Highly Nonstationary Wavelet Packets

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Abstract

We introduce a new class of basic wavelet packets, called highly nonstationary wavelet packets, and show how to obtain uniformly bounded basic wavelet packets with support contained in some fixed interval using a sequence of Daubechies filters with associated filterlengths $\{d_n\}_{n=0}^{\infty}$ satisfying $d_n \geq Cn^{2+\varepsilon}$ for some constants $C, \varepsilon > 0$. We define the periodic Shannon wavelet packets and show how to obtain perturbations of this system using periodic highly nonstationary wavelet packets. Such perturbations provide examples of periodic wavelet packets that do form a Schauder basis for $L^p[0,1)$ for $1 < p < \infty$. We also consider the representation of the differentiation operator in such periodic wavelet packets.

1 Introduction

Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short-time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals. It corresponds to a decomposition of phase space in which the trade-off between time and frequency localization has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. In fact the wavelet $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k)$ has a frequency resolution proportional to 2^j, which follows by taking the Fourier transform:

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}.$$

This makes the analysis well adapted to the study of transient phenomena and has proven a very successful approach to many problems in signal processing, numerical analysis, and quantum mechanics. Nevertheless, for stationary signals wavelet analysis is outperformed by short-time Fourier analysis. Wavelet packets were introduced by R. Coifman, Y. Meyer, and M. V. Wickerhauser to improve the poor frequency localization of wavelet bases for large j and thereby provide a more efficient decomposition of signals containing both transient and stationary components.

A problem noted by Coifman, Meyer, and Wickerhauser in [3], and generalized by Hess-Nielsen in [6], is that the L^1 norms of the Fourier transforms of the wavelet packets are not uniformly bounded (except for wavelet packets generated using certain idealized filters) indicating a loss of frequency resolution at high frequencies. Hess-Nielsen introduced nonstationary wavelet packets in [6] as a way to minimize the loss of frequency resolution by using a sequence of Daubechies filters with increasing filter length to generate the basic wavelet packets.

In the present paper we generalize the definition of nonstationary wavelet packets to what we call highly nonstationary wavelet packets. The new wavelet packets still live entirely within the multiresolution structure and we have an associated discrete algorithm to calculate the expansion of a given function in the wavelet packets. One advantage of the new functions is that we have better control of the frequency resolution. As an example of this we show how to obtain wavelet packets with uniformly bounded L^1-norm of their Fourier transforms and with support contained in some fixed compact interval. We use a sequence of Daubechies filters with associated filterlengths $\{d_n\}$ that grow at least as fast as $Cn^{2+\varepsilon}$ for some $C, \varepsilon > 0$. Using the same methods we are also able to improve the result on frequency resolution by Hess-Nielsen in [6] for nonstationary wavelet packets.

Another application is to obtain Schauder bases for $L^p[0,1)$, $1 < p < \infty$, consisting of periodic wavelet packets. The author proves in [9] that periodic wavelet packets associated with the classical wavelet packet construction can fail to be Schauder bases for such spaces. The method we use in the present paper to obtain bases is to define the periodic wavelet packets associated with the Shannon wavelet packets and then obtain perturbations of this system using periodic highly nonstationary wavelet packets.

Finally, we consider the representation of the operator d/dx in periodic wavelet packets and show that for certain systems the matrix representing the operator is almost diagonal.

2 Wavelet Packets. Definitions and Properties

In the original construction by Coifman, Meyer and Wickerhauser ([1, 2]) of wavelet packets the functions were constructed by starting from a multiresolution analysis and then generating the wavelet packets using the associated filters. However, it was observed by Hess-Nielsen ([5, 6]) that it is an unnecessary constraint to use the multiresolution filters to do the frequency decomposition. We present his, more general, definition of so-called nonstationary wavelet packets here. We assume the reader is familiar with the concept of a multiresolution analysis (see e.g. [8]), and we will use the Meyer indexing convention for such structures.

Definition 1. *Let (ϕ, ψ) be the scaling function and wavelet associated with a multiresolution analysis, and let $(F_0^{(p)}, F_1^{(p)})$, $p \in \mathbb{N}$, be a family of bounded operators on $\ell^2(\mathbb{Z})$ of the form*

$$(F_\varepsilon^{(p)} a)_k = \sum_{n\in\mathbb{Z}} a_n h_\varepsilon^{(p)}(n-2k), \qquad \varepsilon = 0, 1,$$

with $h_1^{(p)}(n) = (-1)^n h_0^{(p)}(1-n)$ a real-valued sequence in $\ell^1(\mathbb{Z})$ such that

$$\begin{aligned} F_0^{(p)*} F_0^{(p)} + F_1^{(p)*} F_1^{(p)} &= I \\ F_0^{(p)} F_1^{(p)*} &= 0 \end{aligned} \tag{1}$$

We define the family of basic nonstationary wavelet packets $\{w_n\}_{n=0}^\infty$ recursively by letting $w_0 = \phi$, $w_1 = \psi$, and then for $n \in \mathbb{N}$

$$\begin{aligned} w_{2n}(x) &= 2 \sum_{q\in\mathbb{Z}} h_0^{(p)}(q) w_n(2x - q) \\ w_{2n+1}(x) &= 2 \sum_{q\in\mathbb{Z}} h_1^{(p)}(q) w_n(2x - q), \end{aligned} \tag{2}$$

where $2^p \leq n < 2^{p+1}$.

Remarks. The wavelet packets obtained from the above definition using only the filters associated with the multiresolution analysis on each scale are called classical wavelet packets. They are the functions introduced by Coifman, Meyer, and Wickerhauser in [3].

Any pair of operators $(F_0^{(p)}, F_1^{(p)})$ of the type discussed in the definition above will be referred to as a pair of conjugate quadrature mirror filters (CQFs).

Associated with each filter sequence $\{h_\varepsilon^{(p)}\}$ is the symbol of the filter, the 2π-periodic function given by

$$m_\varepsilon^{(p)}(\xi) = \sum_{k\in\mathbb{Z}} h_\varepsilon^{(p)}(k) e^{ik\xi}.$$

The symbol $m_\varepsilon^{(p)}$ determines the filter sequence uniquely so we will also refer to the symbol $m_\varepsilon^{(p)}$ as the filter.

The following is the fundamental result about nonstationary wavelet packets. We have included the proof since it will be used in the construction of highly nonstationary wavelet packets presented in the following section.

Theorem 2 ([6, 7]). *Let $\{w_n\}_{n=0}^{\infty}$ be a family of nonstationary wavelet packets associated with the multiresolution analysis $\{V_j\}$ with scaling function and wavelet (ϕ, ψ). The functions $\{w_n\}$ satisfy the following*

- $\{w_0(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0
- $\{w_n(\cdot - k)\}_{k\in\mathbb{Z}, 0\le n<2^j}$ *is an orthonormal basis for* V_j.

In particular, $\{w_n(\cdot - k)\}_{k\in\mathbb{Z}, n\in\mathbb{N}_0}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Proof. Since $w_0 = \phi$ and $w_1 = \psi$ we get the first statement, and the second in the case $j = 0$, immediately. Next step is to prove that $\{w_{2n}(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\{w_{2n+1}(\cdot - k)\}_{k\in\mathbb{Z}}$ are orthonormal systems. Suppose the result is true for all indices j with $j < n$ with n such that $2^p \le n < 2^{p+1}$. We have,

$$\begin{aligned}
\langle w_{2n}, w_{2n}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_0^{(p)}(\ell)h_0^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_0^{(p)}(\ell)h_0^{(p)}(\ell-2k) \\
&= \delta_{0,k}, \\
\langle w_{2n+1}, w_{2n+1}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_1^{(p)}(\ell)h_1^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_1^{(p)}(\ell)h_1^{(p)}(\ell-2k) \\
&= \delta_{0,k},
\end{aligned}$$

and

$$\begin{aligned}
\langle w_{2n}, w_{2n+1}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_0^{(p)}(\ell)h_1^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_0^{(p)}(\ell)h_1^{(p)}(\ell-2k) \\
&= 0.
\end{aligned}$$

Thus, a simple additional induction argument using the above shows that $\{w_n(\cdot - k)\}_{n\in\mathbb{N}_0, k\in\mathbb{Z}}$ is an orthonormal system.

Let $\Omega_n = \overline{\text{Span}}\{w_n(\cdot - k)\}_{k\in\mathbb{Z}}$. Define $\delta f(x) = \sqrt{2}f(2x)$. Since $\{w_n(\cdot - k)\}_k$ is an orthonormal system so is $\{\delta w_n(\cdot - k)\}_k$, and it follows from the exact reconstruction property of the filters (see (1)) that for $2^p \le n < 2^{p+1}$,

$$w_n(x-k) = \frac{1}{2}\sum_{q\in\mathbb{Z}} h_0^{(p)}(k-2q)w_{2n}(x/2-q) + \frac{1}{2}\sum_{q\in\mathbb{Z}} h_1^{(p)}(k-2q)w_{2n+1}(x/2-q).$$

Hence, by (2),

$$\overline{\mathrm{Span}}\{\sqrt{2}w_n(2\cdot -k)\}_k = \overline{\mathrm{Span}}\{w_{2n}(\cdot - k)\}_k \oplus \overline{\mathrm{Span}}\{w_{2n+1}(\cdot - k)\}_k,$$

i.e. $\delta\Omega_n = \Omega_{2n} \oplus \Omega_{2n+1}$. Thus,

$$\begin{aligned}
\delta\Omega_0 \ominus \Omega_0 &= \Omega_1 \\
\delta^2\Omega_0 \ominus \delta\Omega_0 &= \delta\Omega_1 = \Omega_2 \oplus \Omega_3 \\
\delta^3\Omega_0 \ominus \delta^2\Omega_0 &= \delta\Omega_2 \oplus \delta\Omega_3 = \Omega_4 \oplus \Omega_5 \oplus \Omega_6 \oplus \Omega_7 \\
&\vdots \\
\delta^k\Omega_0 \ominus \delta^{k-1}\Omega_0 &= \Omega_{2^{k-1}} \oplus \Omega_{2^{k-1}+1} \oplus \cdots \oplus \Omega_{2^k-1}.
\end{aligned}$$

By telescoping the above equalities we finally get the wanted result

$$\delta^k\Omega_0 \equiv \delta^k V_0 = V_k = \Omega_0 \oplus \Omega_1 \oplus \cdots \oplus \Omega_{2^k-1},$$

and $\cup_{k\geq 0} V_k$ is dense in $L^2(\mathbb{R})$ by the definition of a multiresolution analysis. ■

3 Frequency Resolution of Wavelet Packets

The author shows in [9] that basic classical wavelet packets associated with some of the most widely used filters, such as the Daubechies filters, are not uniformly bounded functions. In this section we prove that using the nonstationary construction of wavelet packets one can obtain uniformly bounded basic wavelet packets. The price we have to pay is that we have to use a sequence of filters with an increasing number of nonzero coefficients. A consequence is that the diameter of the support of the basic wavelet packets grows with frequency. We propose a new construction of wavelet packets in the next section to avoid such support problems. It should be noted that Theorem 5 below is somewhat stronger than the frequency localization result obtained by Hess-Nielsen in [6, Theorem 8] for the same sequence of filters. Let us recall that the Daubechies filter of length $2N$ is given by

$$m_0^N(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)\mathcal{L}^N(\xi),$$

with

$$|\mathcal{L}^N(\xi)|^2 = \sum_{j=0}^{N-1} \binom{N-1+j}{j} \sin^{2j}(\xi/2).$$

We extract $\mathcal{L}^N(\xi)$ from $|\mathcal{L}^N(\xi)|$ by the Riesz factorization (see [4]).

The following two lemmas give us some basic information on the geometry of the Daubechies filters.

Lemma 3. *Let* m_0^N *be the Daubechies filter of length* $2N$*. Then*

$$|m_0^N(\xi)| \le |\sin(\xi)|^{N-1}, \qquad \text{for } \pi/2 \le |\xi| \le \pi.$$

Moreover,

$$S(\xi) = |m_0^N(\xi)| + |m_0^N(\xi+\pi)| \le 1 + |\sin(\xi)|^{N-1}, \qquad \xi \in \mathbb{R},$$

and

$$\|S\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})} = 1 + O(1/\sqrt{N}).$$

Proof. We have, for $\pi/2 \le |\xi| \le \pi$,

$$|m_0^N(\xi)|^2 = \cos^{2N}(\xi/2)|P_N(\xi)|^2,$$

where

$$\begin{aligned}
|P_N(\xi)|^2 &= \sum_{j=0}^{N-1} \binom{N-1+j}{j} \sin^{2j}(\xi/2) \\
&= \sum_{j=0}^{N-1} \binom{N-1+j}{j} [2\sin^2(\xi/2)]^j 2^{-j} \\
&\le [2\sin^2(\xi/2)]^{N-1} \sum_{j=0}^{N-1} \binom{N-1+j}{j} 2^{-j} \\
&= [2\sin^2(\xi/2)]^{N-1} |P_N(\pi/2)|^2 \\
&= [4\sin^2(\xi/2)]^{N-1},
\end{aligned}$$

so

$$|m_0^N(\xi)|^2 \le \cos^{2N}(\xi/2)[4\sin^2(\xi/2)]^{N-1} \le [4\cos^2(\xi/2)\sin^2(\xi/2)]^{N-1} = |\sin(\xi)|^{2(N-1)}.$$

To get the second part, we just notice that for $\pi/2 \le |\xi| \le \pi$:

$$|m_0^N(\xi)| \le |\sin(\xi)|^{N-1}, \quad \text{and} \quad |m_0^N(\xi+\pi)| \le 1.$$

For $|\xi| \le \pi/2$ we have, using $|\sin(\xi \pm \pi)| = |\sin(\xi)|$,

$$|m_0^N(\xi)| \le 1, \quad \text{and} \quad |m_0^N(\xi+\pi)| \le |\sin(\xi)|^{N-1}.$$

Finally,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\xi)^2\, dx \le 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} [|\sin(\xi)|^{2N-2} + 2|\sin(\xi)|^{N-1}]\, d\xi.$$

Assume N is odd (the case N even is similar). We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{(2N-2)}(\xi)\,d\xi = \frac{1\cdot 3\cdot 5\cdots(2N-3)}{2\cdot 4\cdot 6\cdots(2N-2)} \le \frac{1}{\sqrt{(N-1)\pi}},$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{(N-1)}(\xi)\,d\xi = \frac{1\cdot 3\cdot 5\cdots(N-2)}{2\cdot 4\cdot 6\cdots(N-1)} \le \frac{1}{\sqrt{\pi(N-1)/2}}$$

so, using $\sqrt{1+\alpha^2} \le 1+\alpha^2/2$ we get the estimate we want. ■

Moreover,

Lemma 4. *Let $\{m_0^{(p)}\}_{p=1}^{\infty}$ be a family of Daubechies low-pass filters. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that $d_p \equiv \deg(m_0^{(p)}) \ge Cp^{2+\varepsilon}$. Then there exists a constant $B < \infty$ such that*

$$\int_{-\pi}^{\pi} |m_{\varepsilon_1}^{(1)}(\xi)m_{\varepsilon_2}^{(2)}(2\xi)\cdots m_{\varepsilon_j}^{(j)}(2^{j-1}\xi)|\,d\xi \le B2^{-j}, \qquad j = 1,2,,\ldots,$$

for any choice of $(\varepsilon_k) \in \{0,1\}^{\mathbb{N}}$.

Proof. Fix $\varepsilon \in \{0,1\}^{\mathbb{N}}$, and define $I_{J,K} = I_{J,K}^{\varepsilon}$, $J > K$, by

$$I_{J,K}(\xi) \equiv 2^{K+1}|m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|.$$

It suffices to find a constant A such that $\int_{-\pi}^{\pi} I_{J,J-1}(\xi)\,d\xi \le A$, independent of J and the choice of ε. Let $S_K(\xi) = |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)| + |m_{\varepsilon_{J-K}}^{(J-K)}(\xi+\pi)|$ (note that S_K is independent of the value of ε_{J-K} which follows from the CQF conditions). Then

$$\begin{aligned}
\int_{-\pi}^{\pi} I_{J,K}(\xi)\,d\xi &= 2^{K+1}\int_{-\pi}^{\pi} |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&= 2^{K+1}\int_{-\pi}^{0} |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&\qquad + 2^{K+1}\int_{0}^{\pi} |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&= 2^{K}\int_{-\pi}^{\pi} S_K(\xi/2)|m_{\varepsilon_{J-K+1}}^{(J-K+1)}(\xi)m_{\varepsilon_{J-K+2}}^{(J-K+2)}(\xi)\cdots m_{\varepsilon_J}^{(J)}(2^{K-1}\xi)|\,d\xi \\
&= \int_{-\pi}^{\pi} S_K(\xi/2)I_{I,K-1}(\xi)\,d\xi
\end{aligned} \tag{3}$$

We have

$$2\pi \le I_{J,0} \le I_{J,1} \le \cdots \le I_{J,K},$$

which follows from (3) and the fact that $S_K(\xi) \geq |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)|^2 + |m_{\varepsilon_{J-K}}^{(J-K)}(\xi+\pi)|^2 = 1$ for $K = 1, 2, \ldots$. Thus, using Lemma 3 and Hölder's inequality,

$$
\begin{aligned}
\|I_{J,K}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} &= \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \\
&\leq \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|_{L^{4/3}([-\pi,\pi],\frac{dx}{2\pi})} \\
&\leq \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|^2_{L^{4/3}([-\pi,\pi],\frac{dx}{2\pi})} \\
&\leq \|I_{J,K-1}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})}\|S_K(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}.
\end{aligned}
$$

Hence,

$$
\|I_{J,J-1}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \leq \|I_{J,0}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \cdot \prod_{j=1}^{J-1} \|S_j(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}.
$$

Clearly $\|I_{J,0}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \leq 2$, so it suffices to prove that $\prod_{j=1}^{J-1} \|S_j(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}$ is uniformly bounded in J. By Lemma 3,

$$
\|S_K(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})} = 1 + O(1/\sqrt{d_{J-K}}),
$$

and by assumption

$$
\sum_{j=1}^{J-1} \frac{1}{\sqrt{d_{J-j}}} \leq \sum_{j=1}^{\infty} \frac{1}{\sqrt{d_j}} \leq C \sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon/2}} < \infty.
$$

The claim now follows from the Weierstrass product test. ■

We use the above Lemma to obtain the following result.

Theorem 5. *Let $\{h^{(p)}\}_{p=0}^{\infty}$ be a family of Daubechies CQF's with associated transfer functions $\{m_0^{(p)}\}$. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that length$(h^{(p)}) \geq Cp^{2+\varepsilon}$. If $|\hat{w}_0(\xi)| \leq B(1+|\xi|)^{-1-\varepsilon}$ for some constant B then the Fourier transforms of associated non-stationary wavelet packets are uniformly bounded in L^1-norm and the wavelet packets are consequently uniformly bounded.*

Proof. Take $n : 2^{J+1} \leq n < 2^{J+2}$. Then

$$
\hat{w}_n(\xi) = m_{\varepsilon_1}^{(J)}(\xi/2)m_{\varepsilon_2}^{(J-1)}(\xi/4)\cdots m_{\varepsilon_{J+1}}^{(0)}(\xi/2^{J+1})\hat{\phi}(\xi/2^{J+1}).
$$

Also, since $|\hat{\phi}(\xi)| \leq B(1+|\xi|)^{-1-\varepsilon}$ we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\hat{w}_n(\xi)|\, d\xi &= \sum_{k\in\mathbb{Z}} \int_{-2^{J+1}\pi+k2^{J+2}\pi}^{2^{J+1}\pi+k2^{J+2}\pi} |\hat{w}_n(\xi)|\, d\xi \\
&\leq \int_{-2^{J+1}\pi}^{2^{J+1}\pi} |m_{\varepsilon_1}^{(J)}(\xi/2) m_{\varepsilon_2}^{(J-1)}(\xi/4)\cdots m_{\varepsilon_{J+1}}^{(0)}(\xi/2^{J+1})|\, d\xi \sum_{k\in\mathbb{Z}} C(1+2\pi|k|)^{-1-\varepsilon}. \\
&\leq B2^{J+1} \int_{-\pi}^{\pi} |m_{\varepsilon_{J+1}}^{(0)}(\xi) m_{\varepsilon_J}^{(1)}(2\xi)\cdots m_{\varepsilon_1}^{(J)}(2^J\xi)|\, d\xi,
\end{aligned}
$$

and the claim follows from Lemma 4. ■

Remark. It is an unfortunate consequence of the above nonstationary construction that the diameter of support for the nonstationary wavelet packets grows just as fast as the filterlength. This problem will be eliminated in the next section using a generalized construction of wavelet packets.

4 Highly Nonstationary Wavelet Packets

This section contains a generalization of stationary and nonstationary wavelet packets. The new definition introduces more flexibility into the construction, and thus allows for construction of functions with better properties than the corresponding nonstationary construction. We have named the new functions highly nonstationary wavelet packets (HNWPs) and the definition is the following

Definition 6 (Highly Nonstationary Wavelet Packets). *Let (ϕ, ψ) be the scaling function and wavelet associated with a multiresolution analysis, and let $\{m_0^{p,q}\}_{p\in\mathbb{N},1\leq q\leq p}$ be a family of CQFs. Let $w_0 = \phi$ and $w_1 = \psi$ and define the functions w_n, $n \geq 2$, $2^J \leq n < 2^{J+1}$, by*

$$
\hat{w}_n(\xi) = m_{\varepsilon_1}^{J,1}(\xi/2) m_{\varepsilon_2}^{J,2}(\xi/4)\cdots m_{\varepsilon_J}^{J,J}(\xi/2^J)\hat{\psi}(\xi/2^J),
$$

where $n = \sum_{j=1}^{J+1} \varepsilon_j 2^{j-1}$ is the binary expansion of n. We call $\{w_n\}_{n=0}^{\infty}$ a family of basic highly nonstationary wavelet packets (HNWPs).

Remarks. It is obvious that the definition of highly nonstationary wavelet packets includes the basic classical and basic nonstationary wavelet packets as special cases.

The new basic wavelet packets are generated by a nonstationary wavelet packet scheme that changes at each scale so it is still possible to use the discrete algorithms associated with the nonstationary wavelet packet construction. The complexity of the algorithm depends entirely on the choice of filters.

The following result shows that the integer translates of the basic HNWPs do give us an orthonormal basis for $L^2(\mathbb{R})$, just like the basic nonstationary wavelet packets.

Theorem 7. *Let $\{w_n\}_{n=0}^{\infty}$ be a family of highly nonstationary wavelet packets. Then*

$$\{w_n(\cdot - k)\}_{n\geq 0, k\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Proof. Recall that

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{j=0}^{\infty} W_j \right),$$

and by definition $w_n \in W_J$ for $2^J \leq n < 2^{J+1}$ so it suffices to show that

$$\{w_n(\cdot - k)\}_{2^J \leq n < 2^{J+1}, k\in\mathbb{Z}}$$

is an orthonormal basis for W_J. However, this follows at once from the first J steps of the induction argument in the proof of Theorem 2 using the filters $m_0^{(p)} = m_0^{J,J-p+1}$, for $p = 1, \cdots, J$. ■

The following corollary to Lemma 4 shows how the added flexibility in the definition of highly nonstationary wavelet packets allows one to get better joint time-frequency localization.

Corollary 8. *Let $\{h^{(p)}\}_{p=0}^{\infty}$ be a family of Daubechies CQFs with associated transfer functions $\{m_0^{(p)}\}$. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that*

$$C^{-1}p^{2+\varepsilon} \leq \mathit{length}(h^{(p)}) \leq Cp^{-1-\varepsilon}2^p.$$

Let $\{w_n\}_n$ be the highly nonstationary wavelet packets associated with $m_0^{p,q} = m_0^{(q)}$ for $p \geq 1$, $q \leq p$, and some pair (ϕ, ψ). If $|\hat{w}_0(\xi)| \leq B(1 + |\xi|)^{-1-\varepsilon}$ for some constant B then the Fourier transforms of associated nonstationary wavelet packets are uniformly bounded in L^1-norm and the wavelet packets are consequently uniformly bounded. Moreover, if w_1 has compact support then there is a $K < \infty$ such that $\mathit{supp}(w_n) \subset [-K, K]$ for all $n \geq 1$.

Proof. The first statement follows directly from the proof of Theorem 4. The second follows from the fact that the distribution defined as the inverse Fourier transform of the product $\prod_{j=1}^{J} m_{\varepsilon_j}^{(j)}(\xi/2)\hat{\psi}(\xi/2^J)$ has support contained in

$$\alpha\big[-\textstyle\sum_{j=1}^{J} \mathrm{length}(m_{\varepsilon_j}^{(j)})2^{-j}, \sum_{j=1}^{J} \mathrm{length}(m_{\varepsilon_j}^{(j)})2^{-j}\big] \subset [-\tilde{K}, \tilde{K}],$$

whenever $w_1 = \psi$ has compact support ($\alpha < \infty$ depends on the support of w_1). ■

5 Periodic HNWPs With Near Perfect Frequency Localization

It is proved in [8] that by periodizing any (reasonable) orthonormal wavelet basis associated with a multiresolution analysis one obtains a multiresolution analysis for $L^2[0,1)$. The same procedure works equally well with highly nonstationary wavelet packets,

Definition 9. *Let $\{w_n\}_{n=0}^{\infty}$ be a family of highly nonstationary basic wavelet packets satisfying $|w_n(x)| \leq C_n(1+|x|)^{-1-\varepsilon_n}$ for some $\varepsilon_n > 0$, $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$ we define the corresponding periodic wavelet packets $\widetilde{w_n}$ by*

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} w_n(x-k).$$

Note that the hypothesis about the pointwise decay of the wavelet packets w_n ensures that the associated periodic wavelet packets are well defined functions contained in $L^p[0,1)$ for every $p \in [1,\infty]$.

The following easy Lemma shows that the above definition is useful.

Lemma 10. *The family $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2[0,1)$.*

Proof. Note that $\widetilde{w_n} \in \widetilde{W_j}$ for $2^{j-1} \leq n < 2^j$ ($\widetilde{W_j}$ is the periodized version of the wavelet space W_j) and that $\widetilde{W_j}$ is 2^{j-1} dimensional (see [8] for details), so it suffices to show that $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal system. We have, using Fubini's Theorem,

$$\begin{aligned}
\int_0^1 \widetilde{w_n}(x)\overline{\widetilde{w_m}(x)}\,dx &= \int_0^1 \sum_{q\in\mathbb{Z}} w_n(x-q)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \sum_{q\in\mathbb{Z}} \int_0^1 w_n(x-q)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \int_{-\infty}^{\infty} w_n(x)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \sum_{r\in\mathbb{Z}} \int_{-\infty}^{\infty} w_n(x)\overline{w_m(x-r)}\,dx \\
&= \delta_{m,n}.
\end{aligned}$$

∎

We are interested in periodic wavelet packets obtained from wavelet packets with very good frequency resolution. The idealized case is the Shannon wavelet packets. The Shannon wavelet packets are defined by taking

$$m_0^S(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\pi/2,\pi/2]}(\xi - 2\pi k)$$

and

$$m_1^S(\xi) = e^{i\xi} m_0^S(\xi + \pi)$$

in Definition 1. There is a nice explicit expression for $|\hat{w}_n|$. We define a map $G : \mathbb{N}_0 \to \mathbb{N}_0$ in the following way. Let $n = \sum_{k=1}^{\infty} n_k 2^{k-1}$ be the binary expansion of $n \in \mathbb{N}_0$. Then we let $G(n)_i = n_i + n_{i+1} \pmod 2$, and put $G(n) = \sum_{k=1}^{\infty} G(n)_k 2^{k-1}$. The map G is the so-called Gray-code permutation. We have the following simple formulas for the Shannon wavelet packets, which show that they have perfect frequency resolution. See [11] for a proof.

Theorem 11 ([11]). *Let $\{w_n\}_n$ be the Shannon wavelet packets. Then*

$$|\hat{w}_{G(n)}(\xi)| = \chi_{[n\pi,(n+1)\pi]}(|\xi|).$$

We define a new system by letting $\omega_n = w_{G(n)}$ for $n \in \mathbb{N}_0$. We call the reordered system $\{\omega_n\}_{n=0}^{\infty}$ the Shannon wavelet packets in frequency order.

The Shannon wavelet packets are not contained in $L^1(\mathbb{R})$ so one has to be careful trying to periodize the functions. We can avoid this problem by viewing the Shannon filter as the limit of a sequence of Meyer filters. The Meyer filter with resolution ε is defined to be a non-negative CQF, $m_0^{M,\varepsilon}$, for which

$$m_0^{M,\varepsilon}|_{(-\pi/2+\varepsilon,\pi/2-\varepsilon)} = 1.$$

We always assume that $m_0^{M,\varepsilon} \in C^1(\mathbb{R})$. As usual, we take $m_1^{M,\varepsilon}(\xi) = e^{i\xi} m_0^{M,\varepsilon}(\xi + \pi)$.

For Meyer filters, Hess-Nielsen observed that periodic wavelet packets in frequency ordering are just shifted sine and cosines at the low frequencies. More precisely, for $n \in \mathbb{N}$ we use the binary expansion $2n = \sum_{\ell=0}^{\infty} \varepsilon_\ell 2^\ell$ to define a sequence $\{\kappa_n\}$ by

$$\kappa_n = \sum_{\ell=0}^{\infty} |\varepsilon_\ell - \varepsilon_{\ell+1}| 2^{-\ell-1}.$$

Then the result is

Theorem 12 ([7]). *Choose ε such that $\pi/6 > \varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $\varepsilon \le 2^{-N}$. For m_0 a Meyer filter with resolution $\varepsilon/(\pi - \varepsilon)$ we consider the periodized wavelet packets $\{\widetilde{w_n}\}_n$ in frequency order generated using m_0 and the associated high-pass filter. They fulfill*

$$\widetilde{w_{2n}}(x) = \sqrt{2}\cos[2\pi n(x - \kappa_n)]$$
$$\widetilde{w_{2n-1}}(x) = \sqrt{2}\sin[2\pi n(x - \kappa_n)],$$

for each n, $0 < n < 2^{N-1}$.

The periodized version of the Shannon wavelet packet system should correspond to the limit of the above results as we let $\varepsilon \to 0$. This consideration leads us to the following definition:

Definition 13. *We define the periodic Shannon wavelet packets $\{\widetilde{S_n}\}$ (in frequency order) by $\widetilde{S}_0 = 1$ and for $n \in \mathbb{N}$:*

$$\widetilde{S_{2n}}(x) = \sqrt{2}\cos[2\pi n(x-\kappa_n)]$$
$$\widetilde{S_{2n-1}}(x) = \sqrt{2}\sin[2\pi n(x-\kappa_n)].$$

This system has all the useful properties one can hope for:

Theorem 14. *The system $\{\widetilde{S_n}\}_n$ is an orthonormal basis for $L^2[0,1)$ and a Schauder basis for $L^p[0,1)$, $1<p<\infty$.*

Proof. The L^2 result follows from the fact that any finite subsystem of $\{\widetilde{S_n}\}_n$ is a subset of the orthonormal basis considered in Theorem 12 for sufficiently small ε. To get the L^p result it suffices to notice that for any sequence $(\delta_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$, $\{e^{2\pi ik(x-\delta_k)}\}_k$ is a Schauder basis for $L^p[0,1)$, which follows easily by calculating the associated partial sums

$$\begin{aligned}\sum_{|k|\le N}\langle f, e^{-2\pi ik\delta_k}e^{2\pi ik\cdot}\rangle e^{2\pi ik(x-\delta_k)} &= \sum_{|k|\le N} e^{2\pi ik\delta_k}\langle f, e^{2\pi ik\cdot}\rangle e^{-2\pi ik\delta_k}e^{2\pi ikx}\\ &= \sum_{|k|\le N}\langle f, e^{2\pi ik\cdot}\rangle e^{2\pi ikx},\end{aligned}$$

where we have used that the coefficient functional of $e^{2\pi in(x-\delta_n)}$ is just $e^{2\pi in(x-\delta_n)}$ since $\{e^{2\pi ik(x-\delta_k)}\}_k$ is an orthonormal system in $L^2[0,1)$. ∎

5.1 Periodic Shannon Wavelets

Our goal in this section is to construct periodic HNWPs that are equivalent in $L^p[0,1)$ to small perturbations of the periodic Shannon wavelet packets. To get such results we need some results on the periodic Shannon wavelets. The Shannon wavelets are not contained in $L^1(\mathbb{R})$ so we have to use the same type of limit precedure as we did for the Shannon wavelet packets to defined the periodized version of the functions. We obtain

Definition 15. *Let $\Sigma_0 = 1$. For $n = 2^J + k$, $0 \le k < 2^J$, $J \ge 0$, we define Σ_n by*

$$\Sigma_n(x) = f_J(x - 2^{-J}k),$$

where

$$f_J(x) = 2^{-J/2}\sum_{\ell=2^{J-1}}^{2^J} b(\ell)\big[e^{2\pi i\ell/2^{J+1}}e^{-2\pi i\ell x} + e^{-2\pi i\ell/2^{J+1}}e^{2\pi i\ell x}\big],$$

and

$$b(\ell) = \begin{cases} 1/\sqrt{2}, & \text{if } \ell \in \{2^j\}_{j\ge 0},\\ 1, & \text{otherwise.}\end{cases}$$

We call $\{\Sigma_n\}_{n=0}^{\infty}$ the family of periodic Shannon wavelets.

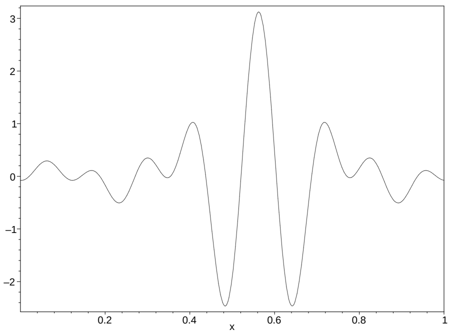

Figure 1: The function $f_3(\cdot - 1/2)$.

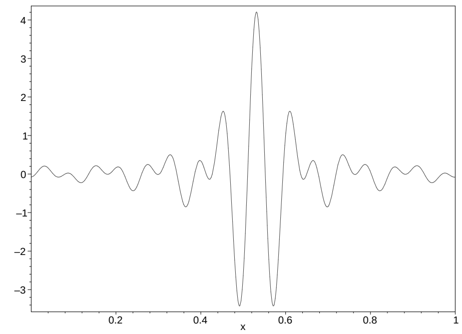

Figure 2: The function $f_4(\cdot - 1/2)$.

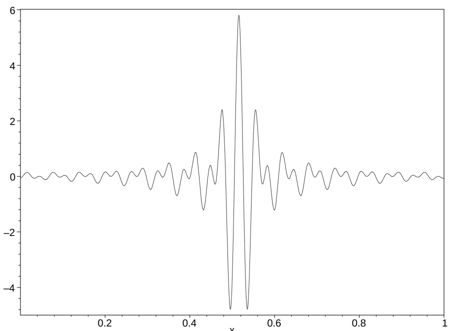

Figure 3: The function $f_5(\cdot - 1/2)$.

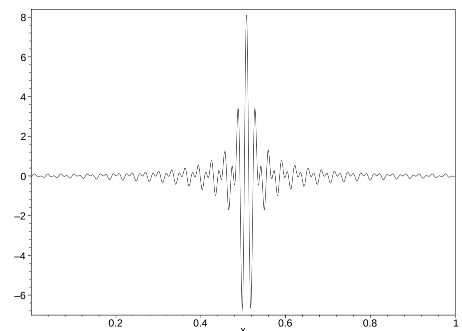

Figure 4: The function $f_6(\cdot - 1/2)$.

Since any finite subset of $\{\Sigma_n\}_{n\geq 0}$ is a subsystem of a periodized Meyer wavelet system (the Meyer wavelet needed depends on the subset of $\{\Sigma_n\}_{n\geq 0}$, of course), it follows that the system is indeed an orthonormal basis for $L^2[0,1)$. First, let us show that the periodic Shannon wavelets are equivalent to the Haar system in $L^p[0,1)$. The Haar system $\{h_n\}_{n=0}^{\infty}$ on $[0,1)$ is defined by letting $h_0 = \chi_{[0,1)}$ and, for $k = 0,1,\dots,\ell = 1,2,\dots,2^k$,

$$h_{2^k+\ell}(x) = \begin{cases} 2^{k/2} & \text{if } x \in [(2\ell-2)2^{-k-1}, (2\ell-1)2^{-k-1}) \\ -2^{k/2} & \text{if } x \in [(2\ell-1)2^{-k-1}, 2\ell \cdot 2^{-k-1}) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that this system is the periodic version of the Haar wavelet system with the numbering introduced in [8].

We will need the following lemma by P. Wojtaszczyk,

Lemma 16 ([14]). *Let f be a trigonometric polynomial of degree n. Then there exists a constant*

$C > 0$ *such that*

$$Mf(x) \geq C \sup_{|t-x| \leq \pi/n} |f(t)|,$$

where M is the classical Hardy-Littlewood maximal operator,

to get the following Theorem. The proof is in the spirit of Wojtaszczyk's work [14].

Theorem 17. *The periodic Shannon wavelets are equivalent to the (periodic) Haar wavelets in $L^p[0,1]$, $1 < p < \infty$.*

Proof. First, we have to introduce and analyze some auxiliary functions. For $n = 2^J + k$, $0 \leq k < 2^J$ we define

$$\Phi_n(x) = 2^{-(J-1)/2} \sum_{s=2^{J-1}}^{2^J-1} \exp\left\{2\pi is\left(x - \frac{k+1/2}{2^J}\right)\right\}.$$

Note that

$$e^{-2^{J-1}2\pi ix}\Phi_n(x) = e^{-\pi i(k+1/2)}2^{-(J-1)/2} \sum_{s=0}^{2^{J-1}-1} \exp\left\{2\pi is\left(x - \frac{k+1/2}{2^J}\right)\right\}. \tag{4}$$

In particular, $\{\Phi_{2n}\}_{n\geq 0}$ and $\{\Phi_{2n-1}\}_{n\geq 1}$ are both orthonormal systems, since each of the blocks

$$\{\Phi_{2n}\}_{2^J \leq 2n < 2^{J+1}} \qquad \text{and} \qquad \{\Phi_{2n-1}\}_{2^J \leq 2n-1 < 2^{J+1}}$$

is a unitary image of the orthonormal system

$$\left\{2^{-(J-1)/2} \sum_{s=0}^{2^{J-1}-1} e^{2\pi is(x-k/2^{J-1})}\right\}_{k=0}^{2^{J-1}-1}.$$

Moreover, it is easy to check that

$$\operatorname{span}\{\Phi_{2n}\}_{0\leq 2n<2^J} = \operatorname{span}\{\Phi_{2n-1}\}_{0\leq 2n-1<2^J} = \operatorname{span}\{e^{2\pi inx}\}_{n=0}^{2^{J-1}-1}.$$

Let $\{a_k\}_{k\geq 0} \subset \mathbb{C}$ and define

$$\begin{aligned} f(x) &= e^{-2^{J-1}2\pi ix} \sum_{2^J \leq 2\ell < 2^{J+1}} a_{2\ell}\Phi_{2\ell}(x) \\ &= 2^{-(J-1)/2}e^{-i\pi/2} \sum_{0\leq 2k<2^J} a_{2^J+2k}\left\{\sum_{s=0}^{2^{J-1}-1} \exp\left\{2\pi is\left(x - \frac{k}{2^{J-1}} - \frac{1}{2^{J+1}}\right)\right\}\right\}. \end{aligned} \tag{5}$$

In particular,

$$|f(\tfrac{\ell}{2^{J-1}} + \tfrac{1}{2^{J+1}})| = \tfrac{1}{\sqrt{2}}2^{J/2}|a_{2^J+2\ell}|,$$

since

$$\sum_{s=0}^{2^{J-1}-1} e^{2\pi i(\ell-k)s/2^{J-1}} = 2^{J-1}\delta_{\ell,k}.$$

It follows from Lemma 16 and (5) that

$$M\Big(\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big)(x) \ge C \sum_{2^J\le 2\ell<2^{J+1}} |a_{2\ell}|\Big(|h_{2\ell}(x)|+|h_{2\ell+1}(x)|\Big). \tag{6}$$

Hence, by using the Littlewood-Paley theorem and the Fefferman-Stein inequality for vector valued maximal functions,

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty} a_{2n}\Phi_{2n}\Big\|_p &= \Big\|a_0\Phi_0 + \sum_{J=0}^{\infty}\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big\|_p \\
&\ge \Bigg(\int_0^1 \Bigg(|a_0\Phi_0|^2 + \sum_{J=0}^{\infty}\Big|\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big|^2\Bigg)^{p/2} dx\Bigg)^{1/p} \\
&\ge C_p\Bigg(\int_0^1 \Bigg(|a_0|^2 + \sum_{J=0}^{\infty}\Big|M\Big(\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big)\Big|^2\Bigg)^{p/2} dx\Bigg)^{1/p},
\end{aligned}$$

and by (6),

$$\begin{aligned}
&\ge C_p\Bigg(\int_0^1 \Bigg(|a_0|^2 + \sum_{J=0}^{\infty}\Big(\sum_{2^J\le 2\ell<2^{J+1}} |a_{2\ell}|[|h_{2\ell}|+|h_{2\ell+1}|]\Big)^2\Bigg)^{p/2} dx\Bigg)^{1/p} \\
&\ge C_p\Big\|\sum_{l=0}^{\infty} a_{2n}[h_{2n}+h_{2n+1}]\Big\|_p \\
&\ge C_p\Big\|\sum_{l=0}^{\infty} a_{2n}h_{2n}\Big\|_p,
\end{aligned}$$

where we have used the unconditionality of the Haar system (in particular, the projection onto the even numbered Haar functions is bounded on $L^p[0,1)$, $1<p<\infty$). A similar proof shows that

$$\Big\|\sum_{n=1}^{\infty} a_{2n-1}\Phi_{2n-1}\Big\|_p \ge C_p\Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p.$$

Actually, it is the opposite inequalities we really need. However, since $\mathrm{span}\{\Phi_{2n}\}_{n\ge 0}$ is dense in

$H^q[0,1)$, for $f = \sum a_{2n}\Phi_{2n}$ and $\varepsilon > 0$ there is a $g = \sum b_{2n}\Phi_{2n}$ with $\|g\|_q \le 1+\varepsilon$ such that

$$\begin{aligned}\|f\|_p - \varepsilon &\le \left|\left\langle \sum a_{2n}\Phi_{2n}, \sum b_{2n}\Phi_{2n}\right\rangle\right| \\ &= \left|\left\langle \sum a_{2n}h_{2n}, \sum b_{2n}h_{2n}\right\rangle\right| \\ &\le \left\|\sum a_{2n}h_{2n}\right\|_p \left\|\sum b_{2n}h_{2n}\right\|_q \\ &\le C\left\|\sum a_{2n}h_{2n}\right\|_p \left\|\sum b_{2n}\Phi_{2n}\right\|_q.\end{aligned}$$

Since ε was arbitrary, we have

$$\left\|\sum a_{2n}\Phi_{2n}\right\|_p \le C\left\|\sum a_{2n}h_{2n}\right\|_p,$$

and similarly,

$$\left\|\sum a_{2n+1}\Phi_{2n+1}\right\|_p \le C\left\|\sum a_{2n+1}h_{2n+1}\right\|_p.$$

Finally, we can prove the theorem. Let $\mathcal{R}$ denote the Riesz projection, i.e. the projection onto $\operatorname{span}\{e^{2\pi inx}\}_{n\ge 0}$. Then for any finite sequence $\{a_k\}_{k\ge 0} \subset \mathbb{C}$ we have

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} a_n\Sigma_n\right\|_p &\le \left\|\sum_{n=0}^{\infty} a_n\mathcal{R}\Sigma_n\right\|_p + \left\|\sum_{n=0}^{\infty} a_n(1-\mathcal{R})\Sigma_n\right\|_p \\ &\le \left\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\right\|_p + \left\|\sum_{n=1}^{\infty} a_{2n-1}\mathcal{R}\Sigma_{2n-1}\right\|_p \\ &\qquad + \left\|\sum_{n=0}^{\infty} a_{2n}(1-\mathcal{R})\Sigma_{2n}\right\|_p + \left\|\sum_{n=1}^{\infty} a_{2n-1}(1-\mathcal{R})\Sigma_{2n-1}\right\|_p. \qquad (7)\end{aligned}$$

Let $P : L^p[0,1) \to L^p[0,1)$ denote the projection onto the frequencies $\{e^{2\pi i2^j x}\}_{j\ge 0}$. The operator P is bounded on $L^p[0,1)$ since for $2 \le p < \infty$, $\{c_k\} \subset \mathbb{C}$,

$$\left\|\sum_{j\ge 0} c_{2^j} e^{i2^j x}\right\|_{L^p[0,1)} \le C\left\|\sum_{j\ge 0} c_{2^j} e^{i2^j x}\right\|_{L^2[0,1)} \le C\left\|\sum_{k\in\mathbb{Z}} c_k e^{ikx}\right\|_{L^2[0,1)} \le C\left\|\sum_{k\in\mathbb{Z}} c_k e^{ikx}\right\|_{L^p[0,1)},$$

where we have used Khintchine's inequality for lacunary Fourier series (see [12, I.B.8]). The case $1 < p < 2$ follows by duality. We have,

$$\begin{aligned}\left\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\right\|_p &\le \left\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\right\|_p + \left\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\right\|_p \\ &\le C\left(\left\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\right\|_2 + \left\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\right\|_p\right).\end{aligned}$$

A direct calculation shows that

$$P\Big(\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\mathcal{R}\Sigma_{2^J+2\ell}\Big) = 2^{-(J+1)/2}\Big\{\Big(e^{-i\pi/2}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^{J-1}x} + \Big(e^{-i\pi}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^J x}\Big\},$$

whereas,

$$P\Big(\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Phi_{2^J+2\ell}\Big) = 2^{-(J-1)/2}\Big(e^{-i\pi/2}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^{J-1}x}.$$

Thus,

$$\Big\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\Big\|_2 \le \Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n}\Big\|_2,$$

and we get

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\Big\|_p &\le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n}\Big\|_2 + \Big\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p\Big)\\
&\le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n} + \sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p\Big)\\
&= C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n} + \sum_{n=0}^{\infty} a_{2n}(1-P)\Phi_{2n}\Big\|_p\Big)\\
&= C\Big\|\sum_{n=0}^{\infty} a_{2n}\Phi_{2n}\Big\|_p\\
&\le C\Big\|\sum_{n=0}^{\infty} a_{2n}h_{2n}\Big\|_p.
\end{aligned}$$

Similarly, we obtain

$$\Big\|\sum_{n=1}^{\infty} a_{2n-1}\mathcal{R}\Sigma_{2n-1}\Big\|_p \le C\Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p.$$

The remaining two terms in (7) can easily be estimated by assuming (w.l.o.g., of course) that $\{a_k\}\subset\mathbb{R}$ and taking complex conjugates of the above estimates (note, the coefficients at negative frequencies of Σ_n are just the conjugate of the coefficients at positive frequencies). We conclude that

$$\Big\|\sum_{n=0}^{\infty} a_n\Sigma_n\Big\|_p \le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}h_{2n}\Big\|_p + \Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p\Big) \le C\Big\|\sum_{n=0}^{\infty} a_n h_n\Big\|_p,$$

where we have used that the projection onto the even numbered Haar functions is bounded on $L^p[0,1)$. To obtain the opposite inequality, we let $\varepsilon>0$ and let $f=\sum a_n h_n$. The Haar system is dense in $L^q[0,1]$ so there is a function $g = \sum b_n h_n \in \operatorname{span}(h_n)$ with $\|g\|_q \le 1+\varepsilon$ such that

$$\begin{aligned}
\|f\|_p - \varepsilon &\le |\langle \sum a_n h_n, \sum b_n h_n \rangle| \\
&= |\langle \sum a_n \Sigma_n, \sum b_n \Sigma_n \rangle| \\
&\le \Big\| \sum a_n \Sigma_n \Big\|_p \Big\| \sum b_n \Sigma_n \Big\|_q \\
&\le C \Big\| \sum a_n \Sigma_n \Big\|_p \Big\| \sum b_n h_n \Big\|_q \\
&\le C(1+\varepsilon) \Big\| \sum a_n \Sigma_n \Big\|_p,
\end{aligned}$$

where we have used the orthonormality of the system Σ_n. Since ε was arbitrary we have

$$\Big\| \sum a_n h_n \Big\|_p \le C \Big\| \sum a_n \Sigma_n \Big\|_p,$$

and we are done. ■

The following Theorem is due to Y. Meyer, but the proof is new.

Theorem 18. *Let $\{\Psi_n\}_n$ be a periodic wavelet system associated with a wavelet ψ satisfying $|\psi(x)| \le C(1+|x|)^{-2-\varepsilon}$. Then $\{\Psi_n\}_n$ is equivalent to the (periodic) Haar wavelets in $L^p[0,1]$.*

Proof. By duality, it suffices to prove that

$$\Big\| \sum_{n=0}^{\infty} a_n \Psi_n \Big\|_p \ge C \Big\| \sum_{n=0}^{\infty} a_n h_n \Big\|_p.$$

We have, by the Fefferman-Stein inequality,

$$\begin{aligned}
\Big\| \sum_{n=0}^{\infty} a_n \Psi_n \Big\|_p &= \Big\| a_0 \Psi_0 + \sum_{J=0}^{\infty} \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big) \Big\|_p \\
&\ge C \Big(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty} \Big| \sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big|^2 \Big)^{p/2} dx \Big)^{1/p} \\
&\ge C \Big(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty} \Big| M \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big) \Big|^2 \Big)^{p/2} dx \Big)^{1/p}
\end{aligned}$$

It follows from [13, p. 208] that for $n = 2^J + k$,

$$|\Psi_n(x)| \le C 2^{J/2} (1 + 2^J |x - k/2^J|)^{-1-\varepsilon}. \tag{8}$$

Hence, for $x \in [k2^{-J}, (k+1)2^{-J})$ (see [10, pp. 62-63]),

$$|a_n| = \left| \int_0^1 \Big(\sum_{\ell=2^J}^{2^{J+1}-1} a_\ell \Psi_\ell(y) \Big) \overline{\Psi_n(y)}\, dy \right| \le C 2^{-J/2} M \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big)(x),$$

where we have used the estimate (8), which shows that $2^{J/2}|\Psi_n|$ is an approximation of the identity centered at $k2^{-J}$. Thus

$$M \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big) \ge C \sum_{k=2^J}^{2^{J+1}-1} |a_k||h_k|,$$

and we have

$$\begin{aligned} \Big\| \sum_{n=0}^{\infty} a_n \Psi_n \Big\|_p &\ge C \Big(\int_0^1 \Big(|a_0|^2 + \sum_{s=0}^{\infty} \Big| \sum_{k=2^J}^{2^{J+1}-1} |a_k||h_k| \Big|^2 \Big)^{p/2} dx \Big)^{1/p} \\ &\ge C \Big\| \sum_{n=0}^{\infty} a_n h_n \Big\|_p . \end{aligned}$$

■

The following corollary is immediate

Corollary 19. *Let $\{\Psi_n\}_n$ be a periodic wavelet packet system associated with a wavelet ψ satisfying $|\psi(x)| \le C(1+|x|)^{-2-\varepsilon}$. Then $\{\Psi_n\}_n$ is equivalent to the periodic Shannon wavelets in $L^p[0,1]$, $1 < p < \infty$.*

We let $\{w_n\}_n$ be a HNWP system for which $|w_1(x)| \le C(1+|x|)^{-2-\varepsilon}$, and let $\{\widetilde{w}_n\}_n$ be the corresponding periodic system. For $2^J \le n < 2^{J+1}$ write

$$\widetilde{w}_n(x) = \sum_{s=2^J}^{2^{J+1}-1} c_{n,s} \Psi_s(x),$$

where Ψ_n is the corresponding periodic wavelet. Define a new system $\{\widetilde{w}_n^S\}$ by

$$\widetilde{w}_n^S(x) = \sum_{s=2^J}^{2^{J+1}-1} c_{n,s} \Sigma_s(x),$$

where Σ_s is the periodic Shannon wavelets. Then we have the following result

Corollary 20. *The systems $\{\widetilde{w}_n\}_n$ and $\{\widetilde{w}_n^S\}_n$ are equivalent in $L^p[0,1)$, $1 < p < \infty$, in the sense that there exists an isomorphism Q on $L^p[0,1)$ such that*

$$Q\widetilde{w}_n = \widetilde{w}_n^S.$$

Proof. Take Q to be the isomorphism from Corollary 19 defined by $Q\Psi_n = \Sigma_n$. ■

Remark. The significance of the previous Corollary is that when dealing with periodic HNWPs $\{\widetilde{w}_n\}_n$ in $L^p[0,1)$, we may assume that the wavelet $\psi = w_1$ is a Meyer wavelet $\psi^{M,\delta}$ with arbitrarily good frequency localization, i.e. $\psi(\xi) = 1$ for $|\xi| \in (\pi+\delta, 2\pi-\delta)$ for a small number δ. To see this, let $\{\widetilde{w}_n^{M,\delta}\}_n$ be the periodic HNWP system obtained using the same filters that generated $\{\widetilde{w}_n\}_n$ but with $\psi^{M,\delta}$ as the wavelet. From the previous discussion of the periodic Meyer wavelets we see that by periodizing $\psi_{n,0}^{M,\delta}$ we get exactly Σ_{2^n} for $n \leq N(\delta)$, where $N(\delta) \to \infty$ as $\delta \to 0$. Hence, $\widetilde{w}_n^S = \widetilde{w}_n^{M,\delta}$ for $n < 2^{N(\delta)+1}$, and $\widetilde{w}_n^S$ can be mapped onto $\widetilde{w}_n$ by the isomorphism of Corollary 20.

5.2 Perturbation of Periodic Shannon Wavelet Packets

We need the following perturbation theorem by Krein and Liusternik (see [15]),

Theorem 21. *Let $\{x_n\}$ be a Schauder basis for a Banach space X and let $\{f_n\}$ be the associated sequence of coefficient functionals. If $\{y_n\}$ is a sequence of vectors in X with dense linear span and if*

$$\sum_{n=1}^{\infty} \|x_n - y_n\|_X \cdot \|f_n\|_{X^*} < \infty$$

then $\{y_n\}$ is a Schauder basis for X equivalent to $\{x_n\}$,

to prove our main theorem on periodic HNWPs;

Theorem 22. *Let $\{d_n\}_{n=0}^{\infty} \subset 2\mathbb{N}$ be such that $d_n \geq Cn4^n \log(n+1)$ for some constant $C > 0$. Let $\{\widetilde{w}_n\}_n$ be a periodic HNWP system (in frequency order) given by the filters $\{m_0^{n,q}\}_{n\geq 1, 1\leq q\leq n}$, where*

$$m_0^{n,q}(\xi) = m_0^{(d_n)}(\xi), \quad q = 1, 2, \ldots, n,$$

is the Daubechies filter of length d_n. Suppose $|w_1(x)| \leq C(1+|x|)^{-2-\varepsilon}$ for some $\varepsilon > 0$. Then $\{\widetilde{w}_n\}_n$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$.

Proof. By the remark at the end of the previous section, we can w.l.o.g. assume that $\tilde{w}_1$ is a periodic Shannon wavelet. We also note that since $\{\widetilde{w}_n\}_n$ is orthonormal in $L^2[0,1)$, a simple duality argument will give us the result for $2 < p < \infty$ if we can prove it for $1 < p < 2$. Fix $1 < p < 2$. Define the phase functions $\eta_n : \mathbb{R} \to [0, 2\pi)$ by

$$|m_0^{(d_n)}(\xi)| = e^{-i\eta_n(\xi)}\, m_0^{(d_n)}(\xi).$$

Define a family of low-pass filters by

$$m_0^{n,q}(\xi) = e^{i\eta_n(\xi)} m_0^{M,\delta_n},$$

where m_0^{M,δ_n} is a Meyer filter with localization δ_n to be chosen as follows. Take ψ^{M,δ_n} as the wavelet and consider the periodic HNWPs $\{\widetilde{w}_n^{M,\delta_n}\}_n$ generated by the filters $\{m_0^{n,q}\}_{n\geq 1}$. For fixed n, there is a $\delta_n > 0$ such that $0 < \delta \leq \delta_n$ implies that $\widetilde{w}_n^{M,\delta} = \widetilde{w}_n^{M,\tilde{\delta_n}}$. Set $\widetilde{w}_n^M = \widetilde{w}_n^{M,\delta_n}$. It follows from Theorem 12 and the proof of Theorem 14 that $\{\widetilde{w}_n^M\}_{n=0}^\infty$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, consisting of shifted sines and cosines (more precisely, $\widetilde{w}_n^M$ is a shifted version of $\widetilde{S}_n$). The property of this new basis we need is that the Fourier coefficients of $\widetilde{w}_n^M$ have the same phase (but not the same size) as the the Fourier coefficients of $\widetilde{w}_n$. We want to apply the perturbation result (Theorem 21). The system $\{\tilde{w}_n\}_n$ is clearly dense in $L^p[0,1)$ since the periodic wavelet packets generate a well behaved periodic multiresolution structure. So all we need to show is that

$$\sum_{n=0}^\infty \|\widetilde{w}_n - \widetilde{w}_n^M\|_p \cdot \|\widetilde{w}_n^M\|_q \simeq \sum_{n=0}^\infty \|\widetilde{w}_n - \widetilde{w}_n^M\|_p < \infty.$$

However,

$$\sum_{n=0}^\infty \|\widetilde{w}_n - \widetilde{w}_n^M\|_p \leq \sum_{n=0}^\infty \|\widetilde{w}_n - \widetilde{w}_n^M\|_2,$$

so it suffices to estimate $\|\widetilde{w}_n - \widetilde{w}_n^M\|_2$. To ensure that

$$\sum_{n=0}^\infty \|\widetilde{w}_n - \widetilde{w}_n^M\|_2 < \infty \tag{9}$$

we will show that for $2^J \leq n < 2^{J+1}$,

$$\|\widetilde{w}_n - \widetilde{w}_n^M\|_2 \leq C 2^{-J} J^{-1} \log(J)^{-2},$$

with C a constant independent of J.

The Fourier series for $\widetilde{w}_n^M$ is particularly simple and by construction it contains only two non-zero terms with the corresponding Fourier coefficients equal to $e^{\pm i\alpha} 2^{-1/2}$, i.e.

$$\widetilde{w}_n^M(x) = 2^{-1/2} e^{i\alpha} e^{2\pi i k_n x} + 2^{-1/2} e^{-i\alpha} e^{2\pi i k_n x}, \tag{10}$$

where $\alpha \in \mathbb{R}$ depends on the phase of the Daubechies filters used to generate $\{\widetilde{w}_n\}_n$ and $k_n \in \mathbb{N}$. We want to estimate the corresponding two coefficients with indices $\pm k_n$ in the Fourier series for $\widetilde{w}_n$. We have, for $2^J \leq n < 2^{J+1}$,

$$\widetilde{w}_n(x) = \sum_{k\in\mathbb{Z}} \hat{w}_n(2\pi k) e^{2\pi i k x},$$

and since w_1 is the Shannon wavelet (limit of Meyer wavelets) this reduces to the following trigonometric polynomial

$$\widetilde{w}_n(x) = \sum_{2^{J-1}\leq |k| \leq 2^J} \hat{w}_n(2\pi k) e^{2\pi i k x}.$$

Recall that

$$\hat{w}_n(\xi) = m_{\varepsilon_1}^{(d_J)}(\xi/2) m_{\varepsilon_2}^{(d_J)}(\xi/4) \cdots m_{\varepsilon_J}^{(d_J)}(\xi/2^J) \hat{\psi}^{M,\delta}(\xi/2^J),$$

where $G(n) = \sum_{j=1}^{J+1} \varepsilon_j 2^{j-1}$ is the binary expansion of of the Gray-code permutation of n. Consider the product

$$\beta_n \equiv m_{\varepsilon_1}^{(d_J)}(2\pi k_n/2) m_{\varepsilon_2}^{(d_J)}(2\pi k_n/4) \cdots m_{\varepsilon_J}^{(d_J)}(2\pi k_n/2^J) \hat{\psi}^{M,\delta}(2\pi k_n/2^J),$$

which equals the k_n'th Fourier coefficient of $\widetilde{w}_n$. We deduce from Theorem 12 that the product has exactly one factor equal to $2^{-1/2}$ in absolute value, namely the factor with argument $2^{-s} 2\pi k_n$ satisfying

$$\frac{2\pi k_n}{2^s} \in \frac{\pi}{2} + 2\pi\mathbb{Z}.$$

The arguments of the remaining factors are at least a distance of $2^{1-J}\pi$ from the set $\pi/2 + 2\pi\mathbb{Z}$. Moreover, Theorem 12 shows that the arguments of the remaining J factors are situated where the respective m_ε's are "big", i.e. in the set $[-\pi/2, \pi/2]$ for the low-pass filters appearing in the product and in the set $[-\pi, -\pi/2] \cup [\pi/2, \pi]$ for the high-pass filters appearing in the product.

Recall that, by construction, the Fourier coefficients of $\widetilde{w}_n$ and $\widetilde{w}_n^M$ have the same phase, i.e. $\beta_n = |\beta_n| e^{i\alpha}$, with the same α as in (10). Also, the Fourier series of $\widetilde{w}_n^M$ contains only two non-zero terms at frequencies $\pm k_n$. From this we deduce that

$$\|\widetilde{w}_n - \widetilde{w}_n^M\|_2^2 = 2|\beta_n - 2^{-1/2} e^{i\alpha}|^2 + \mathrm{err}, \tag{11}$$

and since $\widetilde{w}_n$ is normalized in $L^2[0,1)$, we have

$$\mathrm{err} + 2|\beta_n|^2 = 1. \tag{12}$$

Hence, the requirement that $\|\widetilde{w}_n - \widetilde{w}_n^M\|_2 \leq C 2^{-J} J^{-1} \log(J)^{-2}$, for $2^J \leq n < 2^{J+1}$, gives us the following inequality by substituting (12) in (11):

$$(\sqrt{2}|\beta_n| - 1)^2 + (1 - 2|\beta_n|^2) \leq \left(\frac{C}{2^J J \log^2(J)}\right)^2,$$

from which we obtain

$$|\beta_n| \geq \frac{1}{\sqrt{2}} \left(1 - \frac{1}{2}\left(\frac{C}{2^J J \log^2(J)}\right)^2\right).$$

We therefore have to verify that

$$\begin{aligned} |\beta_n| &= |m_{\varepsilon_1}^{(d_J)}(2\pi k_n/2) m_{\varepsilon_2}^{(d_J)}(2\pi k_n/4) \cdots m_{\varepsilon_J}^{(d_J)}(2\pi k_n/2^J) \hat{\psi}^{M,\delta}(2\pi k_n/2^J)| \\ &\geq \frac{1}{\sqrt{2}} \left(1 - \left(\frac{C}{2^J J \log^2(J)}\right)^2\right), \end{aligned} \tag{13}$$

for some constant C independent of J. The d_J's have already been chosen, so we just have to check the estimates to see that everything works out. We now consider (13) as an inequality in $d_J = N(J)$. Hence, (13) will be satisfied if

$$|m_0^{(N(J))}(\pi/2 - 2^{1-J}\pi)| \geq \left(1 - \left(\frac{C}{2^J J \log^2(J)}\right)^2\right)^{1/J}. \tag{14}$$

By the CQF conditions, (14) is equivalent to

$$|m_0^{(N(J))}(\pi/2 + 2^{1-J}\pi)|^2 \leq 1 - \left(1 - \left(\frac{C}{2^J J \log^2(J)}\right)^2\right)^{2/J}.$$

From lemma 3 we have

$$|m_0^{(N(J))}(\pi/2 + 2^{1-J}\pi)|^2 \leq |\cos(2^{1-J}\pi)|^{2N(J)-2},$$

which gives us an explicit way to pick a sequence $N(J)$ that works. We put

$$\cos(2^{1-J}\pi)^{2(N(J)-1)} \leq 1 - \left(1 - \left(\frac{C}{2^J J \log^2(J)}\right)^2\right)^{2/J}$$

A simple estimate shows that

$$1 - \left(1 - \left(\frac{C}{2^J J \log^2(J)}\right)^2\right)^{2/J} \leq \left(\frac{C}{2^J J \log^2(J)}\right)^2.$$

Hence,

$$2(N(J)-1)\log\cos(2^{1-J}\pi) \leq 2(\log(C) - (J + \log(J) + 2\log\log(J))) \tag{15}$$

Using

$$\log\cos(x) = -\frac{1}{2}x^2 + O(x^4), \quad \text{as } x \to 0,$$

in (15), we see that choosing

$$N(J) \geq CJ2^{2J}\log(J)$$

for any $C > 0$ will work. This is exactly our hypothesis about the d_J's. ■

Remark. It follows from the above estimates that the factor $\log(n+1)$ in the hypothesis about the sequence $\{d_n\}$ can be replaced by α_n with $\{\alpha_n\}$ any positive increasing sequence with $\alpha_n \to \infty$.

6 Representation of $\frac{d}{dx}$ in Periodic HNWPs

We conclude this paper by using some of the estimates obtained in the previous section to get estimates of the differentiation operator represented in certain periodic HNWP bases. First we consider the idealized case of periodic Shannon wavelet packets, in which the matrix for the differentiation operator is almost diagonal. Then we show that the matrix of the operator in the periodic HNWPs of Theorem 22 is a small perturbation of the almost diagonal matrix associated with the periodic Shannon wavelet packets.

Let P_j be the projection onto the closed span of $\{\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_j\}$. We let $\Delta_j = P_j \frac{d}{dx} P_j$. Note that $\widetilde{S}'_{2n} = -2\pi n \widetilde{S}_{2n-1}$ and $\widetilde{S}'_{2n-1} = 2\pi n \widetilde{S}_{2n}$, so if we let Δ be the 2×2-matrix defined by

$$\Delta = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix},$$

then Δ_{2n} is the block diagonal matrix given by

$$\Delta_{2n} = \mathrm{diag}(0, \Delta, 2\Delta, \ldots, n\Delta).$$

For a general periodic HNWP system $\{\widetilde{w}_n\}_n$ in frequency order we let $D_j = P_j \frac{d}{dx} P_j$, with P_j the projection onto the closed span of $\{\widetilde{w}_0, \widetilde{w}_1, \ldots, \widetilde{w}_j\}$. We can write $D_{2n} = \Delta_{2n} + E_{2n}$, where E_{2n} is the "error term" resulting from the fact that the system does not have perfect frequency resolution like the Shannon system. The error term is not necessarily almost diagonal and easy to implement like Δ_{2n} is, and it can be difficult to calculate. However, the following Corollary (to the proof of Theorem 22) shows that we can make this error term as small as we like, and we may therefore disregard it in any implementation.

Corollary 23. *Given $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $\{\widetilde{w}_n\}_{n=0}^{\infty}$ be a periodic HNWP system in frequency order constructed as in Theorem 22 associated with a Meyer wavelet of resolution $\varepsilon/2^N$. Suppose*

$$d_n \geq \frac{\log \varepsilon - \log(12\pi) - N \log 4}{2 \log \cos(2^{1-N}\pi)} + 1, \quad n \leq N,$$

then $D_{2^N} = \Delta_{2^N} + E_{2^N}$ with $\|E_{2^N}\|_{\ell^2 \to \ell^2} \leq \varepsilon$.

Proof. We write $\widetilde{w}_n = \widetilde{w}_n^M + e_n$ with $\widetilde{w}_n^M$ defined as in the proof of Theorem 22. Hence

$$\begin{aligned} \langle \widetilde{w}'_n, \widetilde{w}_m \rangle &= \langle (\widetilde{w}_n^M)' + e'_n, \widetilde{w}_m^M + e_m \rangle \\ &= \langle (\widetilde{w}_n^M)', \widetilde{w}_m^M \rangle + \langle (\widetilde{w}_n^M)', e_m \rangle + \langle e'_n, \widetilde{w}_m^M \rangle + \langle e'_n, e_m \rangle. \end{aligned}$$

The first term is just the nm'th entry in Δ_{2^N}. We then impose the inequality

$$|\langle (\widetilde{w}_n^M)', e_m \rangle| + |\langle e'_n, \widetilde{w}_m^M \rangle| + |\langle e'_n, e_m \rangle| \leq \frac{\varepsilon}{2^N},$$

which will make $\|E_{2^N}\|_{\ell^2\to\ell^2} \leq \varepsilon$ since E_{2^N} is a $(2^N+1)\times(2^N+1)$-matrix with its first row and column both equal to 0. The choice of wavelet ensures that support of the Fourier coefficients of each e_n, $0 \leq n \leq 2^N$, is contained in the set $[-2\pi 2^{N+1}, 2\pi 2^{N+1}]$ so using the Schwartz and Bernstein inequalities we see that it suffices to take $\|e_n\|_2 \leq \dfrac{\varepsilon}{12\pi\cdot 4^N}$. The estimate then follows from similar estimates as those in the proof of Theorem 22. ■

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