EAST JOURNAL ON APPROXIMATIONS
Volume 7, Number 2 (2001), 1-19

SOME REMARKS ON NON-LINEAR APPROXIMATION WITH SCHAUDER BASES

R. Gribonval and M. Nielsen

IMI, Department of Mathematics, University of South Carolina
South Carolina 29208, USA
E-mail: remi.gribonval@inria.fr
nielsen@math.sc.edu

We study the approximation classes $\mathcal{A}_s^\alpha$ and $\mathcal{G}_s^\alpha$ associated with non-linear m-term approximation and greedy approximation by elements from a quasi-normed Schauder basis in a separable Banach space. We show that there always is a two-sided embedding

$$\mathcal{K}_s^{\tau_p} \hookrightarrow \mathcal{A}_s^\alpha \hookrightarrow \mathcal{K}_s^{\tau_q},$$

where $\mathcal{K}_s^\tau$ denotes the associated smoothness space. We provide estimates of τ_p and τ_q in terms of quantitative properties of the basis. The lower and upper estimates are sharp for so-called quasi-greedy bases, but may not coincide with each other to completely characterize $\mathcal{A}_s^\alpha$. For a quasi-greedy and democratic basis, a complete characterization $\mathcal{G}_s^\alpha = \mathcal{K}_s^{1/\alpha}(w)$ is obtained where w is a weight depending on the properties of the basis. For greedy bases, $\mathcal{G}_s^\alpha = \mathcal{A}_s^\alpha$ but the converse is not true. The results in this paper can be considered a generalization of the characterization for an orthonormal basis $\mathcal{B}$ in a Hilbert space $\mathcal{H}$, where it is well known that

$$\mathcal{A}_s^\alpha(\mathcal{B}) = \mathcal{K}_s^\tau(\mathcal{B}),$$

with $\alpha = \frac{1}{\tau} - \frac{1}{2}$ and $s \in (0, \infty]$.

Key words and phrases: non-linear approximation, best m-term approximation, smoothness space, approximation class, Schauder basis, quasi-greedy basis, greedy basis, democratic basis, sandwich property.

1. Introduction

Let X be a separable Banach space, and let $\mathcal{S} = \{g_k\}_{k\geq 1}$ be a quasi-normed Schauder basis of X, i.e., a basis that satisfies $\inf_k \|g_k\|_X > 0$ and $\sup_k \|g_k\|_X < \infty$. For any given $f \in X$, the error associated to the *best m-term* approximation to f from $\mathcal{S}$ is given by

$$\sigma_m(f, \mathcal{S})_X = \inf_{\Lambda\subset\mathbb{N}:|\Lambda|=m, \{c_k\}_{k\in\Lambda}\in\mathbb{C}^m} \big\|f - \sum_{k\in\Lambda} c_k g_k\big\|_X. \tag{1}$$

We are interested in the characterization of *approximation classes*:

$$\mathcal{A}_s^\alpha(\mathcal{S}) = \Big\{ f \in X, \|f\|_{\mathcal{A}_s^\alpha(\mathcal{S})} := \|f\|_X + \|\{\sigma_m(f,\mathcal{S})_X\}_{m\geq 1}\|_{\ell_{1/\alpha,s}} < \infty \Big\} \tag{2}$$

which are defined using the Lorentz (quasi-)norm; for $0 < \tau < \infty$ and $0 < s \leq \infty$:

$$\|\{a_m\}_{m=1}^\infty\|_{\ell_{\tau,s}} := \begin{cases} \left(\sum_{m=1}^{\infty} \frac{[m^{1/\tau}|a_m^\star|]^s}{m} \right)^{1/s}, & 0 < s < \infty \\ \sup_{m\in\mathbb{N}} m^{1/\tau}|a_m^\star|, & s = \infty, \end{cases} \tag{3}$$

where $\{|a_k^\star|\}_k$ denotes a decreasing rearrangement of $\{a_k\}_k$.

Remark 1.1.

1. Notice that $\|\cdot\|_{\ell_{\tau,\tau}} = \|\cdot\|_{\ell_\tau}$.

2. Throughout this paper we will use the notation $V \hookrightarrow W$, where V and W are (quasi-)normed spaces, whenever $V \subset W$ and $\|\cdot\|_W \leq C\|\cdot\|_V$ for some $C < \infty$, which we will denote by $\|\cdot\|_W \lesssim \|\cdot\|_V$. The equivalence of (quasi)-norms, *i.e.* $\|\cdot\|_W \lesssim \|\cdot\|_V$ and $\|\cdot\|_V \lesssim \|\cdot\|_W$, will be denoted by $\|\cdot\|_W \asymp \|\cdot\|_V$. It can be verified [4] that the Lorentz spaces $\ell_{\tau,s}$, defined by

$$\ell_{\tau,s} = \{\{c_k\} : \|\{c_k\}\|_{\ell_{\tau,s}} < \infty\},$$

satisfy the continuous embedding $\ell_{\tau_1,s_1} \hookrightarrow \ell_{\tau_2,s_2}$ provided that $\tau_1 < \tau_2$ or $\tau_2 = \tau_1$ with $s_1 \leq s_2$.

$\mathcal{A}_s^\alpha(\mathcal{S})$ is thus basically the set of functions f that can be approximated at a given rate $\mathcal{O}(m^{-\alpha})$ $(0 < \alpha < \infty)$ by m elements from the basis. The parameter $0 < s \leq \infty$ is auxiliary and gives a finer classification of the approximation rate.

Approximation classes are often related to "smoothness" classes, that is to say classes where the coefficients $\{c_k(f)\}$ have a "fast" decay. The following characterization was proved by Stechkin [8] for the case $\tau = 1$ and for general τ by DeVore and Temlyakov [1] when the basis $\mathcal{S}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$.

Theorem 1.1 ([8, 1]). *If $\mathcal{B} = \{h_k\}_{k\geq 1}$ is an orthonormal basis of $\mathcal{H}$, then*

$$\mathcal{A}_s^\alpha(\mathcal{B}) = \left\{f \in \mathcal{H},\ \|\{\langle f, h_k\rangle\}_{k\geq 1}\|_{\ell_{\tau,s}} < \infty\right\} \tag{4}$$

with $\alpha = \frac{1}{\tau} - \frac{1}{2}$. Moreover, we have the norm equivalence

$$\|f\|_{\mathcal{A}_s^\alpha(\mathcal{B})} \asymp \|\{\langle f, h_k\rangle\}_{k\geq 1}\|_{\ell_{\tau,s}}. \tag{5}$$

The fundamental tools to prove these results are Hardy's inequalities. Less is known when $\mathcal{S}$ is not an orthonormal basis of a Hilbert space, and the purpose of this paper is to generalize Theorem 1.1 to general quasi-normed Schauder bases for a Banach space. This will be done in Section 3. In Section 4 we will show that for quasi-greedy bases the results of Section 3 are the best possible but may not lead to a complete characterization of $\mathcal{A}_s^\alpha(\mathcal{S})$. In Section 5 we consider bases with more structure and characterize completely $\mathcal{A}_s^\alpha(\mathcal{S})$, for $\mathcal{S}$ a greedy basis, in terms of weighted Lorentz spaces. We also define *greedy approximation spaces* $\mathcal{G}_s^\alpha(\mathcal{S})$ and get a similar characterization with the weaker assumption that $\mathcal{S}$ is quasi-greedy and democratic. We consider some specific examples of the main result for Banach spaces which are uniformly smooth and uniformly convex in Section 6.

2. Notations and definitions

We begin by summarizing in this section the notations and definitions which will be used throughout this paper.

Definition 2.1. Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a quasi-normed Schauder basis for the Banach space X. For any $f \in X$ and $m \geq 1$, a *greedy m-term approximant* to f from $\mathcal{S}$ is any vector

$$G_m(f, \mathcal{S}, \pi) := \sum_{k=1}^{m} c_k^\star g_{\pi(k)},$$

where $\{c_k^\star\} = \{c_{\pi(k)}\}$ is a decreasing rearrangement of $\{c_k(f)\}$. The error associated to greedy m-term approximation to f from $\mathcal{S}$ is denoted by

$$\gamma_m(f, \mathcal{S}, \pi)_X := \|f - G_m(f, \mathcal{S}, \pi)\|_X. \tag{6}$$

Note that greedy approximants are sometimes referred to as *thresholding approximants* or *thresholding greedy approximants*. In the following, we will simply denote $G_m(f, \mathcal{S})$ and $\gamma_m(f, \mathcal{S})_X$. Any statement on these quantities will be assumed to hold for all π such that $\{c_{\pi(k)}\}$ is a decreasing rearrangement of $\{c_k(f)\}$.

Definition 2.2. Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a quasi-normed Schauder basis for the Banach space X. We call $\mathcal{S}$ a *quasi-greedy* basis if for each $f \in X$ we have $\gamma_m(f, \mathcal{S})_X \to 0$ as $m \to \infty$.

Remark 2.1. The notion of a quasi-greedy basis was introduced in [6] and studied further in [11]. It is clear that every quasi-normed unconditional basis for X will also be quasi-greedy but it is known that the converse result is false [6, Section 3.4], so being quasi-greedy is a weaker condition than being unconditional.

Definition 2.3. Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a Schauder basis for the Banach space X. We call $\mathcal{S}$ a *greedy* basis if there exists a constant $C < \infty$ such that for each $f \in X$, we have for all m

$$\gamma_m(f, \mathcal{S}) \leq C\sigma_m(f)_X. \tag{7}$$

The following theorem was proved in [6], where the notion of greedy basis was introduced.

Theorem 2.1. *A Schauder basis $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ is a greedy basis of the space X if and only if it is unconditional and democratic.*

By *democratic* we mean the following:

Definition 2.4. A Schauder basis $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ is democratic if there exists a constant $C < \infty$ such that for every two finite sets $\Lambda, \Lambda' \subset \mathbb{N}$ of same cardinality $|\Lambda| = |\Lambda'|$ we have

$$\Big\| \sum_{k\in\Lambda} g_k \Big\| \leq C \Big\| \sum_{k\in\Lambda'} g_k \Big\|.$$

Clearly, democracy implies that the basis is quasi-normed by taking $|\Lambda| = 1$ in the definition. Moreover, unconditionality and democracy imply *super-democracy* [6]:

Definition 2.5. A basis $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ is super-democratic if there exists a constant $C < \infty$ such that for every two finite sets $\Lambda, \Lambda' \subset \mathbb{N}$ of same cardinality $|\Lambda| = |\Lambda'|$ and every choice of signs $\theta_k = \pm 1$ and $\varepsilon_k = \pm 1$ we have

$$\Big\| \sum_{k\in\Lambda} \theta_k g_k \Big\| \leq C \Big\| \sum_{k\in\Lambda'} \varepsilon_k g_k \Big\|.$$

However it was proved in [6, Section 3] that unconditionality does not imply democracy, and neither democracy nor super-democracy imply unconditionality.

For a quasi-greedy basis, it is of interest to characterize at which rate the (thresholding) greedy approximant $G_m(f,\mathcal{S})$ converges to f. This is done by defining *(thresholding) greedy approximation spaces*

$$\mathcal{G}^\alpha_s(\mathcal{S}) := \Big\{ f \in X, \|f\|_{\mathcal{G}^\alpha_s(\mathcal{S})} := \|f\|_X + \big\|\{\gamma_m(f,\mathcal{S})_X\}_{m\geq 1}\big\|_{\ell_{1/\alpha,s}} < \infty \Big\}. \tag{8}$$

For $\mathcal{S}$ a greedy basis, $\mathcal{G}^\alpha_s(\mathcal{S}) = \mathcal{A}^\alpha_s(\mathcal{S})$ with equivalent norms $\|\cdot\|_{\mathcal{G}^\alpha_s(\mathcal{S})} \asymp \|\cdot\|_{\mathcal{A}^\alpha_s(\mathcal{S})}$, because $\sigma_m(f,\mathcal{S}) \asymp \gamma_m(f,\mathcal{S})$. Obviously, if $\mathcal{G}^\alpha_s(\mathcal{S}) = \mathcal{A}^\alpha_s(\mathcal{S})$ for some α and s, then $\mathcal{S}$ is α-greedy [10], that is, for any $f \in X$, if $\sigma_m(f,\mathcal{S}) \lesssim m^{-\alpha}$ then $\gamma_m(f,\mathcal{S}) \lesssim m^{-\alpha}$. However we will see examples in Section 6 showing that $\mathcal{G}^\alpha_s(\mathcal{S}) = \mathcal{A}^\alpha_s(\mathcal{S})$ does not imply that $\mathcal{S}$ is greedy.

Let us now introduce the so-called "smoothness" spaces. For $\tau \in (0,\infty)$ and $s \in (0,\infty]$, we let $\mathcal{K}^\tau_s(\mathcal{S}, M)$ denote the set

$$\mathrm{clos}_X \Big\{ f \in X \mid \exists \Lambda \subset \mathbb{N}, |\Lambda| < \infty,\ f = \sum_{k\in\Lambda} c_k g_k,\ \|\{c_k\}\|_{\ell_{\tau,s}} \leq M \Big\}.$$

Then we define

$$\mathcal{K}^\tau_s(\mathcal{S}) := \cup_{M>0} \mathcal{K}^\tau_s(\mathcal{S}, M), \tag{9}$$

with

$$\|f\|_{\mathcal{K}^\tau_s(\mathcal{S})} = \inf\{M : f \in \mathcal{K}^\tau_s(\mathcal{S}, M)\}.$$

Remark 2.2. In a Hilbert space $\mathcal{H}$, consider $\mathcal{K}^\tau_s(\mathcal{S})$ with $\tau \in (0,2)$ and suppose that the basis $\mathcal{S}$ is hilbertian, i.e., for every ℓ_2 sequence of scalars $\{c_k\}$, the sum $\sum_k c_k g_k$ is convergent in $\mathcal{H}$. As $\ell_{\tau,s} \hookrightarrow \ell_2$ one can check that

$$\mathcal{K}^\tau_s(\mathcal{S}) = \Big\{ f = \sum_k c_k g_k \in \mathcal{H}, \|\{c_k\}\|_{\ell_{\tau,s}} < \infty \Big\}, \tag{10}$$

where Fatou's Lemma can be used to obtain the $\subseteq$ inclusion in (10).

Generalized smoothness spaces $\mathcal{K}^\tau_s(w,\mathcal{S})$ can be defined similarly using weighted Lorentz norms, where the weights $w = \{w_m\}$ form a slowly increasing sequence (i.e., $w_{2m} \leq C w_m$ for all m):

$$\|\{a_m\}_{m=1}^\infty\|_{\ell_{\tau,s}(w)} := \begin{cases} \left(\sum_{m=1}^\infty \dfrac{[w_m m^{1/\tau} |a^\star_m|]^s}{m} \right)^{1/s}, & 0 < s < \infty \\ \sup_{m\in\mathbb{N}} w_m m^{1/\tau} |a^\star_m|, & s = \infty. \end{cases} \tag{11}$$

One can notice that for weights $w_m = m^{1/p}$, the weighted Lorentz spaces reduce to standard ones $\ell_{1/\alpha,s}(\{m^{1/p}\}) = \ell_{\tau_p,s}$ where $1/\tau_p = \alpha + 1/p$.

3. Best m-term approximation with general quasi-normed Schauder bases

Let us consider a quasi-normed Schauder basis $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ for a Banach space X. Since the basis is quasi-normed, it is known (see [12]) that there exist constants $0 < A \leq B < \infty$ such that for every $f = \sum_k c_k(f) g_k \in X$ we have

$$A\|\{c_k(f)\}\|_{\ell_\infty} \leq \|f\|_X \leq B\|\{c_k(f)\}\|_{\ell_1}.$$

For any pair $1 \leq p \leq q \leq \infty$ we can thus ask whether the estimate

$$\|\{c_k(\cdot)\}\|_{\ell_{q,\infty}} \lesssim \|\cdot\|_X \lesssim \|\{c_k(\cdot)\}\|_{\ell_{p,1}} \tag{12}$$

holds. For quasi-normed Schauder bases we have the following result, generalizing Theorem 1.1:

Theorem 3.1. *Let $\mathcal{S}$ be a quasi-normed Schauder basis for a Banach space X. For every pair (p,q), $1 \leq p \leq q \leq \infty$, such that* (12) *is satisfied, we have for $\alpha > 0, s \in (0,\infty]$:*

$$\mathcal{K}_s^{\tau_p}(\mathcal{S}) \hookrightarrow \mathcal{A}_s^{\alpha}(\mathcal{S}) \hookrightarrow \mathcal{K}_s^{\tau_q}(\mathcal{S}), \tag{13}$$

with

$$\frac{1}{\tau_p} = \frac{1}{p} + \alpha \quad \text{and} \quad \frac{1}{\tau_q} = \frac{1}{q} + \alpha.$$

Remark 3.1.

1. For a general quasi-normed Schauder basis we get a "weaker" result than for an orthonormal basis in the sense that the approximation class is not entirely characterized as a smoothness space by Theorem 3.1. Indeed, the only case where the theorem gives an exact characterization is when $p = q$ can be realized in (12), that is, when X can be "sandwiched" between $\ell_{p,1}$ and $\ell_{p,\infty}$. The theorem then reduces to a variant of Hardy's inequality. We will see in Section 4 that such a "sandwich" assumption is also almost necessary for the class of quasi-greedy bases. It is only natural that we have to pay a price to use a less structured basis.

2. For certain Schauder bases with special structure in $L_p(\mathbb{R})$, results similar to Theorem 3.1 are known, see e.g. [3, 7]. Section 5 of [9] contains results in the same spirit as Theorem 3.1 for wavelet type systems in $L_p(\mathbb{R})$. We will prove in Section 5 an extension of the results in [9] for the class of quasi-greedy bases.

3. That we need some structure (and not just a set with dense span) to get a result like Theorem 3.1 will be demonstrated at the end of this section with an explicit example.

We now give the proof of Theorem 3.1. We will use some basic properties of the real interpolation method of Lions and Peetre. The reader can find more information on this topic and the notation used below in [4, Chap. 6].

Proof of Theorem 3.1. Let $1 \le p \le q \le \infty$ be such that (12) is satisfied. Given $\alpha \in (0,\infty)$, we take τ with $\tau < p$ such that $\alpha < \tau^{-1} - p^{-1}$, this choice will be justified later. We put $Y_1 = \mathcal{K}^\tau_\infty(\mathcal{S})$ and $Y_2 = \mathcal{K}^\tau_1(\mathcal{S})$. The proof has two steps. First, we will prove a two-sided embedding of the approximation class in interpolation spaces of the type $(X, Y_i)_{\beta,s}$. Then we will find two-sided embeddings of the interpolation spaces $(X, Y_i)_{\beta,s}$ into spaces that can be identified with sequence spaces.

For $S = \sum_{k=1}^n c_k g_{n_k}$, using (12), we obtain

$$\begin{aligned} \|S\|_{Y_1} &= \sup_{1\le k\le n} k^{\frac{1}{\tau}} c_k^\star = \sup_{1\le k\le n} k^{\frac{1}{\tau}-\frac{1}{q}} k^{\frac{1}{q}} c_k^\star \\ &\le n^{\frac{1}{\tau}-\frac{1}{q}} \sup_{1\le k\le n} k^{\frac{1}{q}} c_k^\star = n^{\frac{1}{\tau}-\frac{1}{q}} \|\{c_k(S)\}\|_{\ell_{q,\infty}} \\ &\le C n^{\frac{1}{\tau}-\frac{1}{q}} \|S\|_X. \end{aligned}$$

Hence, we have the Bernstein inequality with exponent $r_1 := \frac{1}{\tau} - \frac{1}{q} > 0$. Notice that $0 < \alpha < \tau^{-1} - p^{-1} \le r_1$. It follows that for $s \in (0,\infty]$, see [4, Chap. 7],

$$\mathcal{A}^\alpha_s(\mathcal{S}) \hookrightarrow (X, Y_1)_{\alpha/r_1,s}. \tag{14}$$

Now we proceed to get a Jackson type inequality. We will use the following notation: we let $\Theta_m(c)$ be the thresholding operator that keeps only the m largest elements of a sequence $c = \{c_k\}_{k=1}^\infty$, and for $f = \sum_k c_k g_k \in Y_2$ we let $f_m = \sum_k [\Theta_m(c)]_k g_k$. We first notice that by (12), we have

$$\sigma_m(f)_X \le \|f - f_m\|_X \le C\|c - \Theta_m(c)\|_{\ell_{p,1}}.$$

Moreover, a Jackson inequality holds in $\ell_{p,1}$ which is analog to Theorem 1.1, namely,

$$\begin{aligned} \|c - \Theta_m(c)\|_{\ell_{p,1}} &= \sum_{k=m+1}^\infty (k-m)^{\frac{1}{p}-1} c_k^\star \le C m^{\frac{1}{p}-\frac{1}{\tau}} \sum_{k=m+1}^\infty k^{\frac{1}{\tau}-1} c_k^\star \\ &\le C m^{-\left(\frac{1}{\tau}-\frac{1}{p}\right)} \|\{c_k\}\|_{\ell_{\tau,1}} \qquad (15) \end{aligned}$$

and it gives the desired result:

$$\sigma_m(f)_X \le C m^{-r_2} \|f\|_{Y_2} \tag{16}$$

with $r_2 := \frac{1}{\tau} - \frac{1}{p}$. The choice of τ is such that $0 < \alpha < r_2$, and for $s \in (0, \infty]$ we have [4, Chap. 7],

$$(X, Y_2)_{\alpha/r_2,s} \hookrightarrow \mathcal{A}_s^\alpha(\mathcal{S}). \tag{17}$$

Now, we will look closer at the spaces $(X, Y_i)_{\theta,s}$ for $\theta \in (0,1)$. Define the operator T by

$$T\left(\sum_{k=1}^\infty c_k g_k\right) = \{c_k\}_{k=1}^\infty.$$

Notice that T is continuous as a mapping on the following spaces

$$T : X \to \ell_{q,\infty}(\mathbb{N}),$$
$$T : Y_1 \to \ell_{\tau,\infty}(\mathbb{N}).$$

Hence, by interpolation, for $\theta \in (0,1)$ and $s \in (0,\infty]$, the mapping

$$T : (X, Y_1)_{\theta,s} \to (\ell_{q,\infty}(\mathbb{N}), \ell_{\tau,\infty}(\mathbb{N}))_{\theta,s}$$

is continuous. Conversely, we define (formally)

$$U(\{c_k\}_{k=1}^\infty) = \sum_{k=1}^\infty c_k g_k,$$

and we see that U is continuous as a mapping on:

$$U : \ell_{p,1}(\mathbb{N}) \to X,$$
$$U : \ell_{\tau,1}(\mathbb{N}) \to Y_2.$$

Thus, for $\theta \in (0,1)$ and $s \in (0,\infty]$,

$$U : (\ell_{p,1}(\mathbb{N}), \ell_{\tau,1}(\mathbb{N}))_{\theta,s} \to (X, Y_2)_{\theta,s}$$

is continuous. Combining this with (14), (17), and using the characterization of the interpolation classes between $\ell_{p,s}$ spaces, see [2, p. 39], we finally obtain

$$\mathcal{K}_s^{\tau_p}(\mathcal{S}) = U\ell_{\tau_p,s}(\mathbb{N}) = U(\ell_{p,1}(\mathbb{N}), \ell_{\tau,1}(\mathbb{N}))_{\alpha/r_2,s} \hookrightarrow (X, Y_2)_{\alpha/r_2,s} \hookrightarrow \mathcal{A}_s^\alpha(\mathcal{S}),$$

and

$$T\mathcal{A}_s^\alpha(\mathcal{S}) \hookrightarrow T(X, Y_1)_{\alpha/r_1,s} \hookrightarrow (\ell_{q,\infty}(\mathbb{N}), \ell_{\tau,\infty}(\mathbb{N}))_{\alpha/r_1,s} = \ell_{\tau_q,s}(\mathbb{N}), \tag{18}$$

with

$$\frac{1}{\tau_p} := \left(1 - \frac{\alpha}{r_2}\right)\frac{1}{p} + \frac{\alpha}{r_2}\frac{1}{\tau} \quad \text{and} \quad \frac{1}{\tau_q} := \left(1 - \frac{\alpha}{r_1}\right)\frac{1}{q} + \frac{\alpha}{r_1}\frac{1}{\tau},$$

which can be reduced to

$$\frac{1}{\tau_p} = \frac{1}{p} + \alpha \quad \text{and} \quad \frac{1}{\tau_q} = \frac{1}{q} + \alpha.$$

Notice that since $\mathcal{S}$ is a Schauder basis, (18) implies that

$$\mathcal{A}_s^\alpha(\mathcal{S}) \hookrightarrow \mathcal{K}_s^{\tau_q}(\mathcal{S}),$$

which completes the proof. $\square$

Let us conclude this section by considering collections of normalized vectors that do not form Schauder bases. So far we have only considered the relationship between the approximation and smoothness spaces associated with Schauder bases. We can define the approximation classes and smoothness spaces by the analogs of (2) and (9), respectively, for any set $\mathcal{U}$ of unit vectors with dense span in X. One can then pose the question whether it is possible to get results like Theorem 3.1 for general sets with dense span, without assuming they form a Schauder basis. This is not the case, in general, and we conclude this section by giving an example of a spanning non-redundant set $\mathcal{U}$ in a Hilbert space $\mathcal{H}$, which fails to be a Schauder basis for $\mathcal{H}$ and for which the upper embedding of Theorem 3.1 fails to be true no matter which combination of parameters α and τ' one chooses.

Proposition 3.1. *Let $\mathcal{H} = \bigoplus_{j\geq 0} \mathcal{V}_j$ be an orthogonal decomposition of $\mathcal{H}$ into two-dimensional subspaces. Let $\{e_{2j}, e_{2j+1}\}$ be a normalized basis of $\mathcal{V}_j$ such that $\langle e_{2j}, e_{2j+1}\rangle = \cos\phi_j$, $\phi_j > 0$, and $\phi_j \to 0$. Let $\mathcal{U} = \{e_k\}_{k\geq 0}$. Then*

$$\mathcal{A}_s^\alpha(\mathcal{U}) \not\hookrightarrow \mathcal{K}_{s'}^{\tau'}(\mathcal{U})$$

for any combination of parameters $0 < \alpha, \tau' < \infty$, and $0 < s, s' \leq \infty$.

Proof. We define a sequence $\{f_j\}$ for which

$$\|f_j\|_{\mathcal{A}_s^\alpha(\mathcal{U})} \to 0 \quad \text{and} \quad \|f_j\|_{\mathcal{K}_{s'}^{\tau'}(\mathcal{U})} \geq 1,$$

which will prevent any type of continuous embedding of the approximation class into the smoothness space. More precisely, we let

$$f_j = \cos\phi_j e_{2j} - e_{2j+1}$$

and check that $\|f_j\|_{\mathcal{H}} = |\sin \phi_j| \to 0$. Hence it is clear that

$$\|f_j\|_{\mathcal{A}_s^\alpha(\mathcal{U})} \leq 3\,\|f_j\|_{\mathcal{H}} \to 0$$

while

$$\|f_j\|_{\mathcal{K}_{s'}^{\tau'}(\mathcal{U})} \geq \|\{\dots, \cos\phi_j, 1, \dots\}\|_{\ell_\infty} \geq 1.$$

$\square$

4. Sharpness results

It is clear that Theorem 3.1 cannot always give an complete characterization of $\mathcal{A}_s^\alpha(\mathcal{S})$, because it may not be possible to get $p = q$ in (12). In this section we show that Theorem 3.1 is sharp, which shows that the incompleteness of the characterization may come from the fact that the family of classical smoothness spaces $\mathcal{K}_s^\tau(\mathcal{S})$ may not be rich enough to describe the approximation spaces $\mathcal{A}_s^\alpha(\mathcal{S})$. This will be confirmed by our results on quasi-greedy and democratic bases in Section 5.

First we have to specify what we mean by a sharp result of this type. Given a quasi-normed Schauder basis $\mathcal{S}$ for a Banach space X it makes sense to define the following quantities:

$$P(\mathcal{S}) := \sup\{p : \text{upper bound of (12) holds}\}, \tag{19}$$

$$Q(\mathcal{S}) := \inf\{q : \text{lower bound of (12) holds}\}, \tag{20}$$

and we clearly always have $1 \leq P(\mathcal{S}) \leq Q(\mathcal{S}) \leq \infty$. For uniformly smooth and uniformly convex Banach spaces we have better estimates on $P(\mathcal{S})$ and $Q(\mathcal{S})$. The following fundamental result is known about Schauder bases in such Banach spaces:

Theorem 4.1 ([5]). *Let $\mathcal{S}$ be a quasi-normed Schauder basis for a Banach space X which is both uniformly smooth and uniformly convex (i.e., super-reflexive). Then $1 < P(\mathcal{S}) \leq Q(\mathcal{S}) < \infty$.*

This theorem shows that whenever the Banach space X is uniformly convex and uniformly smooth we are guaranteed to get better embedding lines from Theorem 3.1 than the ones for the "worst case" scenario where $P(\mathcal{S}) = 1$ and $Q(\mathcal{S}) = \infty$. How much improvement we get in uniformly smooth and uniformly convex Banach spaces clearly depends on the specific structure of the basis $\mathcal{S}$. In fact, any pair of p and q with $1 < p \leq 2 \leq q < \infty$ is realized by some Schauder basis in some Banach space, as the following theorem shows.

Theorem 4.2 ([5]). *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Given a pair of numbers p and q satisfying $1 < p \leq 2 \leq q < \infty$, there exists a Schauder basis $\mathcal{S}$ for $\mathcal{H}$ with the property that $P(\mathcal{S}) = p$ and $Q(\mathcal{S}) = q$.*

Remark 4.1. It depends on the properties of $\mathcal{S}$ and X whether (12) holds for $p = P(\mathcal{S})$ and $q = Q(\mathcal{S})$. For example, with $\mathcal{S}$ being the canonical basis in the weighted Lorentz space $\ell_{p,\infty}(\{\log(1+m)\})$, $P(\mathcal{S}) = Q(\mathcal{S}) = p$ but $\|\{1\}_{k=1}^m\|_X = m^{1/p}\log(1+m)$ and $\|\{1\}_{k=1}^m\|_{\ell_{p,1}} \asymp m^{1/p}$, hence (12) does not hold for $P(\mathcal{S})$.

Theorem 3.1 says that for any $p \leq P(\mathcal{S})$ and $q \geq Q(\mathcal{S})$ for which (12) holds we have the embedding lines given by $1/\tau_p = 1/p + \alpha$ and $1/\tau_q = 1/q + \alpha$. The sharpness of these embedding lines are in the following sense.

- Suppose that we have the lower embedding line $1/\tau = 1/p + \alpha$, then Proposition 4.1 below will show that $p \leq P(\mathcal{S})$ (but (12) does not necessarily hold with p). If we in addition assume that $\mathcal{S}$ is *greedy*, then Proposition 4.3 will show that indeed (12) holds for p as soon as the embedding holds at *one point* of the line.
- If we assume that $\mathcal{S}$ is *quasi-greedy* and we are given the upper embedding at *one point* of the line $1/\tau = 1/q + \alpha$, then Proposition 4.2 will show that (12) holds for q, hence $q \geq Q(\mathcal{S})$.

First we consider weak sharpness of the lower embedding.

Proposition 4.1. *Let $\mathcal{S}$ be a Schauder basis for X and suppose that $p > 1$ is such that $\mathcal{K}_s^\tau(\mathcal{S}) \hookrightarrow \mathcal{A}_s^\alpha(\mathcal{S})$ for every $\alpha > 0$, $s \in (0,\infty]$, and $\tau := (\alpha + 1/p)^{-1}$. Then $p \leq P(\mathcal{S})$, where $P(\mathcal{S})$ is defined in* (19).

Proof. We notice that $\mathcal{A}_s^\alpha(\mathcal{S}) \hookrightarrow X$ for all $\alpha > 0$. Let $1 < \tau < p$. By taking $\alpha = 1/\tau - 1/p$, we deduce from the embedding

$$\mathcal{K}_\tau^\tau(\mathcal{S}) \hookrightarrow \mathcal{A}_\tau^\alpha(\mathcal{S}) \hookrightarrow X$$

that $\mathcal{K}_\tau^\tau(\mathcal{S}) \hookrightarrow X$, that is to say $\|\cdot\|_X \lesssim \|\{c_k(\cdot)\}\|_{\ell_\tau}$. As this is true for any $1 < \tau < p$, $P(\mathcal{S}) \geq p$. □

Next we consider the upper embedding in Theorem 3.1, but first we need a few technical lemmas about quasi-greedy bases. Strong sharpness in the upper embedding for quasi-greedy bases will be proved in Proposition 4.2.

The following two lemmas were proved in the special case $p = 2$ in [11], and the authors would like to thank D. Kutzarova-Ford and S. Dilworth for pointing out to us that the technique used in [11] also works in the more general setting presented below.

Lemma 4.1. *Let $\mathcal{S} = \{g_k\}$ be a quasi-greedy basis for X and suppose that there is a constant $c > 0$ such that for any finite subset $\Lambda \subset \mathbb{N}$,*

$$c|\Lambda|^{1/q} \leq \Big\| \sum_{k\in\Lambda} \pm g_k \Big\|_X.$$

Then

$$\|\{c_k(\cdot)\}\|_{\ell_{q,\infty}(\mathbb{N})} \lesssim \|\cdot\|_X.$$

Lemma 4.2. *Let $\mathcal{S} = \{g_k\}$ be a quasi-greedy basis for X and suppose that there is a constant $C < \infty$ such that for any finite subset $\Lambda \subset \mathbb{N}$,*

$$\Big\| \sum_{k\in\Lambda} \pm g_k \Big\|_X \leq C|\Lambda|^{1/p}.$$

Then

$$\|\cdot\|_X \lesssim \|\{c_k(\cdot)\}\|_{\ell_{p,1}(\mathbb{N})}.$$

Proof of Lemma 4.1. Let $f = \sum_{k\in\mathbb{N}} c_k g_k \in X$. Since $\mathcal{S}$ is quasi-greedy there is a constant C depending only on $\mathcal{S}$ [6] (see also [11, Theorem 1]) such that

$$\sup_N \| \sum_{k=1}^N c_k^\star g_{\pi(k)} \|_X \leq C\|f\|_X.$$

Using the Abel transform we get for any increasing sequence $\{\alpha_k\}$ of positive numbers that $\sup_N \| \sum_{k=1}^N \alpha_k c_k^\star g_{\pi(k)}\|_X \leq C(\sup_k \alpha_k)\|f\|_X$. Thus, for every $N \geq 1$ for which $c_N^\star \neq 0$ and $\alpha_k = |c_N^\star||c_k^\star|^{-1}$, $k = 1, 2, \ldots, N$, we have

$$|c_N^\star| N^{1/q} \leq c^{-1} \Big\| \sum_{k=1}^N \frac{c_k^\star |c_N^\star|}{|c_k^\star|} g_{\pi(k)} \Big\|_X \leq c^{-1} C\|f\|_X.$$

It follows at once that $\|\{c_k\}\|_{\ell_{q,\infty}(\mathbb{N})} \leq c^{-1}C\|f\|_X$. □

Proof of Lemma 4.2. An extremal point argument shows that for all finite set $\Lambda \subset \mathbb{N}$,

$$\| \sum_{k\in\Lambda} c_k g_k \| \leq \sup_{k\in\Lambda} |c_k| \sup_{\varepsilon_k\in\{-1,1\}} \| \sum_{k\in\Lambda} \varepsilon_k g_k \| \leq C \sup_{k\in\Lambda} |c_k| |\Lambda|^{1/p}.$$

Let $f = \sum_{k\in\mathbb{N}} c_k g_k \in X$ and denote by $\Lambda_j = \{k : |c_k^\star| \geq 2^{-j}\}$. As $\mathcal{S}$ is quasi-greedy we can write

$$\begin{aligned}\|f\|_X &= \Big\| \sum_{j=-\infty}^{\infty} \sum_{k\in\Lambda_j\setminus\Lambda_{j-1}} c_k^\star g_{\pi(k)} \Big\|_X \leq \sum_{j=-\infty}^{\infty} \Big\| \sum_{k\in\Lambda_j\setminus\Lambda_{j-1}} c_k^\star g_{\pi(k)} \Big\|_X \\ &\leq \sum_{j\in\mathbb{Z}} C 2^{-(j-1)} |\Lambda_j \setminus \Lambda_{j-1}|^{1/p} \leq \tilde{C} \sum_j 2^{-j} |\Lambda_j|^{1/p} \leq C' \|\{c_k(f)\}\|_{\ell_{p,1}}.\end{aligned}$$

□

We now turn to the strong sharpness result for the upper embedding for quasi-greedy bases.

Proposition 4.2. *Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a quasi-greedy basis for X and $1/\tau_0 > \alpha_0 > 0$, $s \in (0,\infty]$ be such that $\mathcal{A}_s^{\alpha_0}(\mathcal{S}) \hookrightarrow \mathcal{K}_\infty^{\tau_0}(\mathcal{S})$. Define $q := (1/\tau_0 - \alpha_0)^{-1}$. Then*

$$\|\{c_k(\cdot)\}\|_{\ell_{q,\infty}} \lesssim \|\cdot\|_X$$

i.e., the lower bound in (12) *holds for q.*

Proof. Let $\Lambda \subset \mathbb{N}$ with $|\Lambda| = m$. Take $\varepsilon \in \{-1,1\}^{\mathbb{N}}$, and put $\psi = \sum_{k\in\Lambda} \varepsilon_k g_k$. Then $\|\psi\|_{\mathcal{K}_\infty^{\tau_0}} = m^{1/\tau_0} = m^{\alpha_0} m^{1/q}$, and by using $\mathcal{A}_s^{\alpha_0}(\mathcal{S}) \hookrightarrow \mathcal{K}_\infty^{\tau_0}(\mathcal{S})$ together with the Bernstein inequality for $\mathcal{A}_s^{\alpha}$, see [4, Chap. 7; Theorem 9.3], we obtain

$$m^{\alpha_0} m^{1/q} = \|\psi\|_{\mathcal{K}_\infty^{\tau_0}(\mathcal{S})} \le C\|\psi\|_{\mathcal{A}_s^{\alpha_0}(\mathcal{S})} \le \tilde{C} m^{\alpha_0} \|\psi\|_X.$$

From this we deduce that for $\Lambda \subset \mathbb{N}$, $|\Lambda| = m$, $\left\|\sum_{k\in\Lambda} \pm g_k\right\|_X \ge \tilde{C}^{-1} m^{1/q}$. We conclude using Lemma 4.1. □

To conclude this section, we give a strong sharpness result for the lower embedding for greedy bases.

Proposition 4.3. *Let $\mathcal{S} = \{g_k\}_{k\in\mathbb{N}}$ be a greedy basis for X and $1/\tau_0 > \alpha_0 > 0$, $s \in (0,\infty]$ be such that $\mathcal{K}_{s'}^{\tau_0}(\mathcal{S}) \hookrightarrow \mathcal{A}_s^{\alpha_0}(\mathcal{S})$ for some $0 < s' \le \infty$. Define $p := (1/\tau_0 - \alpha_0)^{-1}$. Then*

$$\|\cdot\|_X \lesssim \|\{c_k(\cdot)\}\|_{\ell_{p,1}},$$

i.e., the upper bound in (12) *holds for p.*

Proof. Let $\Lambda, \Lambda' \subset \mathbb{N}$ with $|\Lambda| = |\Lambda'| = m$ and $\Lambda \cap \Lambda' = \emptyset$. Take $\varepsilon \in \{-1,1\}^{\mathbb{N}}$, and put

$$\phi = \sum_{k\in\Lambda\cup\Lambda'} \varepsilon_k g_k, \quad \psi = \sum_{k\in\Lambda} \varepsilon_k g_k = \phi - \sum_{k\in\Lambda'} \varepsilon_k g_k.$$

Then, by the greediness of $\mathcal{S}$ (see Eq. (7)) and the Jackson inequality for $\mathcal{A}_s^{\alpha}(\mathcal{S})$ (again [4, Chap. 7; Theorem 9.3]) together with $\mathcal{K}_{s'}^{\tau_0}(\mathcal{S}) \hookrightarrow \mathcal{A}_s^{\alpha_0}(\mathcal{S})$, we obtain

$$\begin{aligned}\|\psi\|_X = \|\phi - \sum_{k\in\Lambda'} \varepsilon_k g_k\|_X &\le C\sigma_m(\phi) \le \tilde{C} m^{-\alpha_0} \|\phi\|_{\mathcal{A}_s^{\alpha_0}} \\ &\le C' m^{1/p - 1/\tau_0} \|\phi\|_{\mathcal{K}_{s'}^{\tau_0}} \le \tilde{C}' m^{1/p}.\end{aligned}$$

From this we deduce that for $\Lambda \subset \mathbb{N}$, $|\Lambda| = m$, $\left\|\sum_{k\in\Lambda} \pm g_k\right\|_X \le C m^{1/p}$. We conclude using Lemma 4.2. □

5. Greedy approximation with quasi-greedy and democratic bases

Given the sharpness results for Theorem 3.1 established in the previous section, there are some Schauder bases for which $\mathcal{A}_s^\alpha(\mathcal{S})$ cannot be characterized using the classical Lorentz norms of the coefficients $\{c_k(f)\}$ of its elements. We prove in this section that for the class of quasi-greedy and democratic bases, it is possible to get a complete characterization of $\mathcal{G}_s^\alpha(\mathcal{S})$ in terms of a weighted Lorentz norm (see Eq. (11)) of $\{c_k(f)\}$, where the weight $w(\mathcal{S})$ is a simple function of the basis $\mathcal{S}$. The proof is based on similar results by Temlyakov [9, Section 5] for the special case of wavelet-type systems in $L_p(\mathbb{R})$. A corollary of the main theorem of this section gives a characterization of $\mathcal{A}_s^\alpha(\mathcal{S})$ with the stronger assumption that $\mathcal{S}$ is greedy (that is to say we have to add unconditionality), this will be seen in Section 6.

Remark 5.1. In [6, Section 3.3], an example is given of a quasi-greedy and democratic basis which is not greedy, and [6, Section 3.1] shows that a quasi-greedy basis is not necessarily democratic.

For any basis $\mathcal{S} = \{g_k\}_{k\geq 1}$, one can define a sequence $w(\mathcal{S}) = \{w_n\}_{n\geq 0}$ with $w_0 = 0$ and for any $n \geq 1$:

$$w_n = \max\Big(w_{n-1}, \big\|\sum_{k=1}^{n} g_k\big\|\Big). \tag{21}$$

We have the following lemma.

Lemma 5.1. *If $\mathcal{S}$ is quasi-greedy and democratic, there exists constants $c > 0$ and $C < \infty$ such that for any set Λ_m of cardinality m and any $\{c_k\}$,*

$$c w_m \inf_{k\in\Lambda_m} |c_k| \leq \big\|\sum_{k\in\Lambda_m} c_k g_k\big\| \leq C w_m \sup_{k\in\Lambda_m} |c_k|. \tag{22}$$

Moreover, (22) *implies that the growth of $w(\mathcal{S})$ is "slow", that is to say*

$$w_{2m} \asymp w_m. \tag{23}$$

Proof. Because $\mathcal{S}$ is quasi-greedy, it is unconditional with constant coefficients [11, Proposition 2], hence democracy implies super-democracy. The upper bound in (22) follows by an extremal point argument similar to the one in the proof of Lemma 4.2. An Abel transform argument similar to the one in the proof of Lemma 4.1 gives the lower bound. To get the slow growth, we first remark that $w(\mathcal{S})$ is an increasing sequence, hence $w_m \le w_{2m}$. Next, using (22) we write

$$cw_{2m} \le \|\sum_{k=1}^{m} g_k + \sum_{k=m+1}^{2m} g_k\|_X \le \|\sum_{k=1}^{m} g_k\|_X + \|\sum_{k=m+1}^{2m} g_k\|_X \tag{24}$$

$$\le 2Cw_m. \tag{25}$$

□

Let us now proceed to our main theorem.

Theorem 5.1. *Let $\mathcal{S}$ be a quasi-greedy basis for a Banach space X. The following conditions are equivalent:*

1. *$\mathcal{S}$ is democratic.*
2. *For any $\alpha > 0$ and $s \in (0, \infty]$,*

$$\mathcal{G}_s^\alpha(\mathcal{S}) = \left\{ f \in X, \left\|\{c_k(f)\}\right\|_{\ell_{1/\alpha,s}(w(\mathcal{S}))} < \infty \right\}, \tag{26}$$

with equivalent norms

$$\|\cdot\|_{\mathcal{G}_s^\alpha(\mathcal{S})} \asymp \|\{c_k(\cdot)\}\|_{\ell_{1/\alpha,s}(w(\mathcal{S}))}. \tag{27}$$

3. *Relations* (26) *and* (27) *hold for some slowly growing sequence $w = \{w_m\}$ at some point α, s.*

First we give the proof of our two main lemmas, based on the technique introduced in [9, Section 5] for the special case of wavelet type systems in $L_p(\mathbb{R})$.

Lemma 5.2. *Assume $\mathcal{S}$ is a quasi-greedy and democratic basis in the space X, then there is a constant $C < \infty$ such that for all $f = \sum_k c_k g_k$ and all integers $N < M$,*

$$c_M^\star \le C\gamma_N(f)_X w_{M-N}^{-1}.$$

Lemma 5.3. *Assume $\mathcal{S}$ is quasi-greedy and democratic basis in the space X, then for all $f = \sum_k c_k g_k$ and every increasing sequence of integers $m_0 < m_1 < \cdots < m_j < \cdots$,*

$$\gamma_{m_j}(f)_X \le \sum_{l=j}^{\infty} c_{m_l}^\star w_{m_{l+1}-m_l}.$$

Proof of Lemma 5.2. By the lower bound in (22) and the quasi-greediness of $\mathcal{S}$ [6], we get

$$c^\star_M w_{M-N} \le c^{-1}\Big\| \sum_{k=N+1}^{M} c^\star_k g_{\pi(k)} \Big\|_X \le C \Big\| \sum_{k=N+1}^{\infty} c^\star_k g_{\pi(k)} \Big\|_X = C\gamma_N(f)_X.$$

□

Proof of Lemma 5.3. By the quasi-greediness of $\mathcal{S}$, we can write

$$\gamma_{m_j}(f)_X = \Big\| \sum_{k=m_j+1}^{\infty} c^\star_k g_{\pi(k)} \Big\|_X \le \sum_{l=j}^{\infty} \Big\| \sum_{k=m_l+1}^{m_{l+1}} c^\star_k g_{\pi(k)} \Big\|_X.$$

Then by the upper bound in (22), we get $\gamma_{m_j}(f)_X \le \sum_{l=j}^{\infty} c^\star_{m_l} w_{m_{l+1}-m_l}$. □

Using the nice properties of $w(\mathcal{S})$, we can now prove Theorem 5.1.

Proof of Theorem 5.1.

1.⇒2. We will only prove the result for $s = \infty$, the other cases easily follow from classical arguments using the discrete Hardy inequality (see, *e.g.* [4, Chapter 3, Lemma 3.4]). From the slow growth of $w(\mathcal{S})$ (Lemma 5.1) and from Lemma 5.2 (using $M = 2^{j+1}$ and $N = 2^j$) we get the estimate

$$c^\star_{2^j} w_{2^j} 2^{\alpha j} \le C c^\star_{2^j} w_{2^{j-1}} 2^{\alpha j} \le \tilde{C} \gamma_{2^j}(f)_X 2^{\alpha j}$$

hence (using again Lemma 5.1 to get the result for all m from its statement for $m = 2^j$) if $\gamma_m(f)_X \lesssim m^{-\alpha}$ one gets $c^\star_m \lesssim m^{-\alpha} w_m^{-1}$. Conversely, using $m_j = 2^j$ in Lemma 5.3, we get that if $c^\star_m \lesssim m^{-\alpha} w_m^{-1}$ then

$$\gamma_{2^j}(f)_X \le \sum_{l=j}^{\infty} c^\star_{2^l} w_{2^l} \le C 2^{-\alpha j}$$

which is the desired result.

2.⇒3. is trivial.

3.⇒1. We prove it with arguments similar to the strong sharpness results of Section 4. First, using the notations of Proposition 4.2,

$$c m^\alpha w_m \le \|\psi\|_{\mathcal{K}_s^{1/\alpha}(w,\mathcal{S})} \le C \|\psi\|_{\mathcal{G}_s^\alpha(\mathcal{S})} \stackrel{(B)}{\le} C' m^\alpha \|\psi\|_X.$$

The generalized Bernstein inequality (B) holds because $\gamma_m(f)_X \leq C\|f\|_X$ thanks to the quasi-greediness of $\mathcal{S}$. Conversely, using the notations of Proposition 4.3,

$$\begin{aligned}\|\psi\|_X = \|\phi - G_m(\phi)\|_X &\overset{(J)}{\leq} C\|\phi\|_{\mathcal{G}_s^\alpha(\mathcal{S})} m^{-\alpha} \\ &\leq C'\|\phi\|_{\mathcal{K}_s^{1/\alpha}(w,\mathcal{S})} m^{-\alpha} \leq \tilde{C}' w_m\end{aligned}$$

where the generalized Jackson inequality (J) holds by the definition of $\|\cdot\|_{\mathcal{G}_s^\alpha(\mathcal{S})}$.

□

Remark 5.2. It is easy to check that condition 3 in Theorem 5.1 can indeed be relaxed to a two sided embedding of $\mathcal{G}_s^\alpha(\mathcal{S})$ with $\mathcal{K}_\infty^{1/\alpha}(w,\mathcal{S})$ and $\mathcal{K}_{s'}^{1/\alpha}(w,\mathcal{S})$ for some $\alpha > 0$ and some $s' \in (0,\infty]$.

6. Examples

We will now present some examples of the use of Theorem 3.1 and 5.1, each of which generalizes Theorem 1.1.

First we state an immediate corollary of Theorem 5.1.

Corollary 1. *Assume $\mathcal{S}$ is a greedy basis for X. Then*

$$\mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{G}_s^\alpha(\mathcal{S}) = \mathcal{K}_s^{1/\alpha}(w(\mathcal{S}),\mathcal{S})$$

with equivalent norms.

Then we state a few other corollaries that will show that $\mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{G}_s^\alpha(\mathcal{S})$ can hold even for non-greedy bases.

Corollary 2. *Let $\mathcal{S}$ a quasi-greedy basis for a Banach space X satisfying the sandwich assumption* (12) *with $p = q < \infty$. Then, with $\alpha = \frac{1}{\tau_p} - \frac{1}{p}$,*

$$\mathcal{G}_s^\alpha(\mathcal{S}) = \mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{K}_s^{\tau_p}(\mathcal{S}) \tag{28}$$

with equivalent norms.

Proof. Using Theorem 3.1, we get $\mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{K}_s^{\tau_p}(\mathcal{S})$ with equivalent norms. From the sandwich assumption we get the democracy of the quasi-greedy basis $\mathcal{S}$ with $w_m \asymp m^{1/p}$, hence Theorem 5.1 gives

$$\mathcal{G}_s^\alpha(\mathcal{S}) = \mathcal{K}_s^{1/\alpha}(\{m^{1/p}\},\mathcal{S}) = \mathcal{K}_s^{\tau_p}(\mathcal{S}).$$

□

Remark 6.1. A simple modification of the proof of [11, Theorem 5] shows that the sandwich assumption $\|\{c_k(\cdot)\}\|_{\ell_{p,\infty}} \lesssim \|\cdot\|_X \lesssim \|\{c_k(\cdot)\}\|_{\ell_{p,1}}$ implies that for some constant $C < \infty$, $\gamma_m(.,\mathcal{S}) \le C\sigma_m(.,\mathcal{S})\log(1+m)$. However this is not sufficient to get 2.

Corollary 3. *Suppose $\mathcal{S}$ is a quasi-greedy basis for a Hilbert space $\mathcal{H}$, then for $\alpha = \frac{1}{\tau} - \frac{1}{2}$,*

$$\mathcal{G}_s^\alpha(\mathcal{S}) = \mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{K}_s^\tau(\mathcal{S})$$

with equivalent norms.

Proof. We can use [11, Proposition 2] which gives (from the type and cotype of the Hilbert space, or Khinchin inequality) the sandwich property with $p = 2$. $\square$

Remark 6.2. From Theorem 4.2, one can see that one cannot simply remove the assumption that $\mathcal{S}$ is quasi-greedy in Corollary 3. Moreover, from the existence of quasi-greedy conditional bases in a separable infinite dimensional Hilbert space [11, Corollary 4], it follows that the equality $\mathcal{A}_s^\alpha(\mathcal{S}) = \mathcal{G}_s^\alpha(\mathcal{S})$ (for all α, s) does not imply that $\mathcal{S}$ is greedy.

Acknowledgments. D. Kutzarova-Ford and S. Dilworth brought to our attention helpful results on quasi-greedy bases. The problem of completely characterizing the approximation spaces for greedy bases was formulated by V. Temlyakov after reading the first version of this paper and a talk on this topic in the "Greedy Seminar" at the Industrial Mathematics Institute of the University of South Carolina. We are grateful to D. Kutzarova-Ford, S. Dilworth and V. Temlyakov for their interest in this topic, and we also wish to thank Prof. R. DeVore for his encouragement and the exceptionally good spirit he brings as a director of the IMI.

References

[1] R. A. DeVore and V. N. Temlyakov, Some remarks on greedy algorithms, *Adv. Comput. Math.* **5**, (2-3) (1996), 173–187.

[2] Ronald A. DeVore, Non-linear approximation, pp. 51–150. In: Acta numerica, 1998, Cambridge Univ. Press, Cambridge, 1998.

[3] Ronald A. DeVore, Björn Jawerth and Vasil Popov, Compression of wavelet decompositions, *Amer. J. Math.* **114**, 4 (1992), 737–785.

[4] Ronald A. DeVore and George G. Lorentz, Constructive Approximation, Springer-Verlag, Berlin, 1993.

[5] V. I. Gurariĭ and N. I. Gurariĭ, Bases in uniformly convex and uniformly smooth Banach spaces, *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 210–215.

[6] S. V. Konyagin and V. N. Temlyakov, A remark on greedy approximation in Banach spaces, *East J. Approx.* **5**, 3 (1999), 365–379.

[7] Pencho P. Petrushev, Direct and converse theorems for spline and rational approximation and Besov spaces, pp. 363–377. In: Function spaces and applications (Lund, 1986), Springer, Berlin, 1988.

[8] S. B. Stechkin, On absolute convergence of orthogonal series, *Dokl. Akad. Nauk SSSR* **102** (1955), 37–40.

[9] V. N. Temlyakov, Non-linear m-term approximation with regard to the multivariate Haar system, *East J. Approx.* **4**, 1 (1998), 87–106.

[10] V. N. Temlyakov, Non-linear methods of approximation, Technical Report 0109, Dept of Mathematics, University of South Carolina, Columbia, SC 29208, 2001.

[11] P. Wojtaszczyk, Greedy algorithm for general biorthogonal systems, *J. Approx. Theory* **107**, 2 (2000), 293–314.

[12] Robert M. Young, An Introduction to Non-harmonic Fourier Series, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.

Received May 16, 2001