\leq \frac{2}{\alpha} + |\bar{I} \cap \{\mathcal{G}^d\beta > \alpha\}|. \end{aligned}$$

Fix $x \in \bar{I}$ and using the estimate (6) on the kernel K_N we obtain

$$|S_N\beta(x)| = \left|\int_I K_N(x,y)\beta(y)\,dy\right| \leq \left\{\frac{C}{|x-a|} + \frac{C}{|x-b|}\right\}\|\beta\|_1.$$

Using $\|\beta\|_1 \leq 1$, and the fact that $x \in \bar{I}$ implies that $|x-a|, |x-b| \geq |I|/2$, we finally obtain

$$|\{x \in \bar{I} : \sup_N |S_N\beta(x)| > \alpha\}| \leq \frac{\tilde{C}}{\alpha},$$

with $\tilde{C}$ independent of I and β and the Lemma follows.

We now turn our attention to G itself. Using the result on $\mathcal{G}^d$ we only need to consider the situation "between" dyadic scales. To get such estimates we need to estimate the size of the kernels

$$K_{J,m}(x,y) := \sum_{n=2^J}^{m} w_n(x)w_n(y), \quad m < 2^{J+1},$$

associated with the partial sum operator. We have

LEMMA 2.8. *Let*

$$K_{J,m}(x,y) := \sum_{n=2^J}^{m} w_n(x)w_n(y),$$

for $2^J \le m < 2^{J+1}$. *Then*

$$|K_{J,m}(x,y)| \le \sum_{\ell=-2N}^{2N} \frac{C}{|x-y+2^{K-J}\ell|}.$$

Proof. We expand the kernel, using Lemma 4.2 from the appendix,

$$\begin{aligned}
K_{J,m}(x,y) &= \sum_{n=2^J}^{m} w_n(x)w_n(y) \\
&= \sum_{n=2^J}^{m} \sum_{\ell=0}^{2^{J-K}-1} \sum_{k=0}^{2^{J-K}-1} W_{n-2^{J-K}}(\ell 2^{-(J-K)}) \\
&\qquad \times W_{n-2^{J-K}}(k2^{-(J-K)}) w_{2^K}(2^{J-K}x-\ell) w_{2^K}(2^{J-K}y-k) \\
&= \sum_{\ell=0}^{2^{J-K}-1} \sum_{k=0}^{2^{J-K}-1} \left\{ \sum_{n=2^J}^{m} W_{n-2^{J-K}}(\ell 2^{-(J-K)}) W_{n-2^{J-K}}(k2^{-(J-K)}) \right\} \\
&\qquad \times w_{2^K}(2^{J-K}x-\ell) w_{2^K}(2^{J-K}y-k)
\end{aligned}$$

Hence, by Lemma 4.3,

$$\begin{aligned}
|K_{J,m}(x,y)| &\le {\sum_{\ell=-N}^{N}}' {\sum_{k=-N}^{N}}' \Bigg| \sum_{n=2^J}^{m} W_{n-2^{J-K}}([2^{J-K}(x+2^{K-J}\ell)]2^{-(J-K)}) \\
&\qquad \times W_{n-2^{J-K}}([2^{J-K}(y+2^{K-J}k)]2^{-(J-K)}) \Bigg| \|w_{2^K}\|_\infty^2 \\
&\le {\sum_{\ell=-N}^{N}}' {\sum_{k=-N}^{N}}' \frac{C}{(y+2^{K-J}\ell)\oplus(x+2^{K-J}k)},
\end{aligned}$$

where $\sum'$ indicates that only the terms for which $x + 2^{K-J}\ell \in [0,1)$ and $y + 2^{K-J}k \in [0,1)$, respectively, should be included in the sum. This implies the estimate

$$|K_{J,m}(x,y)| \le \sum_{\ell=-N}^{N} \sum_{k=-N}^{N} \frac{\tilde{C}}{|x-y+2^{K-J}(\ell-k)|},$$

since we have $a \oplus b \geq 2^{-[\log_2 |a-b|]} \geq |a-b|/2$.

PROPOSITION 2.9. *Let $1 < q \leq \infty$. There exists a finite constant C_q such that*

$$|\{G\beta > \alpha\}| \leq \frac{C_q}{\alpha}, \qquad \alpha > 0,$$

for all q-blocks β.

Proof. Fix $\alpha > 0$ and a q-block β supported on the dyadic interval $I \subset [0,1)$. We consider two cases. If $1 < \alpha|I|$ then $|I|^{1-q}/\alpha^q \leq 1/\alpha$. Hence by Theorem 2.5 we have

$$|\{G\beta > \alpha\}| \leq C_q \left(\frac{\|\beta\|_q}{\alpha}\right)^q \leq C_q \frac{|I|^{1-q}}{\alpha^q} \leq \frac{C_q}{\alpha}.$$

Now suppose $1 \geq \alpha|I|$ with $I = [a,b)$. Put

$$\tilde{I} = \left(\bigcup_{j=-1}^{1} (j + [\frac{3a-b}{2}, \frac{3b-a}{2}))\right) \cap [0,1),$$

and define $\bar{I} = [0,1)\backslash\tilde{I}$. Then

$$\begin{aligned} |\{G\beta > \alpha\}| &\leq |\tilde{I}| + |\bar{I} \cap \{G\beta > \alpha\}| \\ &\leq \frac{6}{\alpha} + |\bar{I} \cap \{G\beta > \alpha\}|. \end{aligned}$$

Notice that

$$|\bar{I} \cap \{G\beta > \alpha\}| \leq |\bar{I} \cap \{\mathcal{G}^d\beta > \alpha/2\}| + |\bar{I} \cap \{\limsup_J M_J\beta > \alpha/2\}|,$$

with

$$M_J\beta(x) = \max_{2^J \leq m < 2^{J+1}-1} M_J^m\beta(x) \quad \text{and} \quad M_J^m\beta(x) = \left|\sum_{n=2^J}^{m} \langle\beta, \widetilde{w_n}\rangle\widetilde{w_n}(x)\right|.$$

For $x \in [0,1)$ we have

$$\begin{aligned} \limsup_{J,m} M_J^m\beta(x) &= \limsup_{J,m} \left|\sum_{\ell_1=-N}^{N}\sum_{\ell_2=-N}^{N}\sum_{n=2^J}^{m} \langle\beta, w_n(\cdot-\ell_1)\rangle w_n(x-\ell_2)\right| \\ &\leq \sum_{\ell_1=-N}^{N}\sum_{\ell_2=-N}^{N} \limsup_{J,m}\left|\sum_{n=2^J}^{m} \langle\beta, w_n(\cdot-\ell_1)\rangle w_n(x-\ell_2)\right|. \end{aligned} \tag{7}$$

Hence it suffices to estimate $|E_\alpha^{\ell_1,\ell_2}|$ with

$$E_\alpha^{\ell_1,\ell_2} = \left\{x \in \bar{I} : \limsup_{J,m}\left|\sum_{n=2^J}^{m} \langle\beta, w_n(\cdot-\ell_1)\rangle w_n(x-\ell_2)\right| > \alpha\right\}.$$

Fix $x \in \mathbb{R}\backslash I$. We have

$$\left|\int_{-\infty}^{\infty} K_{J,m}(x-\ell_1, y-\ell_2)\beta(y)\,dy\right| \leq \tilde{C} \sum_{\ell=-2N}^{2N} \int_{-\infty}^{\infty} \frac{\beta(y)\,dy}{|x-y+\ell_2-\ell_1+2^{K-J}\ell|},$$

which implies that whenever $x \in E_\alpha^{\ell_1,\ell_2}$ there is an increasing sequence $J_k \to \infty$ for which

$$\left(\frac{1}{|x-a+\ell_2-\ell_1+2^{K-J_k}\ell|} + \frac{1}{|x-b+\ell_2-\ell_1+2^{K-J_k}\ell|}\right) > C\alpha,$$

for some fixed $C > 0$ and for $k = 1, 2, \ldots$. As $J_k \to \infty$ we see that this implies

$$\left(\frac{1}{|x-a+\ell_2-\ell_1|} + \frac{1}{|x-b+\ell_2-\ell_1|}\right) > C\alpha.$$

Using that $\bar{I} = [0,1)\backslash\tilde{I}$ and the same technique as in the proof of Lemma 2.8, we complete the proof to conclude that $|E_\alpha^{\ell_1,\ell_2}| \lesssim \alpha^{-1}$ and consequently

$$|\bar{I} \cap \{\limsup_J M_J\beta > \alpha/2\}| \leq \frac{\tilde{C}}{\alpha},$$

which completes the proof.

Now we can complete the proof of Theorem 2.3 using standard estimates. The proof of the following lemma can be found in [7, Sec. 6.3].

LEMMA 2.10. *Suppose $\{\beta_k\}_{k=1}^\infty$ is a sequence of q-blocks satisfying*

$$|\{|\beta_k| > \alpha\}| \leq \frac{1}{\alpha}.$$

Then for any sequence $\{c_k\}_{k=1}^\infty \in \ell^1(\mathbb{N})$ and $f = \sum_{k=1}^\infty c_k\beta_k$, we have

$$|\{|f| > \alpha\}| \leq \frac{3|||c|||}{\alpha}.$$

With this result we have:

Proof. (Theorem 2.3) We notice that for $f = \sum_{k=1}^\infty c_k\beta_k \in \mathcal{B}_q$, we have $\Sigma_N f = \sum_{k=1}^\infty c_k \Sigma_N \beta_k$ due to the L^1 convergence of the sum defining f. Hence

$$Gf \leq \sum_{k=1}^\infty |c_k| G\beta_k,$$

and Theorem 2.3 follows by combining Proposition 2.9 and Lemma 2.10.

3. Shannon Wavelet Packets

Now we turn our attention to Shannon wavelet packets. The Shannon filters can be considered the opposite extremal to the Haar filters, and is the only pair of filters with "perfect" frequency localization.

The (stationary) Shannon wavelet packets are defined by taking

$$m_0^S(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\pi/2,\pi/2]}(\xi - 2\pi k)$$

and

$$m_1^S(\xi) = 1 - m_0^S(\xi)$$

in Equations (3) and (4), with w_0 defined by $\hat{w}_0(\xi) = \chi_{[-\pi/2,\pi/2)}(\xi)$. It is possible to finad an explicit expression for $\hat{w}_n$. We define a map $G : \mathbb{N}_0 \to \mathbb{N}_0$ in the following way. Let $n = \sum_{k=1}^{\infty} n_k 2^{k-1}$ be the binary expansion of $n \in \mathbb{N}_0$. Then we let $G(n)_i = n_i + n_{i+1} \pmod 2$, and put $G(n) = \sum_{k=1}^{\infty} G(n)_k 2^{k-1}$. The map G is the so-called Gray-code permutation (one can easily check that G is $1-1$ and onto $\mathbb{N}_0$). The Gray-code permutation relates the Walsh system in Paley order and frequency order, and enters naturally into the frequency localization of more general wavelet packets. We have the following simple formulas for the Shannon wavelet packets. See [11] for a proof.

THEOREM 3.1 ([11]). *Let $\{w_n\}_n$ be the Shannon wavelet packets. Then*

$$\hat{w}_{G(n)}(\xi) = \chi_{[n\pi,(n+1)\pi]}(|\xi|).$$

Note that the Shannon wavelet packets are uniformly bounded just like the Walsh functions due to their perfect frequency localization.

The above result suggests that reordering the Shannon wavelet packets using the inverse Gray-code permutation might improve their convergence properties. We define a new system by letting $\omega_n = w_{G(n)}$ for $n \in \mathbb{N}_0$. We call the reordered system $\{\omega_n\}_{n=0}^{\infty}$ the Shannon wavelet packets in frequency order.

We want to prove that the Shannon wavelet packets form a Schauder basis for the $L^p(\mathbb{R})$-spaces. We need the following sampling theorem. The proof can be found in [4].

THEOREM 3.2 ([4]). *Let* $L_k^\delta(x) = \sin(\pi\delta^{-1}(x-\delta k))/(\pi\delta^{-1}(x-\delta k))$, $0 < \delta \leq 1$, *and let* $\{c_k\}_k \subset \mathbb{C}$. *Then*

$$\|\sum_{k\in\mathbb{Z}} c_k\, L_k^\delta\|_p \simeq \|\{c_k\}\|_{\ell^p(\mathbb{Z})},$$

for $1 < p < \infty$.

REMARK 3.3. *Note that if* $\{c_k\} \in \ell^p(\mathbb{Z})$, $1 < p < \infty$, *then it follows from the Lemma that* $\sum_{k\in\mathbb{Z}} c_k\, L_k^\delta$ *converges unconditionally in* $L^p(\mathbb{R})$.

The following two lemmas will be used to prove the main result, Theorem 3.6. The first is a well known fact and we therefore omit the proof.

LEMMA 3.4. *Let* $f \in L^p(\mathbb{R})$, $1 < p < \infty$. *Define* $f_{a,b} = \mathcal{F}^{-1}\chi_{[a,b]}\mathcal{F}f$, *for* $a, b \in \mathbb{R}$, $a < b$. *Then*

$$\|f_{a,b}\|_p \leq C_p\|f\|_p,$$

for some constant C_p *independent of* a *and* b. *Moreover,*

$$\|f - f_{a,b}\|_p \longrightarrow 0 \qquad \text{as } -a, b \to \infty.$$

$\mathcal{S}(\mathbb{R})$. But
We have the following Lemma which shows that the expansion of each $L^p(\mathbb{R})$-function in the Shannon scaling functions is well behaved.

LEMMA 3.5. *Let*

$$L_k^\delta(x) = \frac{\sin(\pi\delta^{-1}(x-\delta k))}{(\pi\delta^{-1}(x-\delta k))}, \qquad 0 < \delta \leq 1,$$

and suppose $f \in L^p(\mathbb{R})$, $1 < p < \infty$. *Then*

$$\sum_{k\in\mathbb{Z}} \langle f, L_k^\delta\rangle\, L_k^\delta \tag{8}$$

converges unconditionally in $L^p(\mathbb{R})$.

Proof. First, assume that $f \in L^p(\mathbb{R})$ with $\operatorname{supp}(\hat{f}) \subset [-\delta^{-1}\pi, \delta^{-1}\pi]$ (with $\hat{f}$ in the sense of tempered distributions for $2 < p < \infty$.). Note that f is the restriction of an analytic function of exponential type in this special case. We claim that $\sum_k |f(\delta k)|^p \leq C_{p,\delta}\|f\|_p^p$ for some constant $C_{p,\delta}$. Indeed, take $\phi \in \mathcal{S}(\mathbb{R})$ with $\hat{\phi} = 1$ on $[-\delta^{-1}\pi, \delta^{-1}\pi]$. Then, by Plancherel's Theorem,

$$\begin{aligned}\int f(x)\,\bar{\phi}(x-\delta k)\,dx &= \frac{1}{2\pi}\int \hat{f}(\xi)\bar{\hat{\phi}}(\xi)\,\exp(i\delta k\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\delta^{-1}\pi}^{\delta^{-1}\pi} \hat{f}(\xi)\,\exp(i\delta k\xi)\,d\xi \\ &= f(\delta k).\end{aligned}$$

We now apply Hölder's inequality to get

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|f(\delta k)|^p &\le \int |f(x)|^p \sum_{k\in\mathbb{Z}}|\phi(x-\delta k)|\,dx\,\|\phi\|_{p'}^{p/p'}\\ &\le C_{p,\delta}\|f\|_p^p.\end{aligned}$$

Thus, Lemma 3.5 applies to the sequence $f(\delta k)$ and (8) converges unconditionally since $\langle f, L_k^\delta\rangle = f(\delta k)$. For general $f \in L^p(\mathbb{R})$ it suffices to notice that, by Lemma 3.4, the operator $f \to (\chi_{[-\delta^{-1}\pi,\delta^{-1}\pi]}\hat{f})\check{}$ is bounded on $L^p(\mathbb{R})$, and that f and $(\chi_{[-\delta^{-1}\pi,\delta^{-1}\pi]}\hat{f})\check{}$ have the same expansion in the functions $\{L_k^\delta\}$.

Finally, we combine the above Lemmas to get a positive convergence result for expansions in the Shannon wavelet packets in frequency order.

THEOREM 3.6. *The Shannon Wavelet Packet system in frequency order* $\{\omega_n(\cdot - k)\}_{n,k}$ *forms a Schauder basis for* $L^p(\mathbb{R})$, $1 < p < \infty$, *in the sense that*

$$\sum_{n=0}^{N}\sum_{k\in\mathbb{Z}}\langle f, \omega_n(\cdot - k)\rangle\,\omega_n(\cdot - k) \xrightarrow[N\to\infty]{L^p(\mathbb{R})} f,$$

for $f \in L^p(\mathbb{R})$.

Proof. We have $\hat{\omega}_n(\xi) = \chi_{[n\pi,(n+1)\pi)}(|\xi|)$ so

$$\omega_n(x) = (n+1)\frac{\sin((n+1)\pi x)}{(n+1)\pi x} - n\frac{\sin(n\pi x)}{n\pi x} = (n+1)\,L_0^{(n+1)^{-1}} - n\,L_0^{n^{-1}}.$$

Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Lemma 3.5 shows that $\{\langle f, \omega_n(\cdot - k)\rangle\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Hence

$$\sum_{k\in\mathbb{Z}}\langle f, \omega_n(\cdot - k)\rangle\,\omega_n(\cdot - k)$$

converges unconditionally to $P_{\Omega_n}f$, where P_{Ω_n} is the projection onto the closed span of $\{\omega_n(\cdot - k)\}_k$, i.e. $P_{\Omega_n} = \mathcal{F}^{-1}\,\chi_{\{n\pi\le|\xi|<(n+1)\pi\}}\,\mathcal{F}$. So all we have to check is that $\sum_{n=0}^{N} P_{\Omega_n}$ are uniformly bounded in N on $L^p(\mathbb{R})$. But $\sum_{n=0}^{N-1} P_{\Omega_n}$ is just the operator $f \to \mathcal{F}^{-1}\chi_{[-N\pi,N\pi]}\mathcal{F}$, and it is uniformly bounded on $L^p(\mathbb{R})$ by Lemma 3.4.

The above result can also be used to show that the expansion in the Shannon wavelet packets coverges pointwise a.e. Indeed this fact follows directly from the Carleson-Hunt theorem for the line:

THEOREM 3.7 (Carleson-Hunt). *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Define T_R, $R > 0$, by*

$$T_R f(x) = \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(\xi) e^{ix\xi}\, d\xi,$$

and let $(Tf)(x) = \sup_{R>0}(T_R)(x)$. Then T is of strong type (p,p).

We have

COROLLARY 3.8. *Let $\{\omega_n\}_n$ be the Shannon Wavelet Packet system in frequency order Then for $f \in L^p(\mathbb{R})$, $1 < p < \infty$,*

$$\sum_{n=0}^{N} \sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\, \omega_n(x-k) \xrightarrow[N\to\infty]{} f(x), \qquad a.e.$$

Proof. Just note that

$$\left\{ \sum_{n=0}^{N-1} \sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\, \omega_n(x-k) \right\}^{\widehat{}}(\xi) = \hat{f}(\xi)\chi_{[-N\pi,N\pi]}(\xi).$$

3.1. Shannon-Type Wavelet Packets

We now generalize the above results to a class of nonstationary wavelet packets.

DEFINITION 3.9. *Let $\{w_n\}_{n\geq 0}$ be the family of nonstationary wavelet packets constructed using a family $\{h_n^{(p)}\}_{p=1}^{\infty}$ of filters in equations (1) and (2). If there exists a constant $J \in \mathbb{N}$ such that $h^{(p)}$ is the Shannon filter for every $p \geq J$ then we call $\{w_n\}$ a family of Shannon-type wavelet packets.*

In what follows, $[\cdot]$ will denote the function that converts a binary string to the corresponding integer, and $[\cdot]_2$ the function that converts an integer to its binary expansion. For fixed $J \in \mathbb{N}$ we define a permutation $G_J : \mathbb{N} \to \mathbb{N}$ by

$$G_J(n) = \begin{cases} n & \text{if } n \leq 2^J \\ [n_L \cdots n_{L-J+1}[G([n_{L-J}\cdots n_1])]_2] & \text{if } 2^L \leq n < 2^{L+1},\ L \geq J, \end{cases}$$

where $n = [n_L n_{L-1} \cdots n_1]$ is the binary expansion of n. So G_J leaves the J most significant bits unchanged, but performs the Gray-code permutation on the least significant $L - J$ bits.

The frequency ordering of any Shannon-type wavelet packet system $\{w_n\}$ (with J as in definition 3.9) is given by

$$\{\omega_n \equiv w_{G_J(n)}\}_{n=0}^{\infty}.$$

The following result is the analog of Theorem 3.6.

THEOREM 3.10. *Let $\{\omega_n\}_n$ be a system of Shannon-type wavelet packets in frequency order. Then $\{\omega_n(\cdot - k)\}_{n\geq 0, k\in\mathbb{Z}}$ forms a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, in the sense that*

$$\sum_{n=0}^{N} \sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle \omega_n(\cdot - k) \xrightarrow[N\to\infty]{L^p(\mathbb{R})} f,$$

for $f \in L^p(\mathbb{R})$.

Proof. First, let us assume that ω_0 is band limited with $\mathrm{supp}(\hat{\omega}_0) \subset [-K\pi, K\pi]$, $K \in \mathbb{N}$, and that $1 < p < 2$. Define P_j by

$$P_j f(x) = \sum_{k\in\mathbb{Z}} \langle f, \omega_j(\cdot - k)\rangle \omega_j(x - k).$$

We know that the family $\{\sum_{j=0}^{2^L-1} P_j\}_{L\in\mathbb{N}}$ is uniformly bounded on $L^p(\mathbb{R})$ since it is just the projection onto the wavelet space V_L. It therefore suffices to prove that $\sum_{j=2^L}^{m} P_j$ is bounded on $L^p(\mathbb{R})$ with bound independent of $L \in \mathbb{N}$ and $m < 2^{L+1}$. Let J be the scale from which only the Shannon filter is used to generate the wavelet packets. Take $j \in \mathbb{N} : 2^L \leq j < 2^{L+1}$, $L > J$, $G_J(j) = [\epsilon_L \cdots \epsilon_1]$. Then

$$\begin{aligned}
\hat{\omega}_j(\xi) &= m^S_{\epsilon_1}(\xi/2) m^S_{\epsilon_2}(\xi/2^2) \cdots m^S_{\epsilon_{L-J}}(\xi/2^{L-J}) \\
&\qquad \times m^{(J)}_{\epsilon_{L-J+1}}(\xi/2^{L-J+1}) \cdots m^{(1)}_{\epsilon_L}(\xi/2^L)\hat{\omega}_0(\xi/2^L) \\
&\equiv \Big\{ \sum_{s=-2^{J-1}K}^{2^{J-1}K} \chi_{I_j}(\xi - 2^{L-J+1}\pi s) \Big\} \hat{\omega}_{i_j}(\xi/2^{L-J}),
\end{aligned}$$

where $i_j = [1\epsilon_{L-1} \cdots \epsilon_{L-J+1}]_2 \in [0, 2^J - 1]$ and $2^{J-L} I_j \subset [-\pi, \pi]$ is symmetric about 0 and is a union of two intervals, each of length $2^{J-L}\pi$ (follows from Theorem 3.1). For a function g we will use the notation $g|_I^{\mathrm{per}}$ to denote the 2π-periodic function obtained by taking the 2π-periodic extension of the restriction of g to I. Using Plancherel's theorem we have

$$\begin{aligned}
\widehat{P_j f}(\xi) &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}\hat f(t)\sum_{s=-2^{J-1}K}^{2^{J-1}K}\chi_{I_j}(t-2^{L-J+1}\pi s)\times \\
&\qquad\qquad \overline{\hat\omega_{i_j}(t2^{J-L})}e^{ikt}\,dte^{-ik\xi}\hat\omega_j(\xi) \\
&= \frac{1}{2\pi}\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{k\in\mathbb{Z}}\int_{I_j}\hat f(t+2^{L-J+1}\pi s)\times \\
&\qquad\qquad \overline{\hat\omega_{i_j}\left(\frac{t+2^{L-J+1}\pi s}{2^{L-J}}\right)}e^{ikt}\,dte^{-ik\xi}\hat\omega_j(\xi) \\
&= \sum_{s=-2^{J-1}K}^{2^{J-1}K}\left[\hat f(\xi+2^{L-J+1}\pi s)\overline{\hat\omega_{i_j}\left(\frac{\xi+2^{L-J+1}\pi s}{2^{L-J}}\right)}\right]\Bigg|_{I_j}^{\mathrm{per}}\times \\
&\qquad\qquad \sum_{r=-2^{J-1}K}^{2^{J-1}K}\chi_{I_j}(\xi-2^{L-J+1}\pi r)\hat\omega_{i_j}(\xi/2^{L-J}) \\
&= \Bigg\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\left[\hat f(\xi+2^{L-J+1}\pi s)\overline{\hat\omega_{i_j}\left(\frac{\xi+2^{L-J+1}\pi s}{2^{L-J}}\right)}\right]\Bigg|_{I_j}^{\mathrm{per}}\times \\
&\qquad\qquad \chi_{I_j}(\xi-2^{L-J+1}\pi r)\Bigg\}\hat\omega_{i_j}(\xi/2^{L-J}) \\
&= \Bigg\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\hat f(\xi+2^{L-J+1}\pi(s-r))\times \\
&\qquad \overline{\hat\omega_{i_j}\left(\frac{\xi+2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)}\chi_{I_j}(\xi-2^{L-J+1}\pi r)\Bigg\}\hat\omega_{i_j}(\xi/2^{L-J})
\end{aligned}$$

Note that the inverse Fourier transform of each term

$$\hat f(\xi+2^{L-J+1}\pi(s-r))\overline{\hat\omega_{i_j}\left(\frac{\xi+2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)}\chi_{I_j}(\xi-2^{L-J+1}\pi r)$$

is the convolution of f with an L^1 function of norm $\|w_{i_j}\|_1$ and then composed with the bounded operator given by the multiplier $\chi_{I_j}(\xi-2^{L-J+1}\pi r)$. Thus, $P_j f$ is a finite sum of convolutions of $L^p(\mathbb{R})$ functions, all with L^p-norm $\le C\|f\|_p$ (with C independent of j), and a function of fixed L^1-norm. So P_j is bounded on $L^p(\mathbb{R})$. In general, for $m-2^L = k2^{L-J}+d$ with $d\in[0,2^{L-J})$,

$$\sum_{j=2^L}^{m}\widehat{P_j f}(\xi) =$$

$$\sum_{l=0}^{k-1}\Big\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\hat{f}(\xi+2^{L-J+1}\pi(s-r))\hat{\omega}_l\overline{\Big(\frac{\xi+2^{L-J+1}\pi(s-r)}{2^{L-J}}\Big)}$$
$$\times\sum_{j=2^L+l2^{L-J}}^{2^L+(l+1)2^{L-J}-1}\chi_{I_j}(\xi-2^{L-J+1}\pi r)\Big\}\hat{\omega}_l(\xi/2^{L-J})$$

$$+\Big\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\hat{f}(\xi+2^{L-J+1}\pi(s-r))\hat{\omega}_k\overline{\Big(\frac{\xi+2^{L-J+1}\pi(s-r)}{2^{L-J}}\Big)}$$
$$\times\sum_{j=2^L+k2^{L-J}}^{2^L+k2^{L-J}+d}\chi_{I_j}(\xi-2^{L-J+1}\pi r)\Big\}\hat{\omega}_k(\xi/2^{L-J})$$

However,

$$\sum_{j=2^L+l2^{L-J}}^{2^L+(l+1)2^{L-J}-1}\chi_{I_j}(\xi-2^{L-J+1}\pi r)\quad\text{and}\quad\sum_{j=2^L+k2^{L-J}}^{2^L+k2^{L-J}+d}\chi_{I_j}(\xi-2^{L-J+1}\pi r)$$

are each the characteristic function of an interval (follows from Theorem 3.1 and the ordering of the functions). The same argument as above applies and $\sum_{j=2^L}^{m} P_j$ is therefore bounded on $L^p(\mathbb{R})$ with bounds independent of m and L. More generally, if ω_0 is not band limited we can always find an isometry on $L^p(\mathbb{R})$ mapping the wavelet packet system onto a band limited Shannon-type wavelet packet system (see, e.g., [4, Chap. 6]). The case $2 < p < \infty$ follows easily by a duality argument.

4. Appendix: Some Elementary Properties of Walsh Functions

In this appendix we define the Walsh system and prove two lemmas we will need in the following section. The reader can find much more information on the Walsh system in the monographs [2, 7].

We need two equivalent definitions of the Walsh system on $[0,1)$. The first one fit into the wavelet packet scheme

DEFINITION 4.1. *The Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is defined recursively on* $[0,1)$ *by letting* $W_0 = \chi_{[0,1)}$ *and*

$$W_{2n}(x) = W_n(2x) + W_n(2x-1)$$
$$W_{2n+1}(x) = W_n(2x) - W_n(2x-1).$$

We note that the Walsh system is the family of wavelet packets obtained by letting $\phi = \chi_{[0,1)}$, $\psi = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, and using only the Haar filters in the definition of the nonstationary wavelet packets.

The Walsh system is closed under pointwise multiplication. Define the binary operator $\oplus : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by

$$m \oplus n = \sum_{i=0}^{\infty} |m_i - n_i| 2^i,$$

where $m = \sum_{i=0}^{\infty} m_i 2^i$ and $n = \sum_{i=0}^{\infty} n_i 2^i$. Then, see [7],

$$W_m(x) W_n(x) = W_{m \oplus n}(x). \tag{9}$$

We can carry over the operator $\oplus$ to the interval $[0,1]$ by identifying those $x \in [0,1]$ with a unique expansion $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (almost all $x \in [0,1]$ has such a unique expansion) by their associated binary sequence $\{x_i\}$. For two such points $x, y \in [0,1]$ we define

$$x \oplus y = \sum_{j=0}^{\infty} |x_j - y_j| 2^{-j-1}.$$

Notice that the operation $\oplus$ is not defined for every pair $x, y \in [0,1]$ but only for a.a. $x, y \in [0,1]$ which is sufficient for our purposes. With this definition we have [2, p. 11]

$$W_n(x \oplus y) = W_n(x) W_n(y) \tag{10}$$

for every pair x, y for which $x \oplus y$ is defined.

The following lemma was proved in [6]

LEMMA 4.2. *Let $f_1 \in L^2(\mathbb{R})$, and define $\{f_n\}_{n \geq 2}$ recursively by*

$$f_{2n+\epsilon}(x) = f_n(2x) + (-1)^{\epsilon} f_n(2x-1), \qquad \epsilon = 0, 1.$$

Then for $n, J \in \mathbb{N}$, $2^J \leq n < 2^{J+1}$, we have

$$f_n(x) = \sum_{s=0}^{2^J - 1} W_{n-2^J}(s 2^{-J}) f_1(2^J x - s).$$

The final result we need deal with size estimates of Dirichlet kernels for the Walsh system.

LEMMA 4.3. *There exists a finite constant C such that*

$$\left| \sum_{n=2^K}^{m} W_{n-2^K}([2^K x] 2^{-K}) W_{n-2^K}([2^K y] 2^{-K}) \right| \leq \frac{C}{x \oplus y},$$

with $K \geq 1, 2^K \leq m < 2^{K+1}$, and for all pairs $x, y \in [0,1)$ for which $x \oplus y$ is defined.

Proof. The Dirichlet kernel, $D_n(x) = \sum_{k=0}^{n-1} W_k(x)$, for the Walsh system satisfies $|D_n(x \oplus y)| \le (x \oplus y)^{-1}$, see [2, p. 21]. Hence,

$$\begin{aligned}
&\Big|\sum_{n=2^K}^{m} W_{n-2^K}([2^K x]2^{-K}) W_{n-2^K}([2^K y]2^{-K})\Big| \\
&\quad = \Big|\sum_{n=2^K}^{m} W_{n-2^K}([2^K x]2^{-K} \oplus [2^K y]2^{-K})\Big| \\
&\quad = |W_{2^K}([2^K x]2^{-K} \oplus [2^K y]2^{-K})) D_{m+1-2^K}([2^K x]2^{-K} \oplus [2^K y]2^{-K})| \\
&\quad = |D_{m+1-2^K}(x \oplus y)| \\
&\quad \le (x \oplus y)^{-1},
\end{aligned}$$

where we have used (9), (10), and the fact that D_{m+1-2^K} is constant on dyadic intervals of the form $[\ell 2^{-K}, (\ell+1)2^{-K})$.

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