On the Construction and Frequency Localization of Finite Orthogonal Quadrature Filters

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October 6, 1999

Abstract

In this article we introduce a new method to construct finite orthogonal quadrature filters using convolution kernels. We show that every filter with value 1 at the origin can be obtained using an even nonnegative kernel.

We apply the method to estimate the optimal frequency localization of finite filters. The frequency localization γ_p of a finite filter m_0 is given by the distance in L^p-norm between $|m_0|^2$ and the Shannon low-pass filter. For each $N > 0$ there is a filter m_0^N of length $2N$ minimizing the value of γ_p. We prove that for such a minimizing sequence we have $\gamma_p^p(m_0^N) = O(1/N)$, $1 \leq p \leq 2$, and this estimate is optimal. We construct several new families of both MRA and non-MRA filters with optimal asymptotic frequency localization.

1 Introduction

A pair of orthogonal quadrature filters is a pair (m_0, m_1) of 2π-periodic measurable functions for which the matrix

$$\begin{bmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{bmatrix} \tag{1}$$

is unitary a.e. We are interested in the case where m_0 and m_1 are trigonometric polynomials with real valued coefficients. Suppose we have a trigonometric polynomial m_0 with real valued coefficients satisfying

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 \equiv 1. \tag{2}$$

Taking m_1 to be the trigonometric polynomial

$$m_1(\xi) = e^{i\alpha} e^{-i\xi} \overline{m_0(\xi+\pi)},$$

where $\alpha \in \mathbb{R}$ is a constant, will make (1) unitary everywhere. So the construction of pairs of finite quadrature filters reduces to finding "nice" solutions of (2).

In the present paper we introduce a new method to construct such trigonometric polynomials using certain convolution kernels. The most important class of solutions of (2) is the one where the filter m_0 takes on the value 1 at 0. In this case we can show that all such m_0's are given in terms of a nonnegative convolution kernel. All of this will be done in Section 2.

In section 3 we apply the new method to construct filters with optimal frequency resolution. The frequency resolution of a filter m_0 is given by the quantity $\gamma_p(m_0)$ defined by

$$\gamma_p(m_0) = \int_{-\pi}^{\pi} ||\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0(\xi)|^2|^p \, d\xi, \qquad 1 \leq p \leq 2. \tag{3}$$

One important application of quadrature filters, where the frequency localization γ_p matters, is to the construction of nonstationary wavelet packets. A problem one has to deal with in the wavelet packet construction using finite filters is that the wavelet packets lose some of their frequency resolution at high frequencies. This is due to the fact that every finite filter m_0 is only an approximation to the idealized low-pass filter $\chi_{[-\pi/2,\pi/2]}(\xi)$. So it is of interest to find an explicit expression for the filter of length $2N$ that gives the best approximation of $\chi_{[-\pi/2,\pi/2]}(\xi)$ wrt. some appropriate measure, and to find the best possible estimates for the error. Hess-Nielsen proves in [5] that if we are restricted to using filter of length $2N$ at a certain scale in the wavelet packet construction then the filter that gives the optimal frequency localization for the wavelet packets at the following scale is the filter of length $2N$ that minimizes γ_1. He also proved that the there is a unique filter (up to a phase factor) of length $2N$ that minimizes γ_1. From this result we easily deduce the following result

Theorem 1. *For each p, $1 \leq p \leq 2$, and $N \in \mathbb{N}$ there is a quadrature filter m_0 of length $2N$ minimizing (3). The minimizing quadrature filter of length $2N$ is unique up to a phase factor.*

Remarks: The proof for $p = 1$ can be found in [4]. The uniqueness result for $1 < p \leq 2$ is due to the fact that $L^p[-\pi, \pi)$ is strictly convex.

For each p, $1 \leq p \leq 2$, we call a filter of length $2N$ that minimizes γ_p a Hess filter of length $2N$.

The main result we prove on frequency resolution is the following theorem, which will be proved at the end of section 3.

Theorem 2. *Fix p, $1 \leq p \leq 2$. Let m_0^N be a Hess filter of length $2N$. Then there exist finite positive constants c and C (depending only on p) such that*

$$\frac{c}{N} \leq \gamma_p(m_0^N) \leq \frac{C}{N}.$$

To get the upper bound we construct explicit examples of families of quadrature filters with optimal frequency resolution using certain convolution kernels such as Fejér-Korovkin and Jackson kernels. The lower bound applies to all filters of length $2N$, and the constant can be chosen independent of p.

A special class of filters is the family of filters associated with multiresolution analyses (called MRA filters). An MRA filter m_0 generates the scaling function ϕ associated with the MRA by

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi).$$

A sufficient condition for a filter m_0 satisfying (2), and taking on the value 1 at 0, to be an MRA filter is that m_0 not vanish on $[-\pi/2, \pi/2]$ (see [3]). In section 4 we construct two families of finite MRA filters having optimal frequency resolution.

2 Filters Generated By Convolution Kernels

In this section we introduce a new method to construct finite quadrature filters convolution operators of polynomial type. But first we make some general and well known observations about trigonometric polynomials satisfying (2).

Suppose m_0 is a trigonometric polynomial with real coefficients satisfying (2). By shifting the phase of m_0, if necessary, we can always assume m_0 can be written as

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=0}^{N} h_n e^{ik\xi}.$$

Then condition (2) can be written in terms of the coefficients as

$$\sum_{k\in\mathbb{Z}} h_k h_{k+\ell 2} = \delta_{0,\ell},$$

which forces N to be odd. A simple calculation shows that

$$|m_0(\xi)|^2 = \frac{1}{2} + \sum_{\ell=0}^{\frac{N-1}{2}} \big[\sum_{k\in\mathbb{Z}} h_k h_{k+2\ell+1}\big] \cos((2\ell+1)\xi). \tag{4}$$

Conversely, any nonnegative trigonometric polynomial of the form

$$f_N(\xi) = \frac{1}{2} + \sum_{\ell=0}^{N-1} a_\ell \cos((2\ell+1)\xi) \tag{5}$$

satisfies (2) and it follows from the Factorization Theorem of Fejér-Riesz (see [3, p. 83])) that f_N can be written $f_N(\xi) = |m_0(\xi)|^2$, where

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=0}^{2N-1} h_n e^{in\xi}, \qquad h_k \in \mathbb{R},\ k = 0, 1, \dots, 2N-1,$$

is a quadrature filter of length (at most) $2N$.

The Fejér-Riesz Theorem thus reduces the problem to finding nonnegative trigonometric polynomials of the form (5). Therefore we refer to nonnegative function of the form (5) as finite filters with the understanding that one has to apply the Fejér-Riesz Theorem to get the actual filter.

Let us classify the trigonometric polynomials of degree at most n according to the following definition. The first three families in Definition 1 represent different types of filters. The remaining families represent different types of kernels we want to use to generate the filters.

Definition 1. *Let $\mathcal{PE}_n$ be the collection of even trigonometric polynomial of degree at most n. We let*

$$\begin{aligned}
\mathbb{F}^n &= \{f \in \mathcal{PE}_n |\, f \geq 0 \ \textit{and}\ \forall \xi : f(\xi) + f(\xi+\pi) = 1\} \\
\mathbb{F}^n_c &= \{f \in \mathbb{F}^n |\, \forall \xi : f(0) \geq f(\xi)\} \\
\mathbb{F}^n_{c,1} &= \{f \in \mathbb{F}^n_c |\, f(0) = 1\} \\
\mathbb{K}^n &= \{K \in \mathcal{PE}_n |\, \forall \xi : \textstyle\int_{\xi-\pi/2}^{\xi+\pi/2} K(u)\,du \geq 0,\ \textit{and}\ \int_{-\pi}^{\pi} K(u)\,du = 2\pi\} \\
\mathbb{K}^n_p &= \{K \in \mathbb{K}^n |\, K \geq 0\} \\
\mathbb{K}^n_{p,c} &= \{K \in \mathbb{K}^n_p |\, \forall \xi : \textstyle\int_{\xi-\pi/2}^{\xi+\pi/2} K(u)\,du \leq \int_{-\pi/2}^{\pi/2} K(u)\,du\}.
\end{aligned}$$

The following two lemmas reveal some of the relationships between the families of trigonometric polynomials defined above. We Let

$$\mathbb{E}^n = \{f \in \mathcal{PE}_n | f \in \operatorname{span}\{\cos(2\ell\xi),\ \ell = 1, 2, \dots\}\}.$$

Lemma 1. *The map $I : \mathbb{K}^n \to \mathbb{F}^n$ defined by*

$$I(K, \xi) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} K(\xi - u)\,du, \qquad K \in \mathbb{K}^n. \tag{6}$$

induces a 1-1 correspondence between $\mathbb{K}^n/\mathbb{E}^n$ and $\mathbb{F}^n$.

Proof. Let $K \in \mathbb{K}^n$ and set $g(\xi) = I(K, \xi)$. Clearly, $g \geq 0$ and we have

$$g(\xi) + g(\xi + \pi) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} f(\xi - u)\, du + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} f(\xi + \pi - u)\, du = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, du = 1.$$

Also, g is a trigonometric polynomial of degree at most n since every polynomial of degree m induces a convolution operator of polynomial type m, i.e. $g \in \mathbb{F}^n$. Conversely, take any $g \in \mathbb{F}^n$. Then, using (5),

$$g(\xi) = \frac{1}{2} + \sum_{\ell=0}^{N-1} a_\ell \cos((2\ell+1)\xi), \quad 2N - 1 \leq n,$$

for some sequence of coefficients $\{a_\ell\}$. Notice that the Fourier series of $\chi_{[-\pi/2,\pi/2]}$ is given by

$$\chi_{[-\pi/2,\pi/2]}(\xi) = \frac{1}{2} + \frac{2}{\pi} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell+1} \cos((2\ell+1)\xi). \tag{7}$$

Therefore, we are led to define

$$K(\xi) = 1 + \pi \sum_{\ell=0}^{N-1} (-1)^\ell (2\ell+1) a_\ell \cos((2\ell+1)\xi), \tag{8}$$

and from the special form of (7), we deduce that

$$g(\xi) = \frac{1}{2} + \sum_{\ell=0}^{N-1} a_\ell \cos((2\ell+1)\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[-\pi/2,\pi/2]}(u) K(\xi - u)\, du.$$

Then, since $g \in \mathbb{F}^n$,

$$2\pi g(\xi) = \int_{\xi-\pi/2}^{\xi+\pi/2} K(u)\, du \geq 0 \quad \text{and} \quad 2\pi = 2\pi[g(\xi) + g(\xi+\pi)] = \int_{-\pi}^{\pi} K(u)\, du,$$

so $K \in \mathbb{K}^n$. Clearly, the kernel of the map I is $\mathbb{E}^n$ which completes the proof. ∎

We call the kernel associated to $|m_0|^2 \in \mathbb{F}^n$ by (8) the filter generating kernel for $|m_0|^2$. As a simple example, let us consider the generating kernels for two of the Daubechies filters.

Example 1. *The Daubechies filter of length 4 is given by*

$$|m_{0,2}(\xi)|^2 = \frac{1}{2} + \frac{9}{16} \cos(\xi) - \frac{1}{16} \cos(3\,\xi)$$

so its filter generating kernel is given by

$$K_{m_{0,2}}(\xi) = 1 + \frac{9\pi}{16} \cos(\xi) + \frac{3\pi}{16} \cos(3\,\xi).$$

The Daubechies filter of length 8 is given by

$$|m_{0,4}(\xi)|^2 = \frac{1}{2} + \frac{1225}{2048}\cos(\xi) - \frac{245}{2048}\cos(3\,\xi) + \frac{49}{2048}\cos(5\,\xi) - \frac{5}{2048}\cos(7\,\xi).$$

The corresponding generating kernel is given by

$$K_{m_{0,4}}(\xi) = 1 + \frac{1225\pi}{2048}\cos(\xi) + \frac{735\pi}{2048}\cos(3\,\xi) + \frac{245\pi}{2048}\cos(5\,\xi) + \frac{35\pi}{2048}\cos(7\,\xi).$$

The following figures show a plot of $K_{m_{0,2}}$ and $K_{m_{0,4}}$.

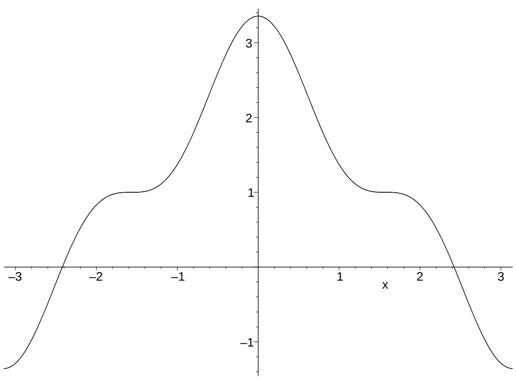

Figure 1: The generating kernel $K_{m_{0,2}}$.

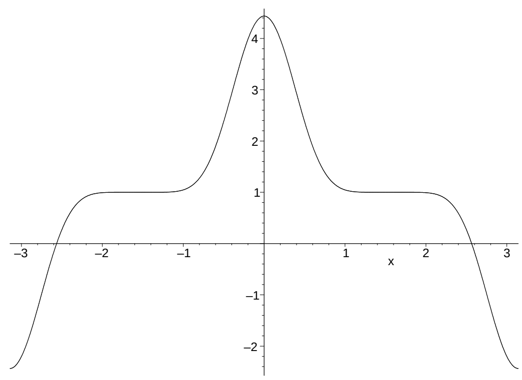

Figure 2: The generating kernel $K_{m_{0,4}}$.

■

It is often preferable to use filters that take on the value 1 at 0, e.g. if we want the filter to generate a multiresolution analysis or induce a convergent subdivision scheme. The following lemma shows that every filter in $\mathbb{F}_c^n$ can mapped onto such a filter using an affine map.

Lemma 2. *We can define a map $U : \mathbb{F}_c^n \to \mathbb{F}_{c,1}^n$ by*

$$Uf(\xi) = 1 + \frac{f(\xi) - f(0)}{f(0) - f(\pi)}, \qquad f \in \mathbb{F}_c^n.$$

Proof. Let $f \in \mathbb{F}_c^n$, and set $g(\xi) = Uf(\xi)$. By the definition of $\mathbb{F}_c^n$ we have $|f(\xi) - f(0)| \leq$

$f(0) - f(\pi)$ so $g \geq 0$, and clearly $g(0) = 1$. Also,

$$\begin{aligned} g(\xi) + g(\xi + \pi) &= 1 + \frac{f(\xi) - f(0)}{f(0) - f(\pi)} + 1 + \frac{f(\xi + \pi) - f(0)}{f(0) - f(\pi)} \\ &= \frac{[2f(0) - 2f(\pi)] + [f(\xi) + f(\xi + \pi)] - 2f(0)}{f(0) - f(\pi)} \\ &= \frac{1 - 2f(\pi)}{f(0) - f(\pi)} \\ &= 1, \end{aligned}$$

since $f(0) - f(\pi) = [1 - f(\pi)] - f(\pi)$. Hence $g \in \mathbb{F}^n_{c,1}$. ■

Remark: The reader should compare the construction in Lemma 2 to the construction of the Daubechies filters. The family of Daubechies filters can be defined using the expression

$$|m_0^N(\xi)|^2 = 1 - c_N \int_0^\xi \sin^{2N-1}(u)\, du,$$

where

$$c_N^{-1} = \int_0^\pi \sin^{2N-1}(u)\, du.$$

This is exactly the same as above with the exception that we let $-f'(\xi)$, $f \in \mathbb{F}^n_c$, play the role of the odd kernel $\sin^{2N-1}(\xi)$.

Finally, we define a map $L : \mathbb{K}^n \to \mathbb{K}^n_p$ by

$$L(K, \xi) = \frac{K(\xi) + \alpha(K)}{1 + \alpha(K)},$$

where $\alpha(K) = \max\{\, 0, -\min_{\xi \in [0,\pi]} K(\xi)\}$.

Let us summarize the results presented above in the following theorem, which tells us how to generate quadrature filters taking the value 1 at 0 using certain even nonnegative convolution kernels. We use the notation I^{-1} to denote the map that associates to a filter from $\mathbb{F}^n$ the kernel in $\mathbb{K}^n$ given by (8).

Theorem 3. *We have the following maps*

$$\mathbb{K}^n_{p,c} \xrightarrow{I} \mathbb{F}^n_c \xrightarrow{U} \mathbb{F}^n_{c,1},$$

and

$$\mathbb{F}^n_{c,1} \xrightarrow{I^{-1}} \mathbb{K}^n \xrightarrow{L} \mathbb{K}^n_{p,c},$$

with $UILI^{-1} : \mathbb{F}^n_{c,1} \to \mathbb{F}^n_{c,1}$ *the identity map.*

Proof. Take $f \in \mathbb{F}^n_{c,1}$. Then $I^{-1}(f,\xi) = K(\xi)$, where

$$f(\xi) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} K(\xi - u)\, du.$$

Clearly, K satisfies $\int_{\xi-\pi/2}^{\xi+\pi/2} K(u)\,du \leq \int_{-\pi/2}^{\pi/2} K(u)\,du$ since $f \in \mathbb{F}^n_{c,1}$. It follows that

$$LI^{-1}(f,\xi) = L(K,\xi) = \frac{K(\xi) + \alpha(K)}{1 + \alpha(K)} \in \mathbb{K}^n_{c,p}.$$

Put $\beta = 1 + \alpha(K)$ and note that

$$ILI^{-1}(f,\xi) = \frac{1}{2\pi\beta}\int_{-\pi/2}^{\pi/2}[K(\xi-u) + \alpha(K)]\,du = \frac{f(\xi) + \alpha(K)/2}{\beta}.$$

Hence

$$\begin{aligned} UILI^{-1}(f,\xi) &= 1 + \frac{1}{\beta}\left[\frac{f(\xi) + \alpha(K)/2 - (f(0) + \alpha(K)/2)}{\frac{1}{\beta}(f(0) - f(\pi))}\right] \\ &= 1 + \frac{f(\xi) - f(0)}{f(0) - f(\pi)} \\ &= f(\xi), \end{aligned}$$

since $f(0) = 1$ and $f(\pi) = 0$. ■

Remark: The important thing to notice about the above Theorem is that it shows that *every* finite quadrature filter of length $2n$ taking the value 1 at 0 can be constructed by the method described in this section using a kernel from $\mathbb{K}^{2n}_{p,c}$.

3 Optimal Asymptotic Frequency Resolution

This section is devoted to a proof of Theorem 2. We begin by obtaining the lower bound appearing in Theorem 2. The bound is a consequence of the fact that the idealized low-pass filter is not continuous and therefore has a Fourier series with slowly decaying coefficients. To get the upper bound we apply the method introduced in the previous section to construct a sequence of filters with optimal resolution using the well known Fejér-Korovkin kernels from approximation theory. Additional examples of filters with optimal resolution are constructed using Jackson kernels and their generalizations.

Lemma 3. *Fix $1 \leq p \leq 2$. Let m_0^N be a Hess filter of length $2N$ minimizing (3). Then there is a constant $c > 0$ (independent of N) such that*

$$\gamma_p(m_0^N) \geq \frac{c}{N}.$$

Proof. Recall that the Fourier series of $\chi_{[-\pi/2,\pi/2]}(\xi)$ is given by

$$\chi_{[-\pi/2,\pi/2]}(\xi) = \frac{1}{2} + \frac{2}{\pi}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\cos((2\ell+1)\xi).$$

Let

$$S_n(\xi) = \frac{1}{2} + \frac{2}{\pi}\sum_{\ell=0}^{n-1}\frac{(-1)^\ell}{2\ell+1}\cos((2\ell+1)\xi).$$

We have

$$\int_{-\pi}^{\pi}|\chi_{[-\pi/2,\pi/2]}(\xi) - S_N(\xi)|^2\,d\xi = \frac{4}{\pi^2}\sum_{\ell=N}^{\infty}\frac{1}{(2\ell+1)^2} \geq \frac{c}{N}$$

for some $c > 0$ [e.g. $c = \pi^{-2}$ will do]. However, S_N is the best L^2 approximation to $\chi_{[-\pi/2,\pi/2]}(\xi)$ by a trigonometric polynomial of degree less than $2N$. Hence,

$$\begin{aligned}\int_{-\pi}^{\pi}|\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0^N(\xi)|^2|^2\,d\xi &\geq \int_{-\pi}^{\pi}|\chi_{[-\pi/2,\pi/2]}(\xi) - S_n(\xi)|^2\,d\xi \\ &\geq \frac{c}{N}.\end{aligned}$$

This proves the case $p = 2$. For $1 \leq p < 2$ we note that

$$\int_{-\pi}^{\pi}|\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0^N(\xi)|^2|^p\,d\xi \geq \int_{-\pi}^{\pi}|\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0^N(\xi)|^2|^2\,d\xi \geq \frac{c}{N},$$

due to the fact that $|\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0^N(\xi)|^2| \leq 1$ by the filter condition (2).

■

The above theorem leads us to the following definition,

Definition 2. *Let $\mathcal{S} = \{m_0^N\}_{N\in\mathbb{N}}$ be a sequence of finite filters with corresponding filterlengths $\{d_N\}$. We say that $\mathcal{S}$ has optimal asymptotic frequency resolution if*

$$\gamma_p(m_0^N) = O(1/d_N), \qquad \forall p \in [1,2].$$

It is not clear that there are any families of finite filters with optimal asymptotic frequency resolution. However, we now know that it suffices to find a sequence of finite polynomial kernels $K_n \in \mathbb{K}_p^{d_n}$ with $d_n = O(n)$ such that

$$\|T_{K_n}(\chi_{[-\pi/2,\pi/2]},\cdot) - \chi_{[-\pi/2,\pi/2]}\|_{L^p[-\pi,\pi)} = O(n^{-1/p}),$$

where $T_{K_n} : L^p[-\pi,\pi) \to L^p[-\pi,\pi)$, $1 \leq p < \infty$, is defined by

$$T_{K_n}(f,\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)K_n(\xi - u)\,du.$$

We need a classical results from approximation theory. First, some additional notation. The $L^p[-\pi,\pi)$-modulus of continuity is defined by

Definition 3. *The $L^p[0,1)$-modulus of continuity of $f \in L^p[0,1)$ is given by*

$$\omega_p(f,\delta) = \sup_{|h|\le\delta} \|f(\cdot - h) - f(\cdot)\|_{L^p[-\pi,\pi)}, \qquad \delta > 0.$$

Example 2. *One easily checks that the $L^p[-\pi,\pi)$-modulus of continuity for the idealized low-pass filter $\chi_{[-\pi/2,\pi/2]}$ is given by*

$$\omega_p(\chi_{[-\pi/2,\pi/2]},\delta) = (4\delta)^{1/p}.$$

We can now state the theorem we need. Suppose the even nonnegative periodic kernel P_n can be written in the form

$$P_n(\xi) = 1 + 2\sum_{k=1}^{d_n} \theta_n(k)\cos(k\xi). \tag{9}$$

Clearly, T_{P_n} maps $L^p[-\pi,\pi)$ into the the set of trigonometric polynomials of degree at most d_n. We have the following result.

Theorem 4. *Let $\{P_n\}$ be a family of positive periodic kernels of the type (9). Fix $1 \le p < \infty$. Suppose $d_n = O(n)$, and $1-\theta_n(1) = O(n^{-2})$, then*

$$\|T_{P_n}(f,\cdot) - f\|_{L^p[-\pi,\pi)} = O(\omega_p(f,n^{-1})), \qquad f \in L^p[-\pi,\pi).$$

The proof can be found in [1]. From Theorem 4 and the above example we see that for every sequence of kernels $\{P_n\}$ of type (9) with $d_n = O(n)$ and $1-\theta_n(1) = O(n^{-2})$, there is a corresponding family $\mathcal{S} = \{m_0^n\}$ of finite filters with optimal asymptotic frequency resolution, given by

$$|m_0^n(\xi)|^2 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} P_n(\xi - u)\,du.$$

We can now complete the proof of our main result, Theorem 2.

Lemma 4. *There is a sequence of filters $\{m_0^n\}_{n=1}^{\infty}$, where m_0^n has length $2n$, such that*

$$\gamma_p(m_0^n) = O(1/n).$$

Proof. Consider the Fejér-Korovkin kernel K_n defined by

$$K_n(\xi) = \begin{cases} \dfrac{2\sin^2(\pi/(n+2))}{n+2}\left[\dfrac{\cos((n+2)x/2)}{\cos(\pi/(n+2)) - \cos(\xi)}\right]^2, & x \notin \pm\dfrac{\pi}{n+2} + 2\mathbb{Z}\pi \\ (n+2)/2, & x \in \pm\dfrac{\pi}{n+2} + 2\mathbb{Z}\pi. \end{cases}$$

One can check that (see [1, Chap. 1])

$$K_n(\xi) = 1 + 2\sum_{k=1}^{n} \theta_n(k)\cos kx,$$

with

$$\theta_n(k) = \frac{1}{2(n+2)\sin(\pi/(n+2))}\left[(n-k+3)\sin\frac{k+1}{n+2}\pi - (n-k+1)\sin\frac{k-1}{n+2}\pi\right].$$

In particular,

$$1-\theta_n(1) = 1-\cos(\pi/(n+2)) = O(n^{-2}).$$

We define the Fejér-Korovkin filters by

$$|m_0^n(\xi)|^2 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} K_n(\xi-u)\,du.$$

Then m_0^n has degree $n+1$ if n is odd and degree n if n is even. It follows from Theorem 4 that

$$\gamma_p(m_0^n) = O(1/n).$$

■

The following figure shows a plot of $|m_0^n|^2$ for $n = 2,4,\ldots,12$. Note that the filters are decreasing on $[0,\pi)$, i.e. the Fejér-Korovkin kernels satisfy $K_{2n} \in \mathbb{K}_{p,c}^{2n}$ for $n = 1,2,\ldots,6$. This will be proved for general n in Lemma 5.

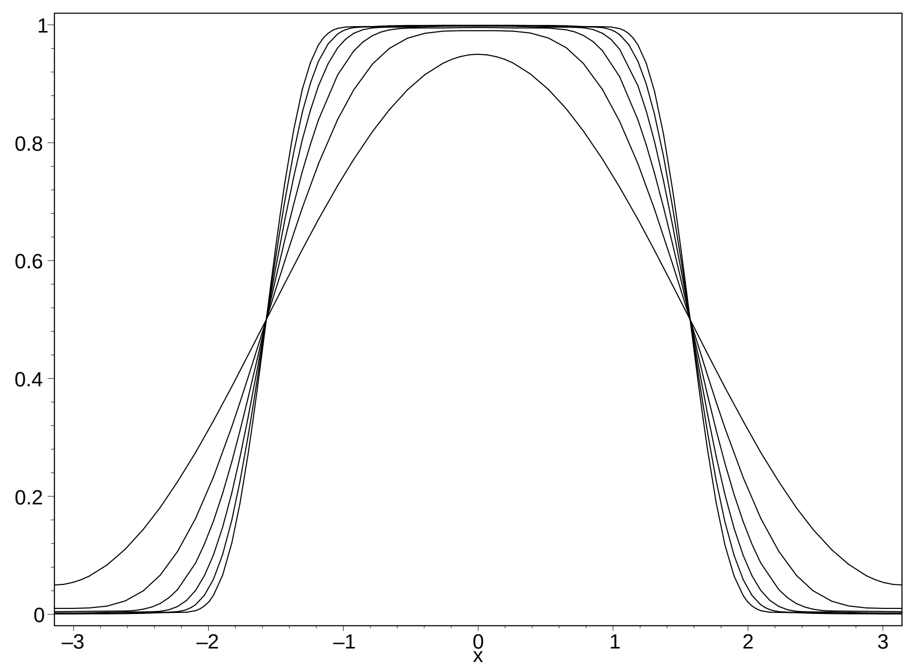

Figure 3: The Fejér-Korovkin filters $|m_0^n|^2$ for $n = 2,4,\ldots,12$.

3.1 Additional Examples

We give two additional examples of families of finite filter constructed using the ideas above. The second family has optimal asymptotic frequency resolution, the first has not.

Fejér Filters

The Fejér kernel F_n is defined by

$$F_n(\xi) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijx} = \frac{1}{n+1} \left\{ \frac{\sin \frac{n+1}{2}\xi}{\sin \xi/2} \right\}^2$$

and we call the corresponding filters Fejér filters. The Fejér filters $T_{F_n}(\chi_{[-\pi/2,\pi/2]}, \cdot)$ give us the arithmetic means of the partial sums of the idealized filter $\chi_{[-\pi/2,\pi/2]}$, however F_n does not satisfy the conditions of Theorem 4 [$1 - \theta_n(1) = O(n^{-1})$ in this case]. An easy calculation shows that

$$\|\chi_{[-\pi/2,\pi/2]} - T_{F_n}(\chi_{[-\pi/2,\pi/2]}, \cdot)\|_{L^1[-\pi,\pi)} = O\big(\tfrac{\log n}{n}\big),$$

and this estimate is optimal. So the Fejér filters are not a family of filters with optimal asymptotic frequency resolution. However, one can verify that

$$\|\chi_{[-\pi/2,\pi/2]} - T_{F_n}(\chi_{[-\pi/2,\pi/2]}, \cdot)\|^p_{L^p[-\pi,\pi)} = O(1/n),$$

for $1 < p < 2$, so the failure to be optimal is only in $L^1[-\pi, \pi)$.

Jackson Filters

The Jackson kernel is defined by

$$j_n(\xi) = \frac{3}{n(2n^2+1)} \left[\frac{\sin(nx/2)}{\sin(\xi/2)}\right]^4 = 1 + 2\sum_{k=1}^{2n-2} \theta_{2n-2}(k) \cos k\xi,$$

with $\theta_{2n-2}(1) = 1 - 3/(2n^2+1)$, see [1, p. 60]. So $\{j_n\}$ satisfies the conditions of Theorem 4. The corresponding family of filters has optimal asymptotic frequency resolution. Note that the filter corresponding to j_n has length $2n - 2$.

Generalized Jackson Filters

The two examples above can be generalized by introducing the generalized Jackson kernel given by

$$j_{n,s}(\xi) = \frac{1}{\Delta_{n,s}} \left(\frac{\sin(n\xi/2)}{\sin(\xi/2)}\right)^{2s}, \quad n, s \in \mathbb{N},$$

where

$$\Delta_{n,s} = 2\pi \int_{-\pi}^{\pi} \left(\frac{\sin(n\xi/2)}{\sin(\xi/2)} \right)^{2s} d\xi.$$

Notice that $j_{n,s}$ is of type (9), which follows from the trigonometric identity

$$1 + 2\sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \cos kx = \frac{1}{n} \left(\frac{\sin(nx/2)}{\sin(\xi/2)} \right)^2.$$

The associated Jackson filter of type s is given by

$$|m_0^{j_{n,s}}(\xi)|^2 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} j_{n,s}(\xi - u)\, du.$$

We notice that the Fejér kernels and Jackson kernels correspond to $s = 1$ and $s = 2$, respectively. The reader can consult [6] for further examples of even nonnegative kernels that generate quadrature filters with optimal frequency localization. The kernels introduced in [6] are based on Jacobi polynomials.

4 MRA Filters With Optimal Frequency Localization

The major "problem" with the filters constructed so far is that they do not take on the value 1 at 0, so they are not associated with multiresolution analyses. In this section we apply the affine map introduced in Lemma 2 to make small corrections to the filters so they do become MRA filters. Moreover, the modifications do not interfere with the fact that each family has optimal asymptotic frequency resolution. However, we first have to show that the corresponding kernels are of type $\mathbb{K}_{p,c}^{2n}$. The following lemma will take care of that.

Lemma 5. *Let $f_{n,s}(\xi) = |m_0^{j_{n,s}}(\xi)|^2$ be the square of the generalized Jackson filters, and let $g_n(\xi) = |m_0^{K_n}(\xi)|^2$ be the square of the Fejér-Korovkin filters. Then $f_{2n,s}$ and g_{2n} are nonincreasing on $[0, \pi)$ for $s, n \in \mathbb{N}$.*

Proof. First we consider the Generalized Jackson Filters. Notice that it suffices to prove that the functions $f_{2n,s}$ are nonincreasing on $[0, \pi/2)$ since they are even and satisfy

$$f_{2n,s}(\xi) + f_{2n,s}(\xi + \pi) = 1.$$

We have, for $0 \le \xi \le \pi/2$,

$$
\begin{aligned}
f'_{2n,s}(\xi) &= \int_{-\pi/2}^{\pi/2} \frac{d}{d\xi} j_{2n,s}(\xi - u)\, du \\
&= j_{2n,s}(\pi/2 + \xi) - j_{2n,s}(\xi - \pi/2) \\
&= j_{2n,s}(\pi/2 + \xi) - j_{2n,s}(\pi/2 - \xi) \\
&= \frac{1}{\Delta_{2n,s}} \left\{ \left(\frac{\sin(n(\pi/2+\xi))}{\sin((\pi/2+\xi)/2)} \right)^{2s} - \left(\frac{\sin(n(\pi/2-\xi))}{\sin((\pi/2-\xi)/2)} \right)^{2s} \right\} \\
&= \frac{\sin(n(\pi/2+\xi))^{2s}}{\Delta_{2n,s}} \left\{ \frac{1}{\sin((\pi/4+\xi/2)^{2s}} - \frac{1}{\sin((\pi/4-\xi/2)^{2s}} \right\} \\
&\le 0.
\end{aligned}
$$

The proof for g_{2n} is similar. ■

Hence, we can apply Lemma 2 to obtain the filters we want.

Corollary 1. *The finite filters* $\{m_0^{j_{2n,s}}\}_{n=1}^{\infty}$ *and* $\{m_0^{K_{2n}}\}_{n=0}^{\infty}$ *for which*

$$|m_0^{j_{2n,s}}(\xi)|^2 = U f_{2n,s}(\xi) = 1 - \frac{f_{2n,s}(\xi) - f_{2n,s}(0)}{f_{2n,s}(0) - f_{2n,s}(\pi)},$$

and

$$|m_0^{K_{2n}}(\xi)|^2 = U g_{2n}(\xi) = 1 - \frac{g_{2n}(\xi) - g_{2n}(0)}{g_{2n}(0) - g_{2n}(\pi)},$$

where $f_{2n,s}$ *and* g_{2n} *are defined as in Lemma 5, are two families of MRA filters.*

Proof. The filters are MRA since they only vanish at the points $\pi\mathbb{Z}$. ■

We need to check that the filters still have optimal asymptotic frequency resolution.

Theorem 5. *The two families of MRA filters corresponding to the Jackson and Fejér-Korovkin kernels, respectively, both have optimal frequency localization.*

Proof. First we consider the Fejér-Korovkin MRA filters defined in Corollary 1. It suffices to prove the estimate for $\gamma_1(m_0^{K_{2n}})$. Note that by the QMF condition (2)

$$\int_{-\pi}^{\pi} |\chi_{[-\pi/2,\pi/2]}(\xi) - |m_0^{K_{2n}}(\xi)|^2|\, dx = 2 \int_{-\pi/2}^{\pi/2} [1 - |m_0^{K_{2n}}(\xi)|^2]\, dx.$$

We have

$$
\begin{aligned}
\int_{-\pi/2}^{\pi/2} [1 - |m_0^{K_{2n}}(\xi)|^2]\, dx &= \int_{-\pi/2}^{\pi/2} \frac{g_{2n}(\xi) - g_{2n}(0)}{g_{2n}(0) - g_{2n}(\pi)} \\
&= \frac{1}{g_{2n}(0) - g_{2n}(\pi)} [\alpha_{2n} - g_{2n}(0)\pi],
\end{aligned}
$$

where $\alpha_{2n} = \int_{-\pi/2}^{\pi/2} g_{2n}(\xi)\, dx$. We know that $|\pi - \alpha_{2n}| = O(1/n)$ since the filters g_{2n} have optimal asymptotic frequency localization. It is also easy to check that $1 - g_{2n}(0) = O(1/n)$ using the expression for the Fejér-Korovkin kernel given in the proof of Lemma 4. Hence, we obtain

$$\int_{-\pi/2}^{\pi/2} [1 - |m_0^{K_{2n}}(\xi)|^2]\, dx = O(1/n).$$

The proof for the Jackson kernel is similar. ■

Jackson and Fejér-Korovkin Wavelets

We conclude this paper by presenting the plots of some of the wavelets corresponding to the two families of MRA filters introduced above. We have fixed the filterlength at 12 which makes it easier to compare the functions. The first plot shows the scaling function and wavelet generated by the Fejér-Korovkin filter of length 12. Notice that the wavelet is more symmetric but less smooth than comparable Daubechies wavelets (see e.g. [2, p. 197]).

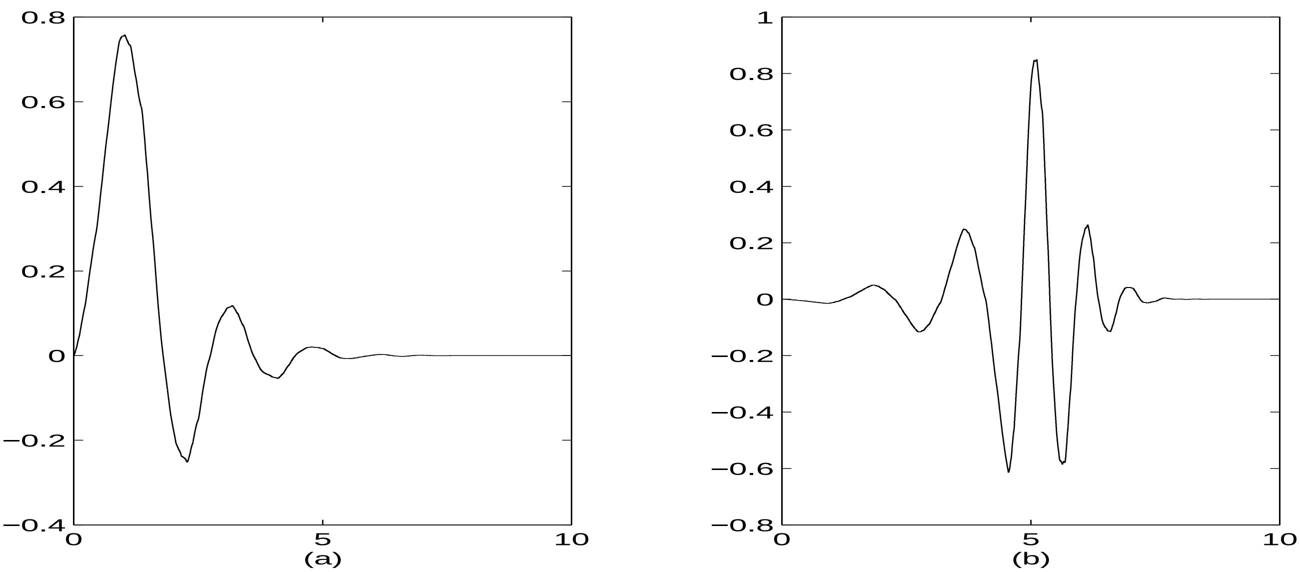

Figure 4: (a) The scaling function generated by the Fejér-Korovkin MRA filter of length 12, (b) is the corresponding wavelet.

The next figure shows some of the Jackson wavelets. One should notice that the regularity of the Jackson wavelet depends primarily on its type, and not on the length of the filter. This can be observed directly from the formula for the Jackson kernel $j_{n,s}$, which shows that $|m_{j_{n,s}}(\xi)|^2 = O(|\xi|^{2s})$ at the origin. We can therefore obtain arbitrarily smooth wavelets by choosing both type s and n sufficiently large.

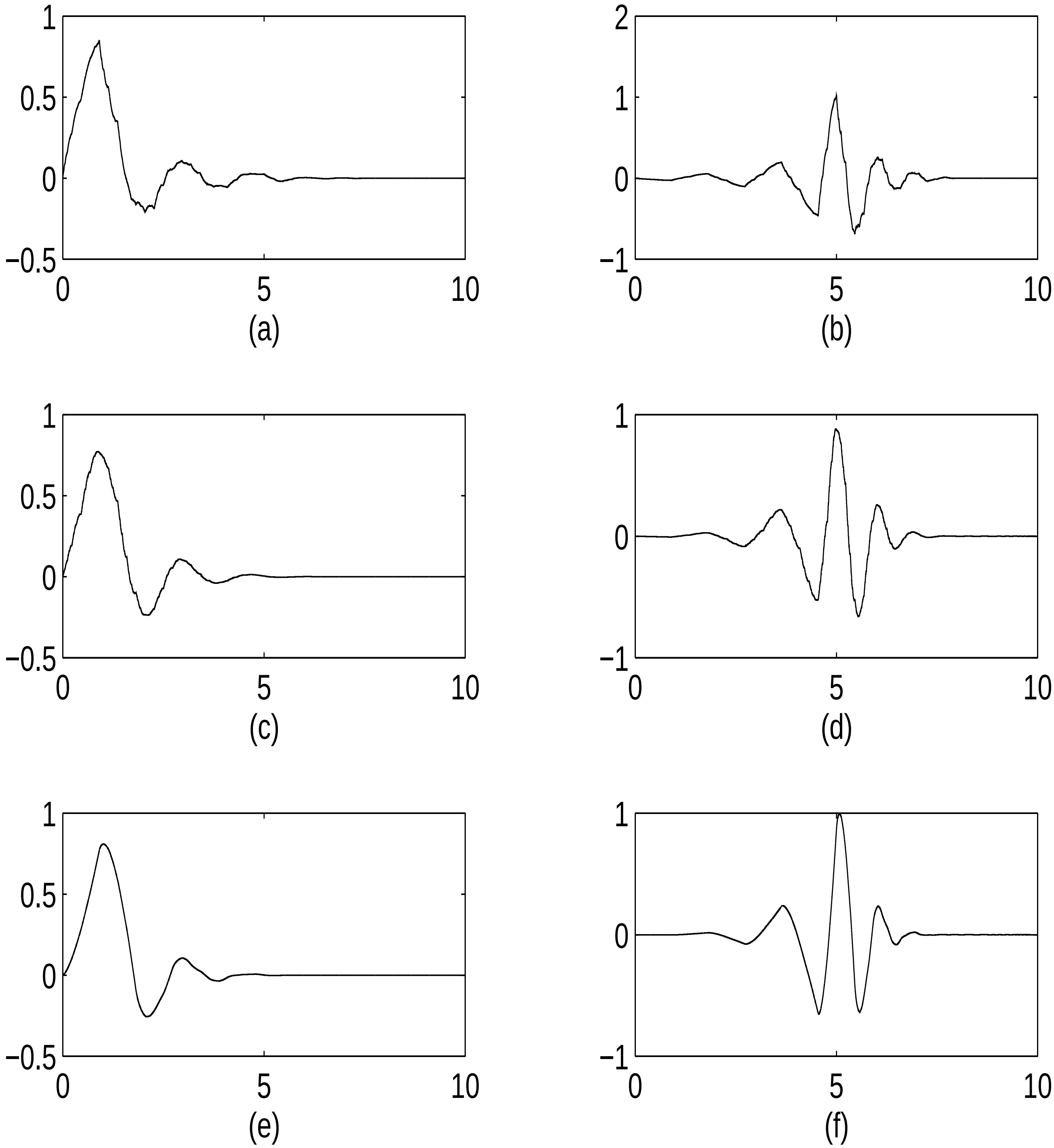

Figure 5: (a), (c), and (e) are the plots of the scaling functions generated by the Jackson MRA filter of length 12 and type 1, 2, and 3, resp. (b), (d), and (f) are plots of the corresponding wavelets.

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