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SIZE PROPERTIES OF WAVELET PACKETS

by

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A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 1999

Saint Louis, Missouri

To my parents

Acknowledgments

Let me begin by thanking my adviser Victor Wickerhauser for his encouragement and support over the past three years, and for his confidence in my ability to complete this dissertation. I owe special thanks to Guido Weiss for substituting as advisor while Victor Wickerhauser was on academic leave in the spring of 1998. I would also like to thank Terri Schneider for her support and professionalism throughout the years.

I am grateful to Lars F. Villemoes of the Royal Institute of Technology in Stockholm for his interest in my work and for inviting me to come and give a talk in Stockholm in December 1998. Thanks to Gerd Grubb of the University of Copenhagen for inviting me to Copenhagen in April 1998.

I would like to specifically acknowledge the help and encouragement I have received from Arne Jensen of Aalborg University over the past five years, and for his support in my job search.

Finally I would like to acknowledge that my three years at Washington University were financed in full by a generous grant from the Danish Research Academy.

Contents

Introduction

In signal analysis one is (among other things) interested in obtaining information about local properties of signals. The Fourier transformation is not very useful for such an analysis, since the Fourier integral decomposes the signal into the stationary signals $e^{ix\xi}$ of infinite duration, which makes it difficult to extract local information.

The traditional solution to the problem of obtaining a local Fourier analysis is due to D. Gabor ([13]). The idea is to analyze the signal through a sliding window, which corresponds to using basis functions of the type

$$g_{ab}(x) = e^{iax} g(x-b), \qquad a, b \in \mathbb{R},$$

where g is a fixed function in $L^2(\mathbb{R})$. The question is, whether it is possible to construct an orthonormal basis for $L^2(\mathbb{R})$ consisting of Gabor basis functions well localized in both time and frequency. The answer is, unfortunately, negative: The Balian-Low Theorem (see [8]) states that whenever

$$g_{m,n}(x) = e^{inp_0 x} g(x - mq_0), \qquad m, n \in \mathbb{Z}$$

is an orthonormal basis for $L^2(\mathbb{R})$ then either

$$\int_{\mathbb{R}} x^2 |g(x)|^2 \, dx = \infty \qquad \text{or} \qquad \int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty,$$

where we define the Fourier transform of $h \in L^1(\mathbb{R})$ by

$$\hat{h}(\xi) = \int_{\mathbb{R}} h(x) e^{-ix\xi} \, dx.$$

It is therefore impossible for the Gabor basis functions to be well localized in both time and frequency.

A new type of basis functions called wavelets was introduced in 1982 by the geophysicist J. Morlet in view of applications for the analysis of seismic data. He considered the family

$$\psi_{a,b}(x) = |a|^{-1/2} \psi\Big(\frac{x-b}{a}\Big), \qquad a, b \in \mathbb{R},$$

of translated and dilated versions of a single function $\psi \in L^2(\mathbb{R})$ (ψ is called the wavelet). Wavelet analysis consists in applying such families of functions to decompose data, functions, or operators. The mathematical justification for using such a decomposition was given by A. Grossmann and J. Morlet ([17]). We are interested in the following discrete version of Morlet's algorithm.

Definition 0.1. *An orthonormal wavelet is a function $\psi \in L^2(\mathbb{R})$ such that the family*

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}},$$

where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, is an orthonormal basis for $L^2(\mathbb{R})$.

The first example of a basis with this structure was given by Alfred Haar in 1910.

Example 0.2 ([18]). *The Haar wavelet h is defined by*

$$h(x) = \begin{cases} 1, & \text{for } x \in [0, 1/2) \\ -1, & \text{for } x \in [1/2, 1) \\ 0, & \text{otherwise.} \end{cases}$$

The family $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ provides an example of an orthonormal wavelet basis for $L^2(\mathbb{R})$.

The Haar wavelet is not continuous and as a consequence has a bad frequency localization in the sense that its Fourier transform is not even integrable.

During 1986, S. Mallat and Y. Meyer introduced a general method for constructing orthonormal wavelets with good time-frequency localization. We give a brief review of this method (called multiresolution analysis) in Chapter 1, and we show how the structure is related to a pair of so-called Conjugate Quadrature Filters. We also consider some special families of compactly supported wavelets, constructed by I. Daubechies, that will be used to construct wavelet packets in Chapters 2 and 3.

A problem with every wavelet basis is that all the high-frequency wavelets have poor frequency localization. Wavelet packets were introduced by R. Coifman, Y. Meyer, and M. V. Wickerhauser in order to improve the frequency resolution and thereby get more efficient algorithms to decompose signals. The idea is to construct a whole library of orthonormal bases for $L^2(\mathbb{R})$ derived from the multiresolution structure, each with distinct time-frequency properties. The orthonormal wavelet basis itself, and the so-called basic wavelet packet basis, are two particular members of the library. The construction of wavelet packets is presented in Chapter 2.

The main results in this thesis are concerned with the behavior of wavelet packets in $L^p(\mathbb{R})$. In chapter 2 we consider the size properties of the basic wavelet packets in $L^p(\mathbb{R})$. One of the main new results is that for a collection of "popular" finite filters one can find a subsequence of the associated basic wavelet packets that grow exponentially in L^p-norm for p large. This result generalizes and refines a result by Coifman, Meyer, and Wickerhauser (see [6]) for the Meyer filters. Another question we consider is whether basic wavelet packets always form a basis for $L^p(\mathbb{R})$ for $1 < p < \infty$. The answer is positive for a select family of basic wavelet packets related to the Walsh system, and we even have pointwise convergence a.e. for expansions in such functions. In general the wavelet packets fail to be a basis. This is true for the basic wavelet packets associated with the "popular" filters mentioned above. We also prove that using so-called nonstationary wavelet packets one can obtain uniformly bounded basic wavelet packet. We introduce a new generalization of wavelet packets, called highly nonstationary wavelet packets, and prove that such basic wavelet packets can be uniformly bounded and have support contained in some fixed compact set.

Chapter 3 contains the generalization of all the results from Chapter 2 to periodic wavelet packets. The tool used to generalize the results is multiplier theory for Fourier series.

It turns out that the trigonometric system and periodic wavelet packets share a number of properties. So it is reasonable to expect that some of the "nice" operators defined using the trigonometric system can be defined using periodic wavelet packets in stead. In chapter 4, we emulate the definition of the Hilbert transform on $\mathbb{T}$ using periodic wavelet packets in place of the trigonometric system. The construction is successful (i.e. the operator is bounded on $L^p[0,1)$, $1 < p < \infty$) as long as we use periodic wavelet packets derived from the Walsh system. For more general periodic wavelet packets, it is not possible to use our method to construct such a bounded Hilbert transform. The negative result is a consequence of the results from Chapter 3.

Chapter 1

Wavelets

This chapter contains a brief review of some basic results about multiresolution analyses, filters, and wavelets. Readers not already familiar with such concepts may find it useful to consult [8], [19], or [44] for some background material.

1.1 Multiresolution Analysis

There exists a very general method for obtaining wavelet bases of various degrees of smoothness. In order to describe this method we introduce the notion of a multiresolution analysis. The multiresolution analysis structure was first introduced by S. Mallat ([26]) and Y. Meyer ([29]) in 1986.

Definition 1.1. *A multiresolution analysis is a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying*

$$V_j \subset V_{j+1}, \qquad j \in \mathbb{Z}, \tag{1.1}$$

$$f \in V_j \Leftrightarrow f(2\,\cdot) \in V_{j+1}, \qquad j \in \mathbb{Z}, \tag{1.2}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \tag{1.3}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{1.4}$$

$$\begin{aligned}&\textit{There exists a } \phi \in V_0 \textit{ such that } \{\phi(\cdot - k)\}_{k\in\mathbb{Z}}\\ &\textit{is an orthonormal basis for } V_0.\end{aligned} \tag{1.5}$$

Given a multiresolution analysis $\{V_j\}$ we want to construct an associated wavelet. To do this we define the 2π-periodic function m_0 by

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\xi},$$

where the sequence $\{c_k\}$ is given by

$$\tfrac{1}{2}\phi(\tfrac{x}{2}) = \sum_{k\in\mathbb{Z}} c_k \phi(x+k).$$

The important property of m_0 is that

$$\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi), \tag{1.6}$$

which under fairly mild hypothesis on the behaviour of m_0 at 0 (see [19]) can be generalized to

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi). \tag{1.7}$$

The function m_0 is called the low-pass filter associated with the multiresolution analysis, and (1.6) is called the two-scale equation for the scaling function ϕ.

An orthonormal wavelet can then be obtained using the following fundamental result

Theorem 1.2 ([29, 26]). *Suppose ϕ is a scaling function for a multiresolution analysis $\{V_j\}$, and m_0 is the associated low-pass filter. Then $\psi \in V_1 \cap V_0^{\perp}$ is an orthonormal wavelet for $L^2(\mathbb{R})$ if and only if*

$$\hat{\psi}(2\xi) = e^{i\xi}\nu(2\xi)\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)$$

a.e., for some 2π-periodic, measurable, and unimodular function ν.

Remark. For simplicity we always take $\nu \equiv 1$, and we let W_j denote the wavelet space $V_{j+1} \cap V_j^{\perp}$.

The procedure used for constructing multiresolution analyses is to find a suitable 2π-periodic function m_0 and then define the associated scaling function using equation (1.7). Clearly, m_0 has to satisfy certain conditions for this construction to work. The main result of A. Cohen's dissertation is the following sufficient condition to ensure that m_0 is the low-pass filter for a multiresolution analysis.

Theorem 1.3 ([3]). *Let m_0 be a 2π-periodic $C^{r+1}(\mathbb{R})$ function, $r = 0, 1, \ldots, \infty$, such that $m_1(0) = 1$. Suppose*

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1.$$

Then m_0 is a low-pass filter for a scaling function associated with a multiresolution analysis if and only if there exists a compact set K with 0 in its interior, such that

$$\sum_{\ell\in\mathbb{Z}} \chi_K(\xi+2\pi\ell) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{1.8}$$

and

$$m_0(2^{-j}\xi) \neq 0 \qquad \text{for all } \xi \in K \text{ and all } j = 1,2\ldots \tag{1.9}$$

The last condition may look rather technical but if we take $K = [-\pi,\pi]$ it just says that m_0 must not vanish on $[-\pi/2, \pi/2]$.

Remark. Only the case $r = \infty$ is proved in [3] but the proof works just as well for finite r. Moreover, if we want a compactly supported scaling function then it suffices to choose a trigonometric polynomial as m_0 (see [8]).

1.2 Conjugate Quadrature Filters

Suppose m_0 is a low-pass filter associated with a multiresolution analysis $\{V_j\}$. We define a new 2π-periodic function, called the high-pass filter associated with $\{V_j\}$, by $m_1(\xi) = e^{i\xi}\overline{m_0(\xi+\pi)}$. One can easily check that the matrix

$$\begin{bmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{bmatrix}$$

is unitary a.e. This fact puts the functions (m_0, m_1) into the context of conjugate quadrature filters (CQFs).

Definition 1.4 (CQFs). *Let $\{h_n\} \in \ell^1(\mathbb{Z})$ be a real-valued sequences, and let $g_k = (-1)^k h_{1-k}$ for $k \in \mathbb{Z}$. Define the operators $H, G : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by*

$$(Ha)_k = \sum_{n\in\mathbb{Z}} a_n h_{n-2k}$$
$$(Ga)_k = \sum_{n\in\mathbb{Z}} a_n g_{n-2k}.$$

The filters H and G are called a pair of CQFs if

$$2HH^* = 2GG^* = I \tag{1.10}$$
$$H\mathbf{1} = \mathbf{1}, \text{ where } \mathbf{1} = (\ldots,1,1,1,\ldots)$$
$$H^*G + G^*H = I \tag{1.11}$$
$$HG^* = GH^* = 0. \tag{1.12}$$

It is not hard to check that the adjoints of the operators H and G are given by

$$(H^*a)_k = \sum_{j\in\mathbb{Z}} a_j h_{k-2j}$$
$$(G^*a)_k = \sum_{j\in\mathbb{Z}} a_j g_{k-2j},$$

so conditions (1.10) and (1.12) can be translated to the following conditions on the filter coefficients

$$\begin{aligned} &\sum_{\ell\in\mathbb{Z}} h_{\ell-2k}h_\ell = \frac{1}{2}\delta_{k,0}, \\ &\sum_{\ell\in\mathbb{Z}} g_{\ell-2k}g_\ell = \frac{1}{2}\delta_{k,0}, \qquad \text{and} \\ &\sum_{\ell\in\mathbb{Z}} h_{\ell-2k}g_\ell = 0 \end{aligned} \tag{1.13}$$

for all $k \in \mathbb{Z}$. To see the connection to the multiresolution filters we introduce the 2π-periodic functions

$$M_0(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{ik\xi} \qquad \text{and} \qquad M_1(\xi) = \sum_{k\in\mathbb{Z}} g_k e^{ik\xi}$$

associated with the CQFs of Definition 1.4 (M_0 and M_1 are the so-called transfer-functions associated with the CQFs). Then the conditions given by (1.13) are equivalent to

$$\begin{bmatrix} M_0(\xi) & M_0(\xi+\pi) \\ M_1(\xi) & M_1(\xi+\pi) \end{bmatrix}$$

being unitary a.e. Using this fact it is not hard to check that the filters (m_0, m_1) associated with a multiresolution analysis is indeed a pair of CQFs (see [8]).

Remark. We will keep up the tradition and abuse notation by referring to the transfer function as the filter.

1.2.1 Some Special FIR Filters

To construct a compactly supported wavelet with N vanishing moments and maximal decay of its Fourier transform one has to use a low-pass filter m_0 of the form

$$m_0(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^N \mathcal{L}(\xi), \qquad N \in \mathbb{N},$$

where $\mathcal{L}$ is a trigonometric polynomial. I. Daubechies proves in [8] that $\mathcal{L}$ must satisfy

$$|\mathcal{L}(\xi)|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \left(\sin\frac{\xi}{2}\right)^{2k} + \left(\sin\frac{\xi}{2}\right)^{2N} R\left(\frac{1}{2} - \sin^2\frac{\xi}{2}\right), \tag{1.14}$$

with R an odd polynomial in order to have

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1.$$

We need a factorization result by F. Riesz to recover $\mathcal{L}$ from (1.14). We present the the proof of the result here since it is essential in order to explain the construction of the Daubechies family of filters.

Lemma 1.5 (Riesz Factorization). *Let A be a positive trigonometric polynomial of the form*

$$A(\xi) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\xi) \tag{1.15}$$

with $a_n \in \mathbb{R}$, $a_N \neq 0$. Then there exists a trigonometric polynomial $B(\xi) = \sum_{n=0}^{N} b_n e^{in\xi}$ with $b_n \in \mathbb{R}$ such that

$$A(\xi) = |B(\xi)|^2.$$

Proof. Define the polynomial P_A by

$$P_A(z) = \frac{1}{2} \sum_{n=-N}^{N} a_{|n|} z^{N+n}.$$

It follows that

$$P_A(e^{i\xi}) = e^{iN\xi} \frac{1}{2} \sum_{n=-N}^{N} a_{|n|} e^{in\xi} = e^{iN\xi} A(\xi).$$

We now factorize P_A. Note that the two polynomials $P_A(z)$ and $z^{2N} P_A(z^{-1})$ agree everywhere. Since $a_N \neq 0$ we have $P_A(0) \neq 0$ so if z_0 is a zero of P_A then so is z_0^{-1}. All a_n are real so we have $\overline{P_A(z)} = P_A(\bar{z})$ and this implies that whenever z_0 is a zero so is $\bar{z}_0$. From these observations it follows that the zeros of P_A come in either complex quadruplets $z_j, \bar{z}_j, z_j^{-1}$, and $\bar{z}_j^{-1}$ or in real doublets r_k and r_k^{-1}. Factorizing P_A then leads to

$$P_A(z) = \frac{1}{2} a_N \left[\prod_{k=1}^{K} (z - r_k)(z - r_k^{-1})\right] \left[\prod_{j=1}^{J} (z - z_j)(z - \bar{z}_j)(z - z_j^{-1})(z - \bar{z}_j^{-1})\right],$$

where we have separated the different kinds of zeros. Note that for $z \neq 0$,

$$|(e^{i\xi} - z)(e^{i\xi} - \bar{z}_j^{-1})| = |z|^{-1} |e^{i\xi} - z|^2,$$

so

$$
\begin{aligned}
A(\xi) &= |A(\xi)| \\
&= |P_A(e^{i\xi})| \\
&= \left[\frac{1}{2}|a_N| \prod_{k=1}^{K} |r_k|^{-1} \prod_{j=1}^{J} |z_j|^{-2}\right] \left| \prod_{k=1}^{K} (e^{i\xi} - r_k) \prod_{j=1}^{J} (e^{i\xi} - z_j)(e^{i\xi} - \bar{z}_j) \right|^2 \\
&= |B(\xi)|^2,
\end{aligned}
$$

where

$$
B(\xi) = \left[\frac{1}{2}|a_N| \prod_{k=1}^{K} |r_k|^{-1} \prod_{j=1}^{J} |z_j|^{-2}\right]^{1/2} \prod_{k=1}^{K} (e^{i\xi} - r_k) \prod_{j=1}^{J} (e^{2i\xi} - 2e^{i\xi}\mathrm{Re}(z_j) + |z_j|^2).
$$

Clearly, B is a trigonometric polynomial of order N with real coefficients. ■

In the next sections we introduce some specific finite filters, constructed by I. Daubechies (see [7, 9]), that will be used in the chapters on wavelet packets.

The Daubechies Filters

First we introduce the "standard" Daubechies filters. We start by letting

$$
|\mathcal{L}^N(\xi)|^2 = \sum_{k=0}^{N-1} \binom{N-1+k}{k} \sin^{2k}(\tfrac{\xi}{2})
$$

to get fewest possible non-zero coefficients for the associated CQF. We find $\mathcal{L}^N(\xi)$ by the Riesz factorization, where we always choose the zero on or within the unit circle. To be more explicit, we choose

$$
\mathcal{L}^N(\xi) = C_N \prod_{k=1}^{N} (e^{i\xi} - r_k) \prod_{j=1}^{J} (e^{2i\xi} - 2e^{i\xi}\mathrm{Re}(z_j) + |z_j|^2),
$$

with $|r_k| \leq 1$ for $k = 1, \ldots K$, $|z_j| \leq 1$ for $j = 1, \ldots, J$. We now let

$$
\tilde{m}_0^N(\xi) = \left(\frac{1 + e^{i\xi}}{2}\right)^N \mathcal{L}^N(\xi).
$$

Since $\tilde{m}_0^N(\pi) = 0$ we have $|\tilde{m}_0(0)| = 1$ and $\mathcal{L}^N(0) \geq 0$ (by our particular choice of $\mathcal{L}^N$). Thus, $\tilde{m}_0^N(0) = 1$. Moreover, $\tilde{m}_0^N(\xi) \neq 0$ on $[-\pi/2, \pi/2]$ so m_0 satisfies the conditions of Theorem 1.3. We let $m_0^N(\xi) = c \cdot \tilde{m}_0^N(\xi)$ where we adjust the phase c such that

$$
m_0^N(\xi) = \sum_{k=0}^{2N-1} c_k^N e^{-ik\xi}.
$$

The sequence $\{c_k^N\}_{k=0}^{2N-1}$ is called the Daubechies low-pass filter of length $2N$. The associated high-pass filter is given, as usual, by

$$m_1^N(\xi) = e^{i\xi}\overline{m_0^N(\xi+\pi)}.$$

The Least Asymmetric Daubechies Filters

A filter with coefficients $\{h_n\}$ has linear phase if the associated transfer function, $H(\xi) = \sum_k h_k e^{ik\xi}$, has linear phase, i.e.

$$H(\xi) = e^{i\alpha\xi}|H(\xi)|,$$

for some $\alpha \in \mathbb{R}$. By inspection, one can check that the Daubechies filters m_0^N do not even come close to having linear phase. The least asymmetric Daubechies filters are constructed like the Daubechies filters with the exception that the roots of $|\mathcal{L}^N|^2$ (in the Riesz factorization) are chosen to make the phase of the transfer function "close" to linear. Note that m_0 is a product of factors of the type

$$(e^{i\xi} - z_j)(e^{i\xi} - \bar{z}_j) = e^{i\xi}[e^{i\xi} - 2R_j \cos\alpha_j + R_j^2 e^{-i\xi}], \tag{1.16}$$

and

$$(e^{i\xi} - r_k) = e^{i\xi/2}[e^{i\xi/2} - r_k e^{-i\xi/2}]. \tag{1.17}$$

Apart from linear terms the phase contributions from such factors are, respectively,

$$\Psi_j(\xi) = \operatorname{arctg}\left(\frac{(1-R_j)^2 \sin\xi}{(1+R_j^2)\cos\xi - 2R_j\cos\alpha_j}\right), \tag{1.18}$$

and

$$\tilde{\Psi}_k(\xi) = \operatorname{arctg}\left(\frac{1+r_k}{1-r_k}\operatorname{tg}\frac{\xi}{2}\right). \tag{1.19}$$

The valuation of arctg is chosen such that Ψ_l is continuous in $[0, 2\pi]$ and $\Psi_l(0) = 0$. Then the least asymmetric filter is the m_0 obtained by minimizing the total nonlinear phase contribution

$$\Psi^{\text{tot}}(\xi) = \sum_{k=1}^{K}(\tilde{\Psi}_k(\xi) - \frac{\xi}{2\pi}\tilde{\Psi}_k(2\pi)) + \sum_{j=1}^{J}(\Psi_j(\xi) - \frac{\xi}{2\pi}\Psi_j(2\pi)),$$

over the $2^{\lfloor N/2 \rfloor}$ different choices of zeros. This is usually done graphically.

The Coiflet Filters

The Coiflet filters (named after R. Coifman, but constructed by Daubechies) are constructed such that we get a fixed number of vanishing moments of both the wavelet ψ and scaling function ϕ, i.e. for some fixed L

$$\int \phi(x)\,dx = 1,$$
$$\int x^k \phi(x)\,dx = 0, \qquad \text{for } k = 1, 2, \ldots, L-1,$$
$$\int x^k \psi(x)\,dx = 0, \qquad \text{for } k = 0, 1, \ldots, L-1.$$

It is not hard to see that the above conditions are equivalent to the following conditions on the low-pass filter m_0,

$$m_0^{(\ell)}(\pi) = 0, \qquad \text{for } \ell = 0, 1, \ldots, L-1,$$
$$m_0(0) = 1, \qquad m_0^{(\ell)}(0) = 0, \qquad \text{for } \ell = 1, 2, \ldots, L-1.$$

For $L = 2K$ we can take

$$m_0(\xi) = \left(\cos^2 \frac{\xi}{2}\right)^K \left[\sum_{k=0}^{K-1} \binom{K-1+k}{k} \left(\sin^2 \frac{\xi}{2}\right)^k + \left(\sin^2 \frac{\xi}{2}\right)^K f(\xi)\right],$$

where $f(\xi) = \sum_{n=0}^{2K-1} f_n e^{-in\xi}$. Using the identity

$$\sum_{j=0}^{K-1} \binom{K-1+j}{j} \left[\cos^{2K} \beta \sin^{2j} \beta + \sin^{2K} \beta \cos^{2j} \beta\right] = 1$$

we get

$$m_0(\xi) = 1 + \left(\sin^2 \frac{\xi}{2}\right)^K \left[-\sum_{k=0}^{K-1} \binom{K-1+k}{k} \left(\cos^2 \frac{\xi}{2}\right)^k + \left(\cos^2 \frac{\xi}{2}\right)^K f(\xi)\right].$$

Thus, m_0 has a zero of order $2K$ at π [use $\cos^2 \xi/2 = \frac{1}{4} e^{-i\xi}(1 + e^{i\xi})^2$]. The coefficients f_n are then chosen appropriately to normalize m_0. The technical details can be found in [7].

The Meyer Filters

The Meyer filter with resolution ε is a non-negative CQF, $m_0^{M,\varepsilon}$, for which

$$m_0^{M,\varepsilon}|_{(-\pi/2+\varepsilon, \pi/2-\varepsilon)} = 1.$$

We always assume that $m_0^{M,\varepsilon} \in C^1(\mathbb{R})$.

Chapter 2

Wavelet Packets

Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short-time Fourier analysis to new algorithms better suited to detect and analyze abrupt changes in signals. It corresponds to a decomposition of phase space in which the trade-off between time and frequency localization has been chosen to provide better and better time localization at high frequencies in return for poor frequency localization. In fact the wavelet $\psi_{j,k}$ has a frequency resolution proportional to 2^j, which follows by taking the Fourier transform:

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}.$$

This makes the analysis well adapted to the study of transient phenomena and has proven a very successful approach to many problems in signal processing, numerical analysis, and quantum mechanics. Nevertheless, for stationary signals wavelet analysis is outperformed by short-time Fourier analysis. Wavelet packets were introduced by R. Coifman, Y. Meyer, and M. V. Wickerhauser to improve the poor frequency localization of wavelet bases for large j and thereby provide a more efficient decomposition of signals containing both transient and stationary components.

2.1 Nonstationary Wavelet Packets

In the original construction by Coifman, Meyer and Wickerhauser ([4, 5]) of wavelet packets the functions were constructed by starting from a multiresolution analysis and then generating the wavelet packets using the associated CQFs. However, it was observed by Hess-Nielsen ([20, 21]) that it is an unnecessary constraint to use the multiresolution filters to do the frequency decomposition. We present his, more general, definition of so-called nonstationary wavelet packets here.

Definition 2.1 (Nonstationary Wavelet Packets). *Let (ϕ, ψ) be the scaling function and wavelet associated with a multiresolution analysis, and let $(F_0^{(p)}, F_1^{(p)})$, $p \in \mathbb{N}$, be a family of bounded operators on $\ell^2(\mathbb{Z})$ of the form*

$$(F_\varepsilon^{(p)} a)_k = \sum_{n \in \mathbb{Z}} a_n h_\varepsilon^{(p)}(n - 2k), \qquad \varepsilon = 0, 1,$$

with $h_1^{(p)}(n) = (-1)^n h_0^{(p)}(1-n)$ a real-valued sequence in $\ell^1(\mathbb{Z})$ such that each $(F_0^{(p)}, F_1^{(p)})$ is a pair of CQFs. We define the family of nonstationary wavelet packets $\{w_n\}_{n=0}^{\infty}$ recursively by letting $w_0 = \phi$, $w_1 = \psi$, and then for $n \in \mathbb{N}$

$$\begin{aligned} w_{2n}(x) &= 2 \sum_{q \in \mathbb{Z}} h_0^{(p)}(q) w_n(2x - q) \\ w_{2n+1}(x) &= 2 \sum_{q \in \mathbb{Z}} h_1^{(p)}(q) w_n(2x - q), \end{aligned} \tag{2.1}$$

where $2^p \le n < 2^{p+1}$.

We are interested in the following special case of Definition 2.1.

Definition 2.2 (Basic Stationary Wavelet Packets). *Let (ϕ, ψ) be the scaling function and wavelet associated with a multiresolution analysis, with associated CQFs $\{h_n\}$ and $\{g_n\}$. The functions $\{w_n\}_n$ generated by Definition 2.1 by letting $\{h_0^{(p)}\} = \{h_n\}$ and $\{h_1^{(p)}\} = \{g_n\}$ for all $p \in \mathbb{N}$ are called basic stationary wavelet packets.*

Definition 2.2 is the original definition of the basic wavelet packets given by Coifman, Meyer, and Wickerhauser. Figure 2.1 shows the basic stationary wavelet packets $w_1, w_2, \dots, w_8$ associated with the Coiflet filter of length 6.

It is an easy consequence of Definition 2.2 that for n, $2^{J-1} \le n < 2^J$, with binary expansion $n = \sum_{j=1}^{J} \varepsilon_j 2^{j-1}$, we have

$$\hat{w}_n(\xi) = \left[\prod_{j=1}^{J} m_{\varepsilon_j}(\xi / 2^j) \right] \hat{\phi}(\xi / 2^J)$$

where (m_0, m_1) are the filters associated with the multiresolution analysis.

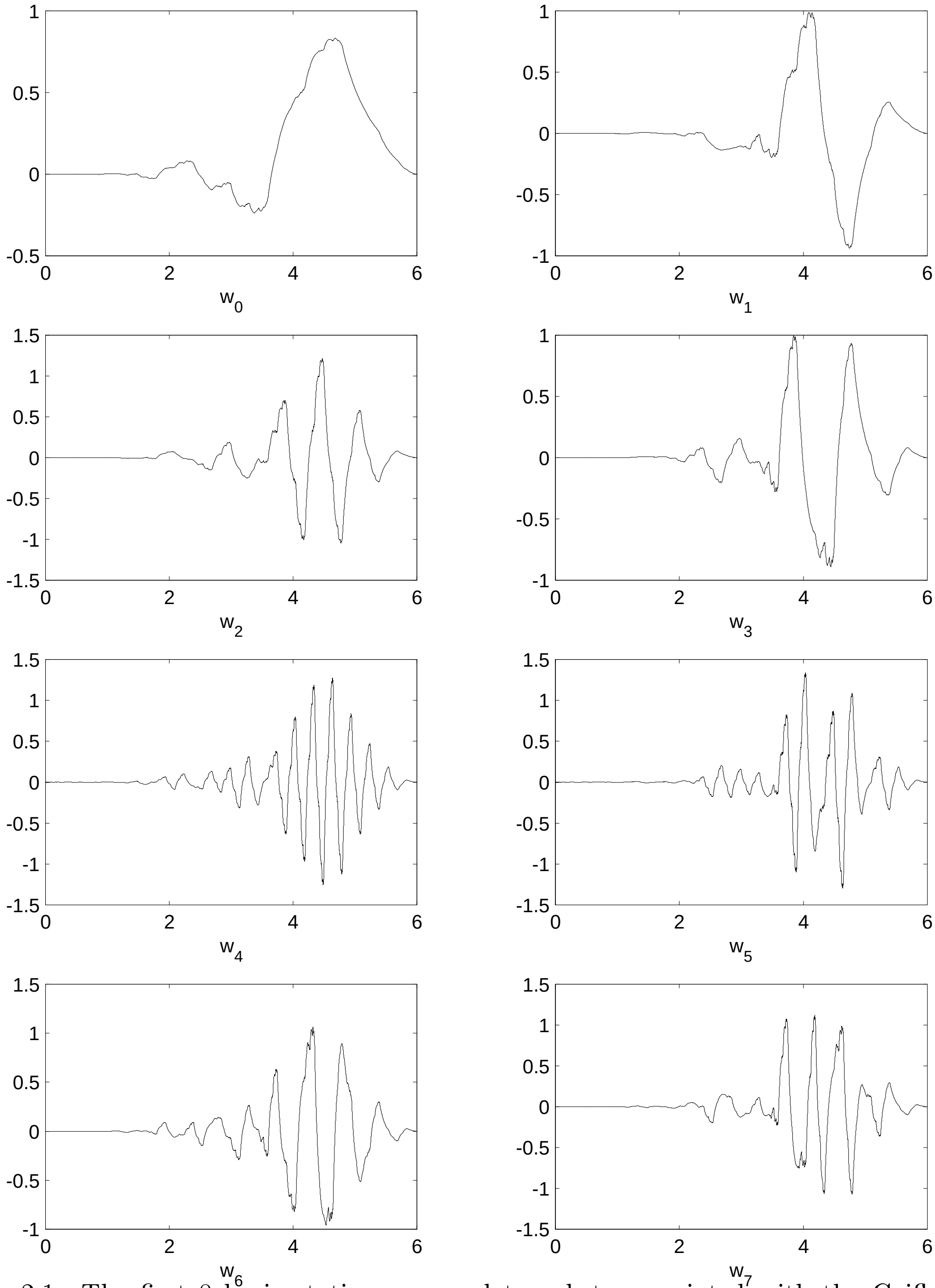

Figure 2.1: The first 8 basic stationary wavelet packets associated with the Coiflet filter of length 6.

The following is the fundamental result about nonstationary wavelet packets.

Theorem 2.3 ([22, 21]). *Let $\{w_n\}_{n=0}^{\infty}$ be a family of nonstationary wavelet packets associated with the multiresolution analysis $\{V_j\}$ with scaling function and wavelet (ϕ, ψ). The functions $\{w_n\}$ satisfy the following*

- $\{w_0(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0
- $\{w_n(\cdot - k)\}_{k\in\mathbb{Z}, 0\le n<2^j}$ *is an orthonormal basis for* V_j.

In particular, $\{w_n(\cdot - k)\}_{k\in\mathbb{Z}, n\in\mathbb{N}_0}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Proof. Since $w_0 = \phi$ and $w_1 = \psi$ we get the first statement, and the second in the case $j = 0$, immediately. Next step is to prove that $\{w_{2n}(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\{w_{2n+1}(\cdot - k)\}_{k\in\mathbb{Z}}$ are orthonormal systems. Suppose the result is true for all indices j with $j < n$ with n such that $2^p \le n < 2^{p+1}$. We have,

$$\begin{aligned}
\langle w_{2n}, w_{2n}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_0^{(p)}(\ell)h_0^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_0^{(p)}(\ell)h_0^{(p)}(\ell-2k) \\
&= \delta_{0,k}, \\
\langle w_{2n+1}, w_{2n+1}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_1^{(p)}(\ell)h_1^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_1^{(p)}(\ell)h_1^{(p)}(\ell-2k) \\
&= \delta_{0,k},
\end{aligned}$$

and

$$\begin{aligned}
\langle w_{2n}, w_{2n+1}(\cdot - k)\rangle &= 4\sum_{\ell\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} h_0^{(p)}(\ell)h_1^{(p)}(q)\int w_n(2t-\ell)\overline{w_n(2(t-k)-q)}\,dt \\
&= 2\sum_{\ell\in\mathbb{Z}} h_0^{(p)}(\ell)h_1^{(p)}(\ell-2k) \\
&= 0.
\end{aligned}$$

Thus, a simple additional induction argument using the above shows that $\{w_n(\cdot - k)\}_{n\in\mathbb{N}_0, k\in\mathbb{Z}}$ is an orthonormal system.

Let $\Omega_n = \overline{\text{Span}}\{w_n(\cdot - k)\}_{k\in\mathbb{Z}}$. Define $\delta f(x) = \sqrt{2}f(2x)$. Since $\{w_n(\cdot - k)\}_k$ is an orthonormal system so is $\{\delta w_n(\cdot - k)\}_k$, and it follows from the exact reconstruction property of the filters (see (1.11)) that for $2^p \le n < 2^{p+1}$,

$$w_n(x-k) = \frac{1}{2}\sum_{q\in\mathbb{Z}} h_0^{(p)}(k-2q)w_{2n}(x/2-q) + \frac{1}{2}\sum_{q\in\mathbb{Z}} h_1^{(p)}(k-2q)w_{2n+1}(x/2-q).$$

Hence, by (2.1),

$$\overline{\text{Span}}\{\sqrt{2}w_n(2\cdot-k)\}_k = \overline{\text{Span}}\{w_{2n}(\cdot-k)\}_k \oplus \overline{\text{Span}}\{w_{2n+1}(\cdot-k)\}_k,$$

i.e. $\delta\Omega_n = \Omega_{2n}\oplus\Omega_{2n+1}$. Thus,

$$\begin{aligned}
\delta\Omega_0 \ominus \Omega_0 &= \Omega_1 \\
\delta^2\Omega_0 \ominus \delta\Omega_0 &= \delta\Omega_1 = \Omega_2\oplus\Omega_3 \\
\delta^3\Omega_0 \ominus \delta^2\Omega_0 &= \delta\Omega_2\oplus\delta\Omega_3 = \Omega_4\oplus\Omega_5\oplus\Omega_6\oplus\Omega_7 \\
&\vdots \\
\delta^k\Omega_0 \ominus \delta^{k-1}\Omega_0 &= \Omega_{2^{k-1}}\oplus\Omega_{2^{k-1}+1}\oplus\cdots\oplus\Omega_{2^k-1}.
\end{aligned}$$

By telescoping the above equalities we finally get the wanted result

$$\delta^k\Omega_0 \equiv \delta^k V_0 = V_k = \Omega_0\oplus\Omega_1\oplus\cdots\oplus\Omega_{2^k-1},$$

and $\cup_{k\ge0}V_k$ is dense in $L^2(\mathbb{R})$ by the definition of a multiresolution analysis. ■

The above theorem can be generalized considerably. The following construction gives us a whole library of orthonormal bases each with different time-frequency properties.

Theorem 2.4. *Let $\{w_n\}$ be a family of nonstationary wavelet packets. For every partition P of $\mathbb{N}_0$ into sets of the form $I_{nj} = \{n2^j,\ldots,(n+1)2^j-1\}$ with $n,j\in\mathbb{N}_0$, the family*

$$\{2^{j/2}w_n(2^j\cdot-k)\}_{k\in\mathbb{Z},I_{nj}\in P}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Proof. An argument similar to the one in Theorem 2.3 shows that

$$\delta^k\Omega_n = \Omega_{2^kn}\oplus\Omega_{2^kn+1}\oplus\cdots\oplus\Omega_{2^k(n+1)-1}.$$

Moreover, the functions $\{2^{j/2}w_n(2^j\cdot-q)\}_{q\in\mathbb{Z}}$ span the space $\delta^j\Omega_n$ and

$$\sum_{I_{nj}\in P}\delta^j\Omega_n = \bigoplus_{q\ge0}\Omega_q = L^2(\mathbb{R}),$$

which proves the theorem. ■

There is an (efficient) algorithm to decompose a given signal in each of the bases given by Theorem 2.4 and another algorithm to find the "best" of such expansions wrt. predetermined criteria. The reader should consult [44] for more material on such applications.

2.2 Wavelet Packets as a Basis for the L^p-spaces

The convergence properties of the expansion of a signal in the wavelet basis have been thoroughly examined, in particular by Y. Meyer ([29]), whereas the convergence properties of the expansion in the basic wavelet packets remain unresolved. In this section we consider the expansion of $L^p(\mathbb{R})$-functions in some special basic wavelet packets, related to the Walsh and Shannon wavelet packets.

Let us recall the definition of a Schauder basis for a separable Banach space.

Definition 2.5. *Let X be a separable Banach space. A collection $\{e_n\}_{n=0}^{\infty} \subset X$ is called a Schauder basis for X if every $f \in X$ has a unique norm-convergent expansion of the form*

$$f = \sum_{n=0}^{\infty} \alpha_n e_n,$$

with $\{\alpha_n\} \subset \mathbb{C}$.

It is easy to check, using the Banach-Steinhaus theorem, that Definition 2.5 is equivalent to the following two requirements, where P_n denotes the projection onto the closed span of $\{e_n\}_{j=0}^{n}$,

- Span$\{e_n\}_{n=0}^{\infty}$ is dense in X.
- $\{P_n\}_{n=0}^{\infty}$ is a uniformly bounded sequence of operators.

It is often much easier to check these two conditions than it is to verify Definition 2.5 directly.

The following two sections contain examples of basic nonstationary wavelet packets that do form Schauder bases for $L^p(\mathbb{R})$, $1 < p < \infty$.

2.2.1 The Walsh System

In this section we consider nonstationary wavelet packets derived from the well known Walsh functions. For the sake of completeness we begin by defining the Walsh function. Further details on the Walsh system can be found in Appendix A. Let us recall the definition of the Haar filter

Definition 2.6. *The CQFs given by $h_0 = h_1 = 1/2$, $h_k = 0$ otherwise, and $g_k = (-1)^k h_{1-k}$ are called the Haar filters.*

The Walsh wavelet packets are defined by using the Haar filter and Haar scaling function $\chi_{[0,1)}$ in Definition 2.2, i.e.

Definition 2.7. *The Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is defined recursively on* $[0,1)$ *by* $W_0(x) = \chi_{[0,1)}(x)$ *and*

$$W_{2n+\varepsilon}(x) = W_n(2x) + (-1)^{\varepsilon} W_n(2x-1), \qquad \varepsilon = 0,1; n = 0,1,\dots .$$

It is a well known result by R. Paley ([30]) that the Walsh system constitutes a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, (see [35] for a nice "Martingale proof") so we have the following positive result

Theorem 2.8. *The functions* $\{W_n(\cdot - k)\}_{n\in\mathbb{N}_0,k\in\mathbb{Z}}$ *constitute a Schauder basis for* $L^p(\mathbb{R})$, $1 < p < \infty$, *in the sense that*

$$\sum_{n=0}^{N} \sum_{|k|\le M} \langle f, W_n(\cdot - k)\rangle\, W_n(\cdot - k) \xrightarrow[M,N\to\infty]{L^p(\mathbb{R})} f.$$

Our goal in the next section is to extend this result to a class of smooth nonstationary wavelet packets that resemble the Walsh system.

2.2.2 Walsh Type Wavelet Packets

We now define a class of nonstationary wavelet packets that can be seen as a natural generalization of the Walsh functions. In particular, each wavelet packet system in the class turns out to be equivalent to the Walsh functions in $L^p(\mathbb{R})$, $1 < p < \infty$.

Definition 2.9 (Walsh Type Wavelet Packets). *Let* $\{w_n\}_{n\ge 0,k\in\mathbb{Z}}$ *be a family of nonstationary wavelet packets constructed by using a family* $\{h_n^{(p)}\}_{p=1}^{\infty}$ *of finite filters in Definition 2.1. If there exists a constant* $J \in \mathbb{N}$ *such that* $h_n^{(p)}$ *is the Haar filter for every* $p \ge J$ *and* w_1 *has compact support then we call* $\{w_n\}_{n\ge 0}$ *a family of Walsh type wavelet packets.*

This definition closely resembles Definition 2.7. To prove the equivalence with the Walsh system we need to generalize the following theorem by Y. Meyer

Theorem 2.10 ([29]). *Let* $\psi \in C^1(\mathbb{R})$ *be a compactly supported wavelet. Then there exists an isomorphism on* $L^p(\mathbb{R})$, $1 < p < \infty$, *taking* $\psi_{j,k}$ *to* $h_{j,k}$, *with* h *the Haar wavelet.*

The generalization we need is the following

Lemma 2.11. *Let* $\{w_n\}_{n\ge 0}$ *be a family of Walsh type wavelet packets with* J *as in Definition 2.9, and let* $\{W_n\}$ *be the Walsh system. Let* $f_{j,k}^n = 2^{j/2} w_n(2^j \cdot -k)$, *and* $g_{j,k}^n = 2^{j/2} W_n(2^j \cdot -k)$. *If* $w_1 \in C^1(\mathbb{R})$ *then there is an isomorphism* $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, *for* $1 < p < \infty$, *such that*

$$Q f_{j,k}^n = g_{j,k}^n, \qquad j,k \in \mathbb{Z}, 2^J \le n < 2^{J+1}.$$

Proof. Let $\{W_n^s\}_n$ be a family of nonstationary wavelet packets generated by taking any compactly supported $C^1(\mathbb{R})$ scaling function and associated wavelet (ϕ, ψ), and letting each $h^{(p)}$ be the Haar filter in definition 2.1. Let $v_{j,k}^n = 2^{j/2} W_n^s(2^j \cdot -k)$. For each $n \geq 1, 2^j \leq n < 2^{j+1}$, we have a finite set $F \subset \mathbb{Z}$ such that

$$W_n = \sum_{s \in F} c_{n,s} h_{j,s}$$
$$W_n^s = \sum_{s \in F} c_{n,s} \psi_{j,s}.$$

Thus, for $2^J \leq n < 2^{J+1}$,

$$g_{p,k}^n = \sum_{s \in F} c_{n,s} h_{p+J, 2^J k + s}$$
$$v_{p,k}^n = \sum_{s \in F} c_{n,s} \psi_{p+J, 2^J k + s}.$$

Let $P : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the isomorphism defined by $Ph_{j,k} = \psi_{j,k}$. It follows that $Pg_{p,k}^n = v_{p,k}^n$. Hence, it suffices to find an isomorphism $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ such that $Qf_{j,k}^n = v_{j,k}^n$. Note that

$$\{f_{j,k}^n\}_{2^J \leq n < 2^{J+1}, j,k \in \mathbb{Z}}, \quad \text{and} \quad \{v_{j,k}^n\}_{2^J \leq n < 2^{J+1}, j,k \in \mathbb{Z}}$$

are both orthonormal bases for $L^2(\mathbb{R})$ (easy consequence of the multiresolution structure). Thus, Q defined by $Qf_{j,k}^n = v_{j,k}^n$, $2^J \leq n < 2^{J+1}, j, k \in \mathbb{Z}$, is unitary. The associated (Schwartz) kernel is given by

$$K(x, y) = \sum_{n=2^J}^{2^{J+1}-1} \sum_{j,k \in \mathbb{Z}} v_{j,k}^n(x) \overline{f_{j,k}^n(y)}.$$

We claim that K is a Calderón-Zygmund kernel. To verify this, choose $N \geq 1$ such that

$$\text{supp}(W_n^s), \text{supp}(w_n) \subset [-N, N]$$

for $2^J \leq n < 2^{J+1}$. We have

$$|K(x, y)| \leq \sum_{n=2^J}^{2^{J+1}-1} \sum_{j,k \in \mathbb{Z}} 2^j |W_n^s(2^j x - k)||w_n(2^j y - k)|.$$

Thus $(x, y) \in \text{supp}(K)$ implies that $|2^j x - k| \leq N$ and $|2^j y - k| \leq N$. Hence, $2^j|x - y| \leq 2N$ so

$$j \leq \log_2 \frac{2N}{|x - y|}.$$

Let

$$j_0 = \left\lfloor \log_2 \frac{2N}{|x - y|} \right\rfloor.$$

We have

$$|K(x,y)| \leq \sum_{n=2^J}^{2^{J+1}-1} \sum_{j\leq j_0} 2^j\,(2N+1)\,\|W_n^s\|_\infty \|w_n\|_\infty$$
$$\leq C2^J\,(2N+1) \sum_{j\leq j_0} 2^j = \frac{2^{J+1}N(2N+1)C}{|x-y|}.$$

Similar estimates give us

$$\left|\frac{\partial}{\partial x}K(x,y)\right| \leq \frac{C}{|x-y|^2}$$
$$\left|\frac{\partial}{\partial y}K(x,y)\right| \leq \frac{C}{|x-y|^2}.$$

It follows that Q is a Calderón-Zygmund operator and thus bounded on $L^p(\mathbb{R})$, $1<p<\infty$ (see [28]). The same type of argument applies to Q^{-1} (the above estimates are symmetric in $f^n_{j,k}$ and $v^n_{j,k}$) and Q is therefore an isomorphism on $L^p(\mathbb{R})$. ■

We can now state and prove the main result of this section, the Walsh type wavelet packets do constitute a Schauder basis for $L^p(\mathbb{R})$ for $1<p<\infty$.

Theorem 2.12. *Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets with J as in Definition 2.9. If $w_1 \in C^1(\mathbb{R})$ then $\{w_n(\cdot - k)\}_{n\geq 0, k\in\mathbb{Z}}$ is a Schauder basis for $L^p(\mathbb{R})$, $1<p<\infty$.*

Proof. We claim that the systems

$$\{w_n(\cdot - k)\}_{n\geq 2^{J+1},k\in\mathbb{Z}} \quad \text{and} \quad \{W_n(\cdot - k)\}_{n\geq 2^{J+1},k\in\mathbb{Z}} \tag{2.2}$$

are equivalent in $L^p(\mathbb{R})$ in the sense that there is an isomorphism Q on $L^p(\mathbb{R})$ mapping one system onto the other. Let $n \geq 2^{J+1}$. Note that

$$\hat{w}_n(\xi) = \prod_{j=1}^{K} m_{\varepsilon_j}(2^{-j}\xi)\cdot \hat{w}_{\tilde{n}}(2^{-K}\xi),$$

for some $2^J \leq \tilde{n} < 2^{J+1}$ and $K\geq 1$. Thus

$$w_n(x-k) = \sum_{s\in F} c_{n,s} f^{\tilde{n}}_{K,s}(x-k), \tag{2.3}$$

with $f^{\tilde{n}}_{j,k} = 2^{j/2} w_{\tilde{n}}(2^j\cdot -k)$ and F a finite set (depending on n). The coefficients $c_{n,s}$ depend only on n and the Haar filter. Thus, W_n has the same expansion:

$$W_n(x-k) = \sum_{s\in F} c_{n,s} g^{\tilde{n}}_{K,s}(x-k), \tag{2.4}$$

with $g_{j,k}^{\tilde{n}} = 2^{j/2}W_{\tilde{n}}(2^j \cdot -k)$. Let $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the isomorphism defined by $Qf_{j,k}^n = g_{j,k}^n$. It follows from (2.3) and (2.4) that

$$Qw_n(\cdot - k) = W_n(\cdot - k),$$

which proves (2.2).

Now we can prove that the wavelet packets form a basis. It is clear that the system

$$\{w_n(\cdot - k)\}_{n \geq 0, k \in \mathbb{Z}}$$

is dense in $L^p(\mathbb{R})$ for $1 < p < \infty$ since the associated wavelets $2^{j/2}\psi(2^j x - k)$, $j \geq 0$, and the translates of the scaling function are all finite linear combinations of the wavelet packets. Hence, it suffices to prove that there exists a finite constant (depending on p) such that for any sequence $(c_{n,k}) \subset \mathbb{C}$ and $M, N \geq 1$ we have

$$\left\| \sum_{0 \leq n \leq N, |k| \leq N} c_{n,k} w_n(\cdot - k) \right\|_p \leq C \left\| \sum_{0 \leq n \leq N+M, |k| \leq M+N} c_{n,k} w_n(\cdot - k) \right\|_p \tag{2.5}$$

First, let us check that

$$\{w_n(\cdot - k)\}_{0 \leq n < 2^{J+1}, k \in \mathbb{Z}}$$

is a Schauder basis for its closed linear span in $L^p(\mathbb{R})$. The kernel for the projection, $P_{N,M}$, onto

$$\{w_n(\cdot - k)\}_{0 \leq n \leq N < 2^{J+1}, |k| \leq M}$$

is given by

$$K_{N,M}(x, y) = \sum_{n=0}^{N} \sum_{|k| \leq M} w_n(x - k)\overline{w_n(y - k)}.$$

Fix K such that $\mathrm{supp}(w_n) \subset [-K, K]$ for $0 \leq n < 2^{J+1}$. Then

$$\begin{aligned} |K_{N,M}(x, y)| &\leq \sum_{n=0}^{2^{J+1}-1} \sum_{k \in \mathbb{Z}} |w_n(x - k)||w_n(y - k)| \\ &\leq 2^{J+1}(2K + 1) \max_{0 \leq n < 2^{J+1}} \{\|w_n\|_\infty^2\} \cdot \chi_{[0,2K]}(|x - y|). \end{aligned}$$

Hence, using Hölder's inequality and Fubini's Theorem,

$$\begin{aligned}
\|P_{N,M}f\|_p^p &= \int \Big| \int K_{N,M}(x,y)\, f(y)\, dy\Big|^p\, dx \\
&\le \int \Big(\int |f(y)||K_{N,M}(x,y)|^{1/p} \cdot |K_{N,M}(x,y)|^{1-1/p}\, dy\Big)^p\, dx \\
&= \int \int |f(y)|^p |K_{N,M}(x,y)|\, dy \Big(\int |K_{N,M}(x,y)|\, dy\Big)^{p/q}\, dx \\
&\le C^{p/q} \int |f(y)|^p \int |K_{N,M}(x,y)|\, dx\, dy \\
&\le C^{1+p/q} \|f\|_p^p,
\end{aligned}$$

which proves the claim. It now follows that whenever $M + N < 2^{J+1}$ then (2.5) holds. Suppose $N < 2^{J+1}$ and $M + N \ge 2^{J+1}$. Then, by the above and the fact that the projection onto the multiresolution space V_{J+1} is bounded on $L^p(\mathbb{R})$,

$$\begin{aligned}
\Big\| \sum_{0\le n\le N, |k|\le N} c_{n,k} w_n(\cdot - k)\Big\|_p &\le C \Big\| \sum_{0\le n< 2^{J+1}, |k|\le M+N} c_{n,k} w_n(\cdot - k)\Big\|_p \\
&\le C\, C_1 \Big\| \sum_{0\le n\le N+M, |k|\le M+N} c_{n,k} w_n(\cdot - k)\Big\|_p .
\end{aligned}$$

Finally, suppose $N \ge 2^{J+1}$. Then, using (2.2), the result for $N + M < 2^{J+1}$, the Schauder basis properties of the Walsh system, and $\|f\|_p \simeq \{\|P_{V_{J+1}} f\|_p + \|(1 - P_{V_{J+1}})f\|_p\}$,

$$\begin{aligned}
\Big\| \sum_{0\le n\le N, |k|\le N} c_{n,k} w_n(\cdot - k)\Big\|_p &\le \Big\{ \Big\| \sum_{0\le n< 2^{J+1}, |k|\le N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\
&\qquad \Big\| \sum_{2^{J+1}\le n\le N, |k|\le N} c_{n,k} w_n(\cdot - k)\Big\|_p \Big\} \\
&\le C \Big\{ \Big\| \sum_{0\le n< 2^{J+1}, |k|\le M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\
&\qquad \Big\| \sum_{2^{J+1}\le n\le N, |k|\le N} c_{n,k} w_n(\cdot - k)\Big\|_p \Big\} \\
&\le C \Big\{ \Big\| \sum_{0\le n< 2^{J+1}, |k|\le M+N} c_{n,k} w_n(\cdot - k)\Big\|_p + \\
&\qquad \Big\| \sum_{2^{J+1}\le n\le N, |k|\le N} c_{n,k} W_n(\cdot - k)\Big\|_p \Big\}
\end{aligned}$$

$$\leq C\Big\{\Big\|\sum_{0\leq n<2^{J+1},|k|\leq M+N} c_{n,k}w_n(\cdot-k)\Big\|_p+$$
$$\Big\|\sum_{2^{J+1}\leq n\leq N+M,|k|\leq M+N} c_{n,k}W_n(\cdot-k)\Big\|_p\Big\}$$
$$\leq C\Big\{\Big\|\sum_{0\leq n<2^{J+1},|k|\leq M+N} c_{n,k}w_n(\cdot-k)\Big\|_p+$$
$$\Big\|\sum_{2^{J+1}\leq n\leq N+M,|k|\leq M+N} c_{n,k}w_n(\cdot-k)\Big\|_p\Big\}$$
$$\leq C\Big\|\sum_{0\leq n\leq N+M,|k|\leq N+M} c_{n,k}w_n(\cdot-k)\Big\|_p.$$

We conclude that (2.5) holds in general, and we are done. ■

From the above proof it is easy to deduce the following Corollary.

Corollary 2.13. *Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets. If $w_1\in C^1(\mathbb{R})$ then there exists an isomorphism $Q : L^p(\mathbb{R})\to L^p(\mathbb{R})$, $1<p<\infty$, such that*

$$Qw_n(\cdot-k)=W(\cdot-k),\qquad n\geq 0, k\in\mathbb{Z}.$$

2.2.3 A Counterexample in $L^1(\mathbb{R})$

It is interesting to know what happens in the "limiting case" $L^1(\mathbb{R})$ of Theorem 2.12. It is well known that the exponentials $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ fail to be a basis for $L^1[0,1)$ whereas the periodic wavelets do form a Schauder basis for $L^1[0,1)$, so it can go both ways. However, the next theorem provides an explicit function in $L^1(\mathbb{R})$ for which the expansion in the Walsh type wavelet packets fails. The construction owes much to a counterexample for the Walsh system in [14] and a construction (unpublished) by N. Hess-Nielsen.

Theorem 2.14. *Let $\{w_n\}_n$ be a family of Walsh type wavelet packet system, and let J be defined as in Definition 2.9. Choose $L\in\mathbb{N}$ such that supp$(w_{2^J+1})\subset[-L+1,L-1]$ and choose $M\in\mathbb{N}$ such that $2^M>2L$. Let $N(k)=k^3+M+1$, and define $K:\mathbb{N}\to\mathbb{N}$ recursively by letting $K(1)=2^J+1$, $K(2n)=2K(n)$, and $K(2n+1)=2K(n)+1$. Define f by*

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{k^2}\Big(\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} w_{K(n)}(x)\Big).$$

Then $f\in L^1(\mathbb{R})$, but the wavelet packet expansion of f diverges in $L^1(\mathbb{R})$-norm.

Proof. Same as for the periodic case (Theorem 3.5). ■

2.2.4 Pointwise Convergence for Walsh Type Wavelet Packet Expansions

In this section, we prove that the Walsh type wavelet packet expansion of an $L^p(\mathbb{R})$-function $(1 < p < \infty)$ converges pointwise almost everywhere. The key step is to analyze the so-called Carleson operator for the Walsh type wavelet packet system.

Definition 2.15 (Carleson Operator). *Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2(M)$. The Carleson operator L is defined by*

$$(Lf)(x) = \sup_{N\geq 0} \sum_{n\leq N} \langle f, f_n\rangle f_n(x),$$

for $f \in L^2(M)$.

The following result shows that the Carleson operator is well-behaved.

Theorem 2.16. *The Carleson operator for any Walsh type wavelet packet system with $w_1 \in C^1(\mathbb{R})$ is of strong type (p,p) for $1 < p < \infty$.*

Proof. Let us start by reducing the problem. Choose $N \in \mathbb{N}$ such that $\operatorname{supp}(w_n) \subset [-N, N]$ for $n \geq 0$. Fix $p \in (1,\infty)$ and take any

$$f(x) = \sum_{n\geq 0, k\in\mathbb{Z}} c_{n,k} w_n(x-k) \in L^p(\mathbb{R}).$$

Define

$$f_k(x) = \sum_{n\geq 0} c_{n,k} w_n(x-k), \qquad g_k(x) = \sum_{n\geq 0} c_{n,k} W_n(x-k).$$

We have $\|f_k\|_p \simeq \|g_k\|_p$ (with bounds independent of k) by the proof of Theorem 2.12. Note that for $l \in \mathbb{Z}$,

$$|\{x \in [l, l+1) : |Lf(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \sum_{k=l-N}^{l+1+N} \int |Lf_k(x)|^p \, dx,$$

so (using the Marcinkiewicz interpolation theorem) it suffices to prove that $\|Lf_k\|_p \leq C\|f_k\|_p$, where C is a constant independent of k, since

$$\sum_{l\in\mathbb{Z}} \sum_{k=l-N}^{l+1+N} \|f_k\|_p^p \leq 2(N+1) \sum_{k\in\mathbb{Z}} \|f_k\|_p^p \leq 2C(N+1) \sum_{k\in\mathbb{Z}} \|g_k\|_p^p \leq \tilde{C}\|f\|_p^p.$$

We can, w.l.o.g., assume that $k = 0$. Let $K \in \mathbb{N}$ be the scale from which only the Haar filter is used to generate the wavelet packets $\{w_n\}_{n\geq 2^{K+1}}$. Let $m \in \mathbb{N}$ and suppose $2^J \leq m < 2^{J+1}$ for some $J > K+1$. Clearly, for each $x \in \mathbb{R}$,

$$\sum_{n=0}^{m} c_{n,0} w_n(x) = \sum_{n=0}^{2^{K+1}-1} c_{n,0} w_n(x) + \sum_{n=2^{K+1}}^{2^J-1} c_{n,0} w_n(x) + \sum_{n=2^J}^{m} c_{n,0} w_n(x),$$

so we have

$$\sup_{m\geq 1}\left|\sum_{n=0}^{m} c_{n,0}w_n(x)\right| \leq$$

$$\sup_{1\leq m<2^{K+1}}\left|\sum_{n=0}^{m} c_{n,0}w_n(x)\right| + \sup_{J>K+1}\left|\sum_{n=2^{K+1}}^{2^J+1} c_{n,0}w_n(x)\right| + \sup_{J>K+1}(M_J f_0)(x), \quad (2.6)$$

where

$$(M_J f_0)(x) = \sup_{2^J\leq m<2^{J+1}}\left|\sum_{n=2^J}^{m} c_{n,0}w_n(x)\right|.$$

We use brute force to estimated the first term of (2.6)

$$\begin{aligned}\sup_{0<m<2^{K+1}}\left|\sum_{n=0}^{m} c_{n,0}w_n(x)\right| &\leq \sum_{n=0}^{2^{K+1}-1}|c_{n,0}|\|w_n(x)\|_\infty\chi_{[-N,N]}(x)\\ &\leq \|f_0\|_p\sum_{n=0}^{2^{K+1}-1}\|w_n\|_{p'}\|w_n(x)\|_\infty\chi_{[-N,N]}(x).\end{aligned}$$

The second term of (2.6) satisfies

$$\left\|\sup_{J>K+1}\left|\sum_{n=2^{K+1}}^{2^J+1} c_{n,0}w_n(x)\right|\right\|_p \leq C\|f_0\|_p$$

since the dyadic partial sums for the wavelet packet expansion for f_0 agree everywhere with the partial sums for the wavelet expansion for f_0 and the Carleson operator for the wavelet expansion is of strong type (p,p) (see [40]). The challenge is to prove that the third term is of strong type (p,p). Fix $x\in\mathbb{R}-\mathcal{D}$, where $\mathcal{D}$ is the family of dyadic rationals. Note that

$$(M_J f_0)(x) \leq \sum_{j=0}^{2^K-1}(M_J^j f_0)(x),$$

where

$$(M_J^j f_0)(x) = \sup_{2^J+j2^{J-K}\leq m<2^J+(j+1)2^{J-K}}\left|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\right|,$$

so it suffices to prove that

$$\|\sup_{J>K+1}(M_J^j f_0)\|_p \leq C\|f_0\|_p$$

for $j=0,1,\ldots 2^K-1$. Fix $J>K+1$, $0\leq j<2^K-1$, and $2^J+j2^{J-K}\leq m<2^J+(j+1)2^{J-K}$.

We have, using Lemma A.4,

$$\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| =$$

$$\Big|\sum_{s=0}^{2^{J-K}-1}\Big\{\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}W_{n-2^J-j2^{J-K}}(s2^{-J+K})\Big\}w_{2^K+j}(2^{J-K}x-s)\Big|.$$

Define

$$F_m(t)=\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}W_{n-2^J-j2^{J-K}}(t), \quad \text{and} \quad F(t)=\sup_{m<2^J+(j+1)2^{J-K}}|F_m(t)|.$$

Then

$$\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| \le \sum_{s=0}^{2^{J-K}-1} F(s2^{-J+K})|w_{2^K+j}(2^{J-K}x-s)|,$$

and using the compact support of the wavelet packets,

$$\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| \le \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} F((\lfloor 2^{J-K}x\rfloor+l)2^{-J+K}).$$

Note that F is constant on dyadic intervals of the type $[l2^{-J+K},(l+1)2^{-J+K})$ so defining $\Delta_l=((\lfloor 2^{J-K}x\rfloor+l)2^{-J+K},(\lfloor 2^{J-K}x\rfloor+l+1)2^{-J+K})$, we have

$$\begin{aligned}\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}w_n(x)\Big| &\le \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} F((\lfloor 2^{J-K}x\rfloor+l)2^{-J+K})\\ &= \|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1}\int_{\Delta_l} F(t)\,dt.\end{aligned}$$

We need an estimate of F that does not depend on J. Note that for k, $0\le k<2^{J-K}$, using (A.1),

$$W_{2^J+j2^{J-K}}(t)W_k(t)=W_{2^J+j2^{J-K}+k}(t),$$

since the binary expansions of 2^J+j2^{J-K} and of k have no 1's in common. Hence,

$$|F_m(t)|=|W_{2^J+j2^{J-K}}(t)F_m(t)|=\Big|\sum_{n=2^J+j2^{J-K}}^{m} c_{n,0}W_n(t)\Big|,$$

so $F(t) \leq 2(Gg_0)(t)$, with G the Carleson operator for the Walsh system. Thus,

$$\Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big| \leq 2\|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1} \int_{\Delta_l} (Gg_0)(t)\, dt.$$

We let $\Delta_l^\star$ be the smallest dyadic interval containing Δ_l and x, and note that $|\Delta_l^\star| \leq (N+1)|\Delta_l|$ since $x \in \Delta_0$ (here we use $x \notin \mathcal{D}$). We have

$$\begin{aligned} \Big| \sum_{n=2^J+j2^{J-K}}^{m} c_{n,0} w_n(x) \Big| &\leq 2\|w_{2^K+j}\|_\infty \sum_{l=-N}^{N+1} |\Delta_l|^{-1} \int_{\Delta_l^\star} (Gg_0)(t)\, dt \\ &\leq 4\|w_{2^K+j}\|_\infty (N+1)^2 (MGg_0)(x), \end{aligned} \tag{2.7}$$

where M is the maximal operator of Hardy and Littlewood. The righthand side of (2.7) does not depend on m nor J so we may conclude that

$$\sup_{J>K+1} (M_J^j f_0)(x) \leq 4\|w_{2^K+j}\|_\infty (N+1)^2 (MGg_0)(x), \quad \text{a.e.}$$

and thus, since M and G are both of strong type (p,p) (see [38]),

$$\| \sup_{J>K+1} (M_J^j f_0)\|_p \leq C\|g_0\|_p \leq C_1 \|f_0\|_p, \qquad j = 0, 1, \ldots 2^K - 1,$$

and we are done. ■

The pointwise convergence result now follows by a standard argument (see [14])

Corollary 2.17. *Let $\{w_n\}_n$ be a Walsh type wavelet packet system for which $w_1 \in C^1(\mathbb{R})$. Then the wavelet packet expansion of every $f \in L^p(\mathbb{R})$, $1 < p < \infty$, converges a.e.*

2.2.5 The Shannon Wavelet Packets

The next well behaved nonstationary wavelet packets we present are related to the Shannon filter. The (stationary) Shannon wavelet packets are defined by taking

$$m_0^S(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\pi/2,\pi/2]}(\xi - 2\pi k)$$

and

$$m_1^S(\xi) = 1 - m_0^S(\xi)$$

in Definition 2.2. We want to find an explicit expression for $\hat{w}_n$. We define a map $G : \mathbb{N}_0 \to \mathbb{N}_0$ in the following way. Let $n = \sum_{k=1}^{\infty} n_k 2^{k-1}$ be the binary expansion of $n \in \mathbb{N}_0$. Then we let

$G(n)_i = n_i + n_{i+1} \pmod 2$, and put $G(n) = \sum_{k=1}^{\infty} G(n)_k 2^{k-1}$. The map G is the so-called Gray-code permutation (one can easily check that G is $1-1$ and onto $\mathbb{N}_0$). The Gray-code permutation relates the Walsh system in Paley order and frequency order, and enters naturally into the frequency localization of more general wavelet packets. We have the following simple formulas for the Shannon wavelet packets. See [43] for a proof.

Theorem 2.18 ([43]). *Let $\{w_n\}_n$ be the Shannon wavelet packets. Then*

$$\hat{w}_{G(n)}(\xi) = \chi_{[n\pi,(n+1)\pi]}(|\xi|).$$

Note that the Shannon wavelet packets are uniformly bounded just like the Walsh functions due to their perfect frequency localization.

The above result suggests that reordering the Shannon wavelet packets using the inverse Gray-code permutation might improve their convergence properties. We define a new system by letting $\omega_n = w_{G(n)}$ for $n \in \mathbb{N}_0$. We call the reordered system $\{\omega_n\}_{n=0}^{\infty}$ the Shannon wavelet packets in frequency order.

We want to prove that the Shannon wavelet packets form a Schauder basis for the $L^p(\mathbb{R})$-spaces. We need the following sampling theorem. The proof can be found in [29].

Theorem 2.19 ([29]). *Let $L_k^\delta(x) = \sin(\pi\delta^{-1}(x-\delta k))/(\pi\delta^{-1}(x-\delta k))$, $0 < \delta \leq 1$, and let $\{c_k\}_k \subset \mathbb{C}$. Then*

$$\Big\| \sum_{k\in\mathbb{Z}} c_k\, L_k^\delta \Big\|_p \simeq \|\{c_k\}\|_{\ell^p(\mathbb{Z})},$$

for $1 < p < \infty$.

Remark. Note that if $\{c_k\} \in \ell^p(\mathbb{Z}), 1 < p < \infty$, then it follows from the Lemma that $\sum_{k\in\mathbb{Z}} c_k\, L_k^\delta$ converges unconditionally in $L^p(\mathbb{R})$.

The following two lemmas will be used to prove the main result, Theorem 2.22. The first is a well known fact and we therefore omit the proof.

Lemma 2.20. *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Define $f_{a,b} = \mathcal{F}^{-1}\chi_{[a,b]}\mathcal{F}f$, for $a, b \in \mathbb{R}$, $a < b$. Then*

$$\|f_{a,b}\|_p \leq C_p \|f\|_p,$$

for some constant C_p independent of a and b. Moreover,

$$\|f - f_{a,b}\|_p \longrightarrow 0 \qquad \text{as } -a, b \to \infty.$$

We have the following Lemma which shows that the expansion of each $L^p(\mathbb{R})$-function in the Shannon scaling functions is well behaved.

Lemma 2.21. *Let*

$$L_k^\delta(x) = \frac{\sin(\pi\delta^{-1}(x-\delta k))}{(\pi\delta^{-1}(x-\delta k))}, \qquad 0 < \delta \le 1,$$

and suppose $f \in L^p(\mathbb{R})$, $1 < p < \infty$. *Then*

$$\sum_{k\in\mathbb{Z}} \langle f, L_k^\delta\rangle \, L_k^\delta \tag{2.8}$$

converges unconditionally in $L^p(\mathbb{R})$.

Proof. First, assume that $f \in L^p(\mathbb{R})$ with $\operatorname{supp}(\hat{f}) \subset [-\delta^{-1}\pi, \delta^{-1}\pi]$ (with $\hat{f}$ in the sense of tempered distributions for $2 < p < \infty$.). Note that f is the restriction of an analytic function of exponential type in this special case. We claim that $\sum_k |f(\delta k)|^p \le C_{p,\delta}\|f\|_p^p$ for some constant $C_{p,\delta}$. Indeed, take $\phi \in \mathcal{S}(\mathbb{R})$ with $\hat{\phi} = 1$ on $[-\delta^{-1}\pi, \delta^{-1}\pi]$. Then, by Plancherel's Theorem,

$$\begin{aligned}\int f(x)\,\bar{\phi}(x-\delta k)\,dx &= \frac{1}{2\pi}\int \hat{f}(\xi)\bar{\hat{\phi}}(\xi)\,\exp(i\delta k\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\delta^{-1}\pi}^{\delta^{-1}\pi} \hat{f}(\xi)\,\exp(i\delta k\xi)\,d\xi \\ &= f(\delta k).\end{aligned}$$

We now apply Hölder's inequality to get

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |f(\delta k)|^p &\le \int |f(x)|^p \sum_{k\in\mathbb{Z}} |\phi(x-\delta k)|\,dx\,\|\phi\|_{p'}^{p/p'} \\ &\le C_{p,\delta}\|f\|_p^p.\end{aligned}$$

Thus, Lemma 2.21 applies to the sequence $f(\delta k)$ and (2.8) converges unconditionally since $\langle f, L_k^\delta\rangle = f(\delta k)$. For general $f \in L^p(\mathbb{R})$ it suffices to notice that, by Lemma 2.20, the operator $f \to (\chi_{[-\delta^{-1}\pi,\delta^{-1}\pi]}\hat{f})\check{}$ is bounded on $L^p(\mathbb{R})$, and that f and $(\chi_{[-\delta^{-1}\pi,\delta^{-1}\pi]}\hat{f})\check{}$ have the same expansion in the functions $\{L_k^\delta\}$. ∎

Finally, we combine the above Lemmas to get a positive convergence result for expansions in the Shannon wavelet packets in frequency order.

Theorem 2.22. *The Shannon Wavelet Packet system in frequency order* $\{\omega_n(\cdot - k)\}_{n,k}$ *forms a Schauder basis for* $L^p(\mathbb{R})$, $1 < p < \infty$, *in the sense that*

$$\sum_{n=0}^{N}\sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\,\omega_n(\cdot - k) \xrightarrow[N\to\infty]{L^p(\mathbb{R})} f,$$

for $f \in L^p(\mathbb{R})$.

Proof. We have $\hat{\omega}_n(\xi) = \chi_{[n\pi,(n+1)\pi)}(|\xi|)$ so

$$\omega_n(x) = (n+1)\frac{\sin((n+1)\pi x)}{(n+1)\pi x} - n\frac{\sin(n\pi x)}{n\pi x} = (n+1)\, L_0^{(n+1)^{-1}} - n\, L_0^{n^{-1}}.$$

Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Lemma 2.21 shows that $\{\langle f, \omega_n(\cdot - k)\rangle\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Hence

$$\sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\, \omega_n(\cdot - k)$$

converges unconditionally to $P_{\Omega_n} f$, where P_{Ω_n} is the projection onto the closed span of $\{\omega_n(\cdot - k)\}_k$, i.e. $P_{\Omega_n} = \mathcal{F}^{-1}\, \chi_{\{n\pi \le |\xi| < (n+1)\pi\}}\, \mathcal{F}$. So all we have to check is that $\sum_{n=0}^{N} P_{\Omega_n}$ are uniformly bounded in N on $L^p(\mathbb{R})$. But $\sum_{n=0}^{N-1} P_{\Omega_n}$ is just the operator $f \to \mathcal{F}^{-1}\chi_{[-N\pi,N\pi]}\mathcal{F}$, and it is uniformly bounded on $L^p(\mathbb{R})$ by Lemma 2.20. ■

The above result can also be used to show that the expansion in the Shannon wavelet packets coverges pointwise a.e. Indeed this fact follows directly from the Carleson-Hunt theorem for the line:

Theorem 2.23 (Carleson-Hunt). *Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Define T_R, $R > 0$, by*

$$T_R f(x) = \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(\xi) e^{ix\xi}\, d\xi,$$

and let $(Tf)(x) = \sup_{R>0}(T_R)(x)$. Then T is of strong type (p,p).

We have

Corollary 2.24. *Let $\{\omega_n\}_n$ be the Shannon Wavelet Packet system in frequency order Then for $f \in L^p(\mathbb{R})$, $1 < p < \infty$,*

$$\sum_{n=0}^{N}\sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\, \omega_n(x-k) \xrightarrow[N\to\infty]{} f(x), \qquad a.e.$$

Proof. Just note that

$$\left\{\sum_{n=0}^{N-1}\sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle\, \omega_n(x-k)\right\}^{\widehat{}}(\xi) = \hat{f}(\xi)\chi_{[-N\pi,N\pi]}(\xi).$$

■

2.2.6 Shannon Type Wavelet Packets

We now generalize the above results to a class of nonstationary wavelet packets.

Definition 2.25 (Shannon Type Wavelet Packets). *Let $\{w_n\}_{n\geq 0}$ be the family of nonstationary wavelet packets constructed using a family $\{h_n^{(p)}\}_{p=1}^{\infty}$ of CQFs in definition 2.1. If there exists a constant $J \in \mathbb{N}$ such that $h^{(p)}$ is the Shannon filter for every $p \geq J$ then we call $\{w_n\}$ a family of Shannon type wavelet packets.*

For fixed $J \in \mathbb{N}$ we define a permutation $G_J : \mathbb{N} \to \mathbb{N}$ by[1]

$$G_J(n) = \begin{cases} n & \text{if } n \leq 2^J \\ [n_L \cdots n_{L-J+1}[G([n_{L-J} \cdots n_1])]_2] & \text{if } 2^L \leq n < 2^{L+1},\ L \geq J, \end{cases}$$

where $n = [n_L n_{L-1} \cdots n_1]$ is the binary expansion of n. So G_J leaves the J most significant bits unchanges, but performs the Gray-code permutation on the least significant $L-J$ bits.

The frequency ordering of any Shannon type wavelet packet system $\{w_n\}$ (with J as in definition 2.25) is given by

$$\{\omega_n \equiv w_{G_J(n)}\}_{n=0}^{\infty}.$$

The following result is the analog of Theorem 2.22.

Theorem 2.26. *Let $\{\omega_n\}_n$ be a system of Shannon type wavelet packets in frequency order. Then $\{\omega_n(\cdot - k)\}_{n\geq 0, k\in\mathbb{Z}}$ forms a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, in the sense that*

$$\sum_{n=0}^{N} \sum_{k\in\mathbb{Z}} \langle f, \omega_n(\cdot - k)\rangle \omega_n(\cdot - k) \xrightarrow[N\to\infty]{L^p(\mathbb{R})} f,$$

for $f \in L^p(\mathbb{R})$.

Proof. First, let us assume that ω_0 is band limited with $\mathrm{supp}(\hat{\omega}_0) \subset [-K\pi, K\pi]$, $K \in \mathbb{N}$, and that $1 < p < 2$. Define P_j by

$$P_j f(x) = \sum_{k\in\mathbb{Z}} \langle f, \omega_j(\cdot - k)\rangle \omega_j(x - k).$$

We know that the family $\{\sum_{j=0}^{2^L-1} P_j\}_{L\in\mathbb{N}}$ is uniformly bounded on $L^p(\mathbb{R})$ since it is just the projection onto the wavelet space V_L. It therefore suffices to prove that $\sum_{j=2^L}^{m} P_j$ is bounded on $L^p(\mathbb{R})$ with bound independent of $L \in \mathbb{N}$ and $m < 2^{L+1}$. Let J be the scale from which only

[1] Here $[\cdot]$ denotes the function that converts a binary string to the corresponding integer, and $[\cdot]_2$ converts an integer to its binary expansion.

the Shannon filter is used to generate the wavelet packets. Take $j \in \mathbb{N} : 2^L \le j < 2^{L+1}$, $L > J$, $G_J(j) = [\varepsilon_L \cdots \varepsilon_1]$. Then

$$\begin{aligned}\hat{\omega}_j(\xi) &= m^S_{\varepsilon_1}(\xi/2)m^S_{\varepsilon_2}(\xi/2^2)\cdots m^S_{\varepsilon_{L-J}}(\xi/2^{L-J})m^{(J)}_{\varepsilon_{L-J+1}}(\xi/2^{L-J+1})\cdots m^{(1)}_{\varepsilon_L}(\xi/2^L)\hat{\omega}_0(\xi/2^L)\\ &\equiv \left\{\sum_{s=-2^{J-1}K}^{2^{J-1}K} \chi_{I_j}(\xi - 2^{L-J+1}\pi s)\right\}\hat{\omega}_{i_j}(\xi/2^{L-J}),\end{aligned}$$

where $i_j = [1\varepsilon_{L-1}\cdots\varepsilon_{L-J+1}]_2 \in [0, 2^J - 1]$ and $2^{J-L}I_j \subset [-\pi, \pi]$ is symmetric about 0 and is a union of two intervals, each of length $2^{J-L}\pi$ (follows from Theorem 2.18). Using Plancherel's theorem we have[2]

$$\begin{aligned}\widehat{P_j f}(\xi) &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}\hat{f}(t)\sum_{s=-2^{J-1}K}^{2^{J-1}K}\chi_{I_j}(t - 2^{L-J+1}\pi s)\overline{\hat{\omega}_{i_j}(t2^{J-L})}e^{ikt}\,dt e^{-ik\xi}\hat{\omega}_j(\xi)\\
&= \frac{1}{2\pi}\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{k\in\mathbb{Z}}\int_{I_j}\hat{f}(t + 2^{L-J+1}\pi s)\overline{\hat{\omega}_{i_j}\left(\frac{t + 2^{L-J+1}\pi s}{2^{L-J}}\right)}e^{ikt}\,dt e^{-ik\xi}\hat{\omega}_j(\xi)\\
&= \sum_{s=-2^{J-1}K}^{2^{J-1}K}\left[\hat{f}(\xi + 2^{L-J+1}\pi s)\overline{\hat{\omega}_{i_j}\left(\frac{\xi + 2^{L-J+1}\pi s}{2^{L-J}}\right)}\right]\Bigg|_{I_j}^{\text{per}} \times\\
&\qquad\qquad \sum_{r=-2^{J-1}K}^{2^{J-1}K}\chi_{I_j}(\xi - 2^{L-J+1}\pi r)\hat{\omega}_{i_j}(\xi/2^{L-J})\\
&= \left\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\left[\hat{f}(\xi + 2^{L-J+1}\pi s)\overline{\hat{\omega}_{i_j}\left(\frac{\xi + 2^{L-J+1}\pi s}{2^{L-J}}\right)}\right]\Bigg|_{I_j}^{\text{per}} \times\right.\\
&\qquad\qquad \left.\chi_{I_j}(\xi - 2^{L-J+1}\pi r)\right\}\hat{\omega}_{i_j}(\xi/2^{L-J})\\
&= \left\{\sum_{s=-2^{J-1}K}^{2^{J-1}K}\sum_{r=-2^{J-1}K}^{2^{J-1}K}\hat{f}(\xi + 2^{L-J+1}\pi(s-r))\overline{\hat{\omega}_{i_j}\left(\frac{\xi + 2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)} \times\right.\\
&\qquad\qquad \left.\chi_{I_j}(\xi - 2^{L-J+1}\pi r)\right\}\hat{\omega}_{i_j}(\xi/2^{L-J})\end{aligned}$$

Note that the inverse Fourier transform of each term

$$\hat{f}(\xi + 2^{L-J+1}\pi(s-r))\overline{\hat{\omega}_{i_j}\left(\frac{\xi + 2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)}\chi_{I_j}(\xi - 2^{L-J+1}\pi r)$$

is just the convolution of f with an L^1 function of norm $\|w_{i_j}\|_1$ and then composed with the bounded operator given by the multiplier $\chi_{I_j}(\xi - 2^{L-J+1}\pi r)$. Thus, $P_j f$ is a finite sum of convolutions of $L^p(\mathbb{R})$ functions all with L^p-norm $\le C\|f\|_p$ (with C independent of j) and a

[2]Notation: $g|_I^{\text{per}}$ is the 2π-periodic function obtained by taking the 2π-periodic extension of the restriction of g to I.

function of fixed L^1-norm. So P_j is bounded on $L^p(\mathbb{R})$. In general, for $m - 2^L = k2^{L-J} + d$ with $d \in [0, 2^{L-J})$,

$$\sum_{j=2^L}^{m} \widehat{P_j f}(\xi) =$$

$$\sum_{l=0}^{k-1} \left\{ \sum_{s=-2^{J-1}K}^{2^{J-1}K} \sum_{r=-2^{J-1}K}^{2^{J-1}K} \hat{f}(\xi + 2^{L-J+1}\pi(s-r)) \overline{\hat{\omega}_l\left(\frac{\xi + 2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)} \right.$$
$$\left. \times \sum_{j=2^L+l2^{L-J}}^{2^L+(l+1)2^{L-J}-1} \chi_{I_j}(\xi - 2^{L-J+1}\pi r) \right\} \hat{\omega}_l(\xi/2^{L-J})$$

$$+ \left\{ \sum_{s=-2^{J-1}K}^{2^{J-1}K} \sum_{r=-2^{J-1}K}^{2^{J-1}K} \hat{f}(\xi + 2^{L-J+1}\pi(s-r)) \overline{\hat{\omega}_k\left(\frac{\xi + 2^{L-J+1}\pi(s-r)}{2^{L-J}}\right)} \right.$$
$$\left. \times \sum_{j=2^L+k2^{L-J}}^{2^L+k2^{L-J}+d} \chi_{I_j}(\xi - 2^{L-J+1}\pi r) \right\} \hat{\omega}_k(\xi/2^{L-J})$$

However,

$$\sum_{j=2^L+l2^{L-J}}^{2^L+(l+1)2^{L-J}-1} \chi_{I_j}(\xi - 2^{L-J+1}\pi r) \quad \text{and} \quad \sum_{j=2^L+k2^{L-J}}^{2^L+k2^{L-J}+d} \chi_{I_j}(\xi - 2^{L-J+1}\pi r)$$

are each the characteristic function of an interval (follows from Theorem 2.18 and the ordering of the functions). The same argument as above applies and $\sum_{j=2^L}^{m} P_j$ is therefore bounded on $L^p(\mathbb{R})$ with bounds independent of m and L. More generally, if ω_0 is not band limited we can always find an isometry on $L^p(\mathbb{R})$ mapping the wavelet packet system onto a band limited Shannon type wavelet packet system (by Theorem 2.10). The case $2 < p < \infty$ follows easily by a duality argument. ■

2.3 Growth in L^p-norm of Wavelet Packets

It was proved in [6] that the family $\{w_n\}_n$ of basic wavelet packets associated with a Meyer filter is not uniformly bounded in $L^p(\mathbb{R})$-norms for p large. The technique used was to prove that the family $\{\hat{w}_n\}_n$ is not bounded in L^1-norm. This works because the Meyer low-pass filter m_0 is a nonnegative functions so each $\hat{w}_n$ is just a modulation of a nonnegative function. It is therefore possible to recover the L^∞-norm of w_n from the L^1-norm of $\hat{w}_n$. However, this technique fails in general since finite filters associated with a multiresolution analysis are *not* nonnegative functions. In this section we use the following fundamental result about multiresolution analyses to calculate the $L^p(\mathbb{R})$-norm of wavelet packets associated with finite filters.

Lemma 2.27 ([29]). *Let $\{V_j\}$ be a multiresolution analysis with associated scaling function ϕ satisfying $|\phi(x)| \leq C(1+|x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Then there exist finite constants $c_p, C_p > 0$ such that for every finite sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ we have*

$$c_p\|\{c_k\}\|_{\ell^p(\mathbb{Z})} \leq \Big\| \sum_{k\in\mathbb{Z}} c_k\phi(x-k)\Big\|_p \leq C_p\|\{c_k\}\|_{\ell^p(\mathbb{Z})}.$$

The following lemma gives us a sharp estimate of the $L^p(\mathbb{R})$ norm of a wavelet packet associated with a multiresolution analysis.

Lemma 2.28. *Let (m_0, m_1) be the filters associated with a multiresolution analysis for which the scaling function ϕ satisfies $|\phi(x)| \leq C(1+|x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Then there exist finite positive constants c_p and C_p such that the $L^p(\mathbb{R})$-norm, $1 \leq p \leq \infty$, of the wavelet packet w_n, defined by*

$$\hat{w}_n(\xi) = \left[\prod_{j=1}^{N} m_{\varepsilon_j}(\xi/2^j)\right]\hat{\phi}(\xi/2^N),$$

is bounded by

$$c_p 2^N 2^{-N/p}\|\{c_k\}\|_{\ell^p(\mathbb{Z})} \leq \|w_n\|_p \leq C_p 2^N 2^{-N/p}\|\{c_k\}\|_{\ell^p(\mathbb{Z})},$$

where

$$m_{\varepsilon_N}(\xi)m_{\varepsilon_{N-1}}(2\xi)\cdots m_{\varepsilon_1}(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\xi}.$$

Proof. We have

$$\hat{w}_n(\xi) = \left[\prod_{j=1}^{N} m_{\varepsilon_j}(\xi/2^j)\right]\hat{\phi}(\xi/2^N),$$

so

$$\hat{w}_n(2^N\xi) = \left[\prod_{j=0}^{N-1} m_{\varepsilon_{N-j}}(2^j\xi)\right]\hat{\phi}(\xi). \tag{2.9}$$

Taking the inverse Fourier Transform of (2.9) shows that $2^{-N}w_n(2^{-N}x)$ is a linear combination of the functions $\{\phi(x-k)\}_k$ and that the expansion coefficients are given by the coefficients of the Fourier series

$$m_{\varepsilon_N}(\xi)m_{\varepsilon_{N-1}}(2\xi)\cdots m_{\varepsilon_1}(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\xi}.$$

Note that $\|2^{-N}w_n(2^{-N}\cdot)\|_p = 2^{-N}2^{N/p}\|w_n\|_p$ for $1 \leq p \leq \infty$. It now follows from Lemma 2.27 that there exist constants c_p and C_p (independent of n) such that

$$c_p 2^N 2^{-N/p} \|\{c_k\}\|_{\ell^p(\mathbb{Z})} \le \|w_n\|_p \le C_p 2^N 2^{-N/p} \|\{c_k\}\|_{\ell^p(\mathbb{Z})}.$$

■

In what follows, we will restrict our attention to subsequences of the form $\{w_{2^n-1}\}_n$. The main reason is that the binary expansion of $2^n - 1$ consists of $n-1$ 1's and nothing else, which simplifies the estimates given by Lemma 2.28. The key to getting good estimates is to consider the following operator.

Definition 2.29. *Let $m_1(\xi) = \sum_{k\in\mathbb{Z}} g_k e^{ik\xi}$ be a finite high-pass filter. We define the bounded operator S on $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, by*

$$(Sc)_i = \sum_{j\in\mathbb{Z}} g_{i-2j} c_j, \qquad i \in \mathbb{Z}. \tag{2.10}$$

S is called the (stationary) subdivision operator associated with the filter m_1. We let $\sigma_p[S]$ denote the spectral radius of S on $\ell^p(\mathbb{Z})$.

Note that S is just the bi-infinite matrix $(g_{i-2j})_{ij}$ considered as a bounded operator on $\ell^p(\mathbb{Z})$. It is also easy to check that S can be represented (formally) as the multiplication operator

$$Sf(\xi) = m_1(\xi) f(2\xi),$$

for $f(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{ik\xi}$.

We are interested in calculating the spectral radius of S on $\ell^p(\mathbb{Z})$. The multiplication representation of S suggests that the product

$$m_1(\xi) m_1(2\xi) \cdots m_1(2^{n-1}\xi)$$

might be useful for that purpose. Indeed, the product is the key to calculating the powers of the matrix S as the following lemma shows

Lemma 2.30. *Let $m_1(\xi) = \sum_{k=-1}^{N} g_k e^{ik\xi}$ be a finite high-pass filter, and let $S = (g_{i-2j})_{ij}$ be the associated subdivision operator. Then*

$$(S^n)_{ij} = g^n_{i-2^n j}, \tag{2.11}$$

where

$$m_1(\xi) m_1(2\xi) \cdots m_1(2^{n-1}\xi) = \sum_{k\in\mathbb{Z}} g^n_k e^{ik\xi}.$$

Proof. We prove this by induction on n. If $n = 1$ then (2.11) is trivially true. Suppose the result is true for n, $n \geq 1$. Then, by the induction hypothesis,

$$
\begin{aligned}
(S^{n+1})_{ij} &= \sum_{k \in \mathbb{Z}} (S^n)_{ik} (S)_{kj} \\
&= \sum_{k \in \mathbb{Z}} g^n_{i-2^n k} g_{k-2j} \\
&= \sum_{\ell \in \mathbb{Z}} g^n_{i-2^n \ell - 2^{n+1} j} g_\ell.
\end{aligned} \tag{2.12}
$$

On the other hand,

$$
\begin{aligned}
m_1(\xi) m_1(2\xi) \cdots m_1(2^n \xi) &= \sum_{p \in \mathbb{Z}} g^n_p e^{ip\xi} \sum_{q \in \mathbb{Z}} g_q e^{i2^n q\xi} \\
&= \sum_{p \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} g^n_p g_\ell e^{i(p+2^n \ell)\xi} \\
&= \sum_{k \in \mathbb{Z}} \Big(\sum_{\ell \in \mathbb{Z}} g^n_{k-2^n \ell} g_\ell \Big) e^{ik\xi}.
\end{aligned}
$$

Thus,

$$
g^{n+1}_{i-2^{n+1}j} = \sum_{\ell \in \mathbb{Z}} g^n_{i-2^{n+1}j-2^n \ell} g_\ell,
$$

so from (2.12) we see that (2.11) is true for $n+1$, and we are done. ■

The spectral radius $\sigma_p[S]$ of S on $\ell^p(\mathbb{Z})$ can be calculated by as follows

Theorem 2.31 ([15]). *Let m_1 be a finite high-pass filter, and let S be defined by (2.10). Define the sequence $\{g^n_k\}_k$ by*

$$
\sum_{k \in \mathbb{Z}} g^n_k e^{in\xi} = m_1(\xi) m_1(2\xi) \cdots m_1(2^{n-1}\xi).
$$

Then

$$
\sigma_p[S] = \lim_{n \to \infty} \|\{g^n_k\}_k\|^{1/n}_{\ell^p(\mathbb{Z})}.
$$

Proof. First, we claim that there is an integer K (independent of n) such that $(S^n)_{ij} = 0$ whenever $|2^{-n}i - j| > K$. To verify this, we note that there is an integer K such that $(S)_{ij} = 0$ if $|i - 2j| > K$ since m_1 is a finite filter. Suppose $(S^n)_{ij} \neq 0$. We have,

$$
(S^n)_{ij} = \sum_{\ell_1, \ldots, \ell_{n-1}} (S)_{i\ell_1} (S)_{\ell_1 \ell_2} \cdots (S)_{\ell_{n-1} j}.
$$

so there are $\ell_1, \ell_2, \ldots, \ell_{n-1}$ such that

$$
|\ell_r - 2\ell_{r+1}| \leq K, \qquad r = 0, 1, \ldots, n-1,
$$

where $\ell_0 = i$ and $\ell_n = j$. Put $\mu_r = \ell_r - 2\ell_{r+1}$ for $r = 0, 1, \ldots, n-1$. Then

$$2^{-n}i - j = 2^{-1}\mu_{n-1} + 2^{-2}\mu_{n-2} + \cdots + 2^{-n}\mu_0.$$

Hence

$$|2^{-n}i - j| \leq \sum_{q=1}^{n} 2^{-q}K \leq K.$$

Next, we let

$$\Omega_r = \{r + (2K+1)j : j \in \mathbb{Z}\},$$

for $|r| \leq K$. Note that whenever j_1 and j_2 are distinct members of Ω_r we have $|j_1 - j_2| \geq 2K+1$. We define the matrices J^r for $|r| \leq K$ by

$$(J^r)_{ij} = \begin{cases} (S^n)_{ij}, & i \in \mathbb{Z}, j \in \Omega_r \\ 0, & i \in \mathbb{Z}, j \notin \Omega_r. \end{cases}$$

We have

$$S^n = \sum_{|r| \leq K} J^r,$$

since $\{\Omega_r\}_{|r|\leq K}$ partitions $\mathbb{Z}$. Take any $(c_k)_k \in \ell^p(\mathbb{Z})$. Then

$$\|S^n c\|_{\ell^p(\mathbb{Z})} \leq (2K+1) \max\{\|J^r c\|_{\ell^p(\mathbb{Z})} : |r| \leq K\}. \tag{2.13}$$

We claim that, by construction, there is no $i \in \mathbb{Z}$ such that for distinct $j_1, j_2 \in \mathbb{Z}$ we have

$$(J^r)_{ij_1}(J^r)_{ij_2} \neq 0. \tag{2.14}$$

Indeed, suppose (2.14) holds. Then j_1 and j_2 must belong to the same Ω_r and thus $|j_1 - j_2| \geq 2K+1$. But we have already seen that $(J^r)_{ij_1}, (J^r)_{ij_2} \neq 0$ implies that

$$|2^{-n}i - j_1|, |2^{-n}i - j_2| \leq K.$$

Hence,

$$\begin{aligned} |j_1 - j_2| &\leq |2^{-n}i - j_1| + |2^{-n}i - j_2| \\ &\leq 2K, \end{aligned}$$

a contradiction. It follows that the columns of J^r, denoted by J_j^r, do not have common nonzero elements. Hence,

$$\begin{aligned} \|J^r c\|_{\ell^p(\mathbb{Z})}^p &= \sum_{j\in\mathbb{Z}} |c_j|^p \|J_j^r\|_{\ell^p(\mathbb{Z})}^p \\ &\leq \sup\{\|J_j^r\|_{\ell^p(\mathbb{Z})}^p : j \in \mathbb{Z}\} \cdot \|c\|_{\ell^p(\mathbb{Z})}^p. \end{aligned}$$

From (2.13) and Lemma 2.30 we get

$$\begin{aligned}\|S^n\|_{\ell^p\to\ell^p} &\le (2K+1)\sup\{\|J_j^r\|_{\ell^p(\mathbb{Z})} : |r|\le K, j\in\mathbb{Z}\}\\ &= (2K+1)\sup\{\|S_j^n\|_{\ell^p(\mathbb{Z})} : j\in\mathbb{Z}\}\\ &= (2K+1)\|\{g_k^n\}_k\|_{\ell^p(\mathbb{Z})}.\end{aligned}$$

Hence,

$$\sigma_p[S] \le \liminf_{n\to\infty} \|\{g_k^n\}\|_{\ell^p(\mathbb{Z})}^{1/n}.$$

To get a lower bound on $\sigma_p[S]$ note that

$$\|\{g_k^n\}_k\|_{\ell^p(\mathbb{Z})} = \|S_j^n\|_{\ell^p(\mathbb{Z})} = \|S^n(\{\delta_{j,k}\}_k)\|_{\ell^p(\mathbb{Z})} \le \|S^n\|_{\ell^p\to\ell^p},$$

from which we have

$$\limsup_{n\to\infty} \|\{g_k^n\}_k\|_{\ell^p(\mathbb{Z})}^{1/n} \le \sigma_p[S].$$

Finally, we combine the above inequalities to get

$$\sigma_p[S] = \lim_{n\to\infty} \|\{g_k^n\}_k\|_{\ell^p(\mathbb{Z})}^{1/n}.$$

■

We combine Theorem 2.31 and Lemma 2.28 to obtain the following useful result about the asymptotic behavior in $L^p(\mathbb{R})$ of the subsequence $\{w_{2^n-1}\}_n$ of a given wavelet packet system $\{w_n\}$.

Corollary 2.32. *Let $\{w_n\}_{n=0}^{\infty}$ be the wavelet packets generated by the finite filters (m_0, m_1) associated with a multiresolution analysis. Define $\tilde{\sigma}_p$, $1\le p\le\infty$, by*

$$\tilde{\sigma}_p = \lim_{n\to\infty} \|w_{2^n-1}\|_p^{1/n}.$$

Then $\tilde{\sigma}_p$ exists and $\tilde{\sigma}_p = 2^{1-1/p}\sigma_p[S]$.

Proof. We have, using the same notation as in Lemma 2.28,

$$c_p 2^n 2^{-n/p}\|\{c_k^n\}\|_{\ell^p(\mathbb{Z})} \le \|w_{2^n-1}\|_p \le C_p 2^n 2^{-n/p}\|\{c_k^n\}\|_{\ell^p(\mathbb{Z})}.$$

The result follows from taking n'th roots of the above inequalities and letting $n\to\infty$. ■

Finding the asymptotic behavior of the subsequence $\{w_{2^n-1}\}_n$ in $L^p(\mathbb{R})$ thus reduces to calculating the spectral radius $\sigma_p[S]$. Unfortunately, there is no general method available to calculate $\sigma_p[S]$. However, the following lemma shows that we only have worry about $\sigma_\infty[S]$ to estimate $\sigma_p[S]$ for p large. Note that the lemma is a Bernstein type inequality.

Lemma 2.33. *Let $\{w_n\}$ be a wavelet packet system associated with a multiresolution analysis $\{V_j\}$. Let $n > 0$, $2^{j-1} \le n < 2^j$. Then there is a finite constant C_p, independent of j, such that for $p \in [1, \infty]$*

$$\|w_n\|_\infty \le C_p 2^{j/p} \|w_n\|_p.$$

Proof. We have $w_n \in V_j$ so

$$w_n(x) = \sum_{k \in \mathbb{Z}} c_k \phi_{j,k},$$

for some finite sequence $\{c_k\}$. Then, using Lemma 2.27,

$$\begin{aligned}
\|w_n\|_\infty &\le C_\infty 2^{j/2} \|\{c_k\}\|_{\ell^\infty(\mathbb{Z})} \\
&\le C_\infty 2^{j/2} \|\{c_k\}\|_{\ell^p(\mathbb{Z})} \\
&= C_\infty 2^{j/p} [2^{j/2 - j/p} \|\{c_k\}\|_{\ell^p(\mathbb{Z})}] \\
&\le C_p 2^{j/p} \|w_n\|_p.
\end{aligned}$$

■

And we have

Corollary 2.34. *Let $\{w_n\}$ be a wavelet packet system associated a multiresolution analysis. Then*

$$\tilde{\sigma}_p \ge 2^{-1/p} \tilde{\sigma}_\infty.$$

2.3.1 Lower Bounds for σ_∞.

We are left with the following problem; how do we obtain a lower bound for $\sigma_\infty[S]$? It turns out that the calculation of $\sigma_\infty[S]$ can be reduced to a finite dimensional problem. We need the following definition and theorem

Definition 2.35. *Let A_0 and A_1 be two $n \times n$-matrices. The joint spectral radius of A_0 and A_1 is given by*

$$\rho_\infty(A_0, A_1) = \limsup_{r \to \infty} \max_{\varepsilon \in \{0,1\}^r} \|A_{\varepsilon_1} A_{\varepsilon_2} \cdots A_{\varepsilon_r}\|^{1/r},$$

where $\|\cdot\|$ is any (matrix) norm on $\mathbb{R}^{n \times n}$.

We have

Theorem 2.36 ([15]). *Let $m_1(\xi) = \sum_{n=-1}^{N} g_n e^{in\xi}$ be a high-pass filter associated with a multiresolution analysis. Form the two matrices*

$$A_0 = (g_{-i+2j})_{i,j=-1}^{N-1}, \qquad A_1 = (g_{1-i+2j})_{i,j=-1}^{N-1}.$$

Then

$$\sigma_\infty[S] = \rho_\infty(A_0, A_1).$$

Proof. Set $\Omega = \{-1, 0, \dots, N-1\}$. Note that if $i \in \Omega$ and $j \in \mathbb{Z}\backslash\Omega$ then $g_{\varepsilon - i + 2j} = 0$ for $\varepsilon = 0, 1$. Fix $i \in \mathbb{Z}$, and write $i = 2i_1 + \varepsilon$ for $\varepsilon \in \{0, 1\}$. Let $\ell \in \Omega$. Then

$$\begin{aligned}(Sc)_{i-\ell} &= \sum_{j\in\mathbb{Z}} g_{i-\ell-2j} c_j \\ &= \sum_{k\in\mathbb{Z}} g_{\varepsilon-\ell+2k} c_{i_1-k} \\ &= \sum_{k\in\Omega} g_{\varepsilon-\ell+2k} c_{i_1-k} \\ &= \sum_{k\in\Omega} (A_\varepsilon)_{\ell k} c_{i_1-k}.\end{aligned} \tag{2.15}$$

Let $J_k : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\Omega)$ be defined by

$$(J_k c)_\ell = c_{k-\ell}, \qquad \ell \in \Omega.$$

It follows from (2.15) that $J_i S = A_\varepsilon J_{i_1}$. Thus, if $i = 2i_1 + \varepsilon_r$, $i_1 = 2i_2 + \varepsilon_{r-1}, \dots, i_{r-1} = 2i_r + \varepsilon_1$ then

$$J_i S^r = A_{\varepsilon_r} \cdots A_{\varepsilon_1} J_{i_r}.$$

Now, we can prove the claim. Given $\eta > 0$, we let R be a positive constant such that for any $\varepsilon \in \{0,1\}^r$ and $r \geq R$, we have $\|A_{\varepsilon_r} \cdots A_{\varepsilon_1}\| \leq (\rho_\infty(A_0, A_1) + \eta)^r$. Fix $i \in \mathbb{Z}$, and let $r \geq R$. We write i on the form

$$i = \varepsilon_r + 2\varepsilon_{r-1} + \cdots + 2^{r-1}\varepsilon_1 + 2^r i_r, \qquad \varepsilon_k \in \{0,1\},\ i_r \in \mathbb{Z}.$$

Hence for $c \in \ell^\infty(\mathbb{Z})$ and $\ell \in \Omega$, we have

$$|(S^r c)_{i-\ell}| \leq \|J_i S^r c\|_{\ell^\infty(\Omega)} = \|A_{\varepsilon_r} \cdots A_{\varepsilon_1} \mu\|_{\ell^\infty(\Omega)},$$

where $\mu = J_{i_r} c$. Since this holds for all $i \in \mathbb{Z}$, we have

$$\|S^r c\|_{\ell^\infty(\mathbb{Z})} \leq (\rho_\infty(A_0, A_1) + \eta)^r \|c\|_{\ell^\infty(\mathbb{Z})}.$$

It follows that $\sigma_\infty[S] \leq \rho_\infty(A_0, A_1)$.

For the other direction, let $\eta > 0$ and choose R so that $\|S^r c\|_{\ell^\infty(\mathbb{Z})} \leq (\sigma_\infty[S] + \eta)^r \|c\|_{\ell^\infty(\mathbb{Z})}$ holds for every $r \geq R$ and $c \in \ell^\infty(\mathbb{Z})$. Take any $\mu \in \ell^\infty(\Omega)$ and write it as $\mu = J_0 \lambda$ for some $\lambda \in \ell^\infty(\mathbb{Z})$ with $\|\mu\|_{\ell^\infty(\Omega)} = \|\lambda\|_{\ell^\infty(\mathbb{Z})}$. For $\varepsilon \in \{0,1\}^r$ define i by

$$i = \varepsilon_r + 2\varepsilon_{r-1} + \cdots + 2^{r-1}\varepsilon_1 + 2^r \cdot 0.$$

Then

$$\|A_{\varepsilon_r}\cdots A_{\varepsilon_1}\mu\|_{\ell^\infty(\Omega)} = \|J_i S^r \lambda\|_{\ell^\infty(\Omega)} \le (\sigma_\infty[S]+\eta)^r \|\lambda\|_{\ell^\infty(\mathbb{Z})}$$

and it follows that $\rho_\infty(A_0, A_1) \le \sigma_\infty[S]$. ■

It is, in general, difficult to calculate the joint spectral radius of the matrices A_0, A_1 introduced in Theorem 2.36. G. Grippenberg gives an example in [16] where one has to perform a very significant number of matrix multiplications to get decent two-sided estimates of the joint spectral radius even for 2×2-matrices. However, we just want a lower bound for σ_∞ so for our purpose it suffices to notice that $\rho_\infty(A_0, A_1) \ge \max\{\rho(A_0), \rho(A_1)\}$. Hence, the spectral radius of the matrix A_0 gives us a lower bound on σ_∞, i.e., we have reduced the problem to a finite dimensional eigenvalue problem that can be solved (numerically, at least) for any finite filter.

2.3.2 Growth in L^p-norm of Some Familiar Wavelet Packets

We now apply this method to some much used filters. We have calculated lower bounds for $\tilde{\sigma}_\infty$ for some of the standard Daubechies filters, least asymmetric Daubechies filters, and Coiflet filters (see Tables 2.1, 2.2, and 2.3). It is interesting to note the difference in the estimates obtained for the Daubechies filters and the least asymmetric Daubechies filters of the same length since their transfer functions agree in absolute value. It suggests that the phase of the transfer function does influence the behavior of the associated wavelet packets in $L^p(\mathbb{R})$.

The following result generalizes the results obtained in [6] for the Meyer wavelets.

Theorem 2.37. *For each wavelet packet system associated with one of the filters listed in Tables 2.1, 2.2, and 2.3 there is a $p_0 > 2$ such that for $p \ge p_0$ we have a constant $r_p > 1$ such that $\|w_{2^n-1}\|_p \ge C_p r_p^n$.*

Figure 2.2 shows the first few elements of the sequence $\{w_{2^n-1}\}$ associated with the Daubechies filter of length 4.

Daub$_N$	Lower bounds for			
	$\tilde{\sigma}_1$	$\tilde{\sigma}_\infty$	$\tilde{\sigma}_1\tilde{\sigma}_\infty$	p_0
2	0.918558	$\frac{\sqrt{11}+\sqrt{3}}{4}$	1.159376	4.687617
3	0.946828	1.182094	1.119240	6.153068
4	0.964076	1.128085	1.087560	8.257957
5	0.975229	1.178557	1.149363	4.979198
6	0.982686	1.120631	1.101229	7.188270
7	0.987780	1.088578	1.075275	9.550474
8	0.991312	1.120338	1.110605	6.607374
9	0.993788	1.081554	1.074836	9.604556
10	0.995538	1.050467	1.045780	15.48460
11	0.996783	1.077456	1.073990	9.710528
12	0.997673	1.053657	1.051206	13.87991
13	0.998313	1.023405	1.021679	32.31807
14	0.998774	1.047230	1.045946	15.42983
15	0.999107	1.034474	1.033551	21.00407
16	0.999349	1.007608	1.006952	100.0505
17	0.999524	1.027401	1.026913	26.10002
18	0.999652	1.021871	1.021515	32.56199
19	0.999745	1.001009	1.000754	919.3268
20	0.999813	1.015251	1.015061	46.36799

Table 2.1: Lower bounds for $\tilde{\sigma}_1$, $\tilde{\sigma}_\infty$, and p_0 for the first 20 Daubechies filters (with filterlength from 4 to 40). The estimates have been calculated using Maple V.3 and checked using the Power Method and Matlab.

Least Asym. Daub_N	Lower bounds for			
	$\tilde{\sigma}_1$	$\tilde{\sigma}_\infty$	$\tilde{\sigma}_1\tilde{\sigma}_\infty$	p_0
4	0.964076	1.192708	1.149862	4.963745
5	0.975229	1.087374	1.060439	11.81179
6	0.982686	1.146192	1.126374	5.825744
7	0.987780	1.133295	1.119446	6.143067
8	0.991312	1.111158	1.101505	7.169679
9	0.993788	1.047619	1.041111	17.20426
10	0.995538	1.084002	1.079118	9.095479

Table 2.2: Lower bounds for $\tilde{\sigma}_1$, $\tilde{\sigma}_\infty$, and p_0 for the least asymmetric Daubechies filters with filterlength 8 to 20. The estimates have been calculated using Maple V.3 and checked using the Power Method and Matlab.

Coiflet_N	Lower bounds for			
	$\tilde{\sigma}_1$	$\tilde{\sigma}_\infty$	$\tilde{\sigma}_1\tilde{\sigma}_\infty$	p_0
3	0.939727	1.075437	1.010617	65.63136
6	0.967122	1.197928	1.158542	4.710071
9	0.984923	1.151143	1.133787	5.520289
12	0.992775	1.114805	1.106750	6.833865
15	0.996445	1.086199	1.082338	8.760274

Table 2.3: Lower bounds for $\tilde{\sigma}_1$, $\tilde{\sigma}_\infty$, and p_0 for the "Coiflet" filters with filterlength 6 to 30. The estimates have been calculated using Maple V.3 and checked using the Power Method and Matlab.

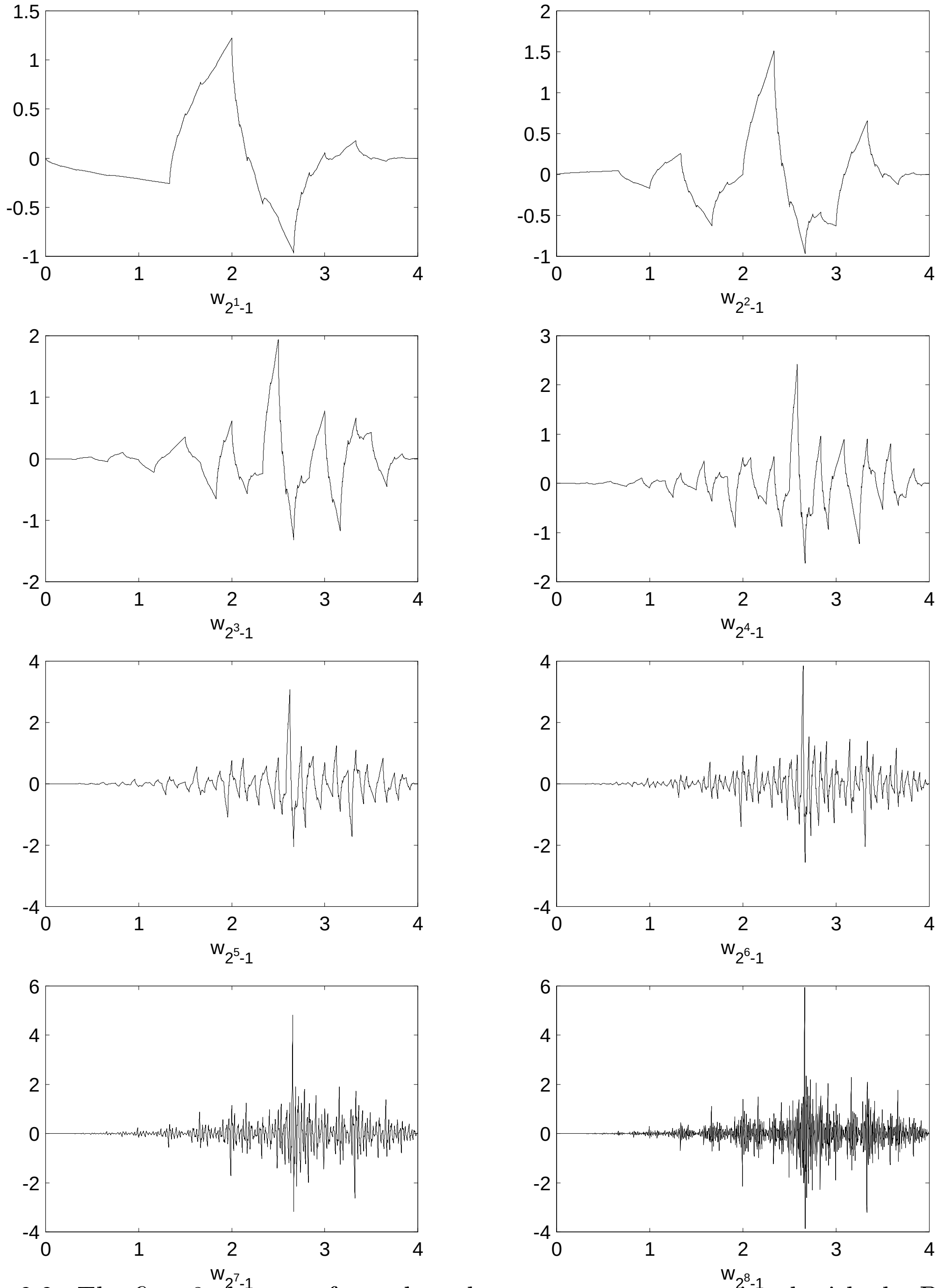

Figure 2.2: The first 8 elements from the subsequence w_{2^n-1} associated with the Daubechies filter of length 4. Note the growth in L^∞-norm.

We would like to know if the previous theorem is sharp in the sense that there is a p, $2 < p < p_0$, such that $\sup_n \|w_{2^n-1}\|_p < \infty$. The answer is, in general, negative as the following result shows.

Theorem 2.38. *Let m_0 be the Daubechies filter of length 4 and let $\{w_n\}$ be the associated wavelet packets. Then*

$$\|w_{2^n-1}\|_p \xrightarrow{n\to\infty} \infty$$

for every $p > 2$.

Proof. If we can prove that $\|w_{2^n-1}\|_1 \xrightarrow{n\to\infty} 0$ then the result will follow by Hölder's inequality since $\|w_{2^n-1}\|_2 = 1$. It suffices to show that $\sigma_1[S] < 1$. Note that if we can find an N such that $\sum_k |c_k^N| = \alpha < 1$, where

$$m_1(\xi)\cdots m_1(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k^N e^{ik\xi},$$

then $\sigma_1[S] \leq \alpha^{1/N} < 1$. But one can check that

$$\sum_{k\in\mathbb{Z}} |c_k^7| = \frac{9517 + 13043\sqrt{3}}{32768} < 0.98.$$

■

2.3.3 Wavelet Packets Generated By Nonnegative Low-Pass Filters

We have included the following result for the sake of completeness. Note that Theorem 2.40 tells us that the only way to avoid growth in L^∞-norm of wavelet packets associated with nonnegative filters is to choose the filter to be the characteristic function of some measurable set.

Theorem 2.39 ([21]). *Let $\{w_n\}$ be the wavelet packet basis associated with the filters (m_0, m_1). Suppose*

$$|\hat{w}_1(\xi)| \geq \delta_0 > 0 \text{ on } [a-\varepsilon, a+\varepsilon]$$
$$0 < |m_0(\xi)| < 1 \text{ on } \mathcal{M} \subset [-\pi, \pi]$$

holds for some $\varepsilon > 0$, point $a \in \mathbb{R}$, and set $\mathcal{M}$ of positive measure. Then there exist constants $C > 0$ and $r > 1$ such that

$$2^{-J} \sum_{2^J \leq n < 2^{J+1}} \|\hat{w}_n\|_1 \geq C r^J.$$

Theorem 2.40. *Let $\{w_n\}$ be the wavelet packet basis associated with the filters (m_0, m_1). Suppose the system satisfies the hypothesis of Theorem 2.39 and m_0 is a nonnegative function then there exist constants $C > 0$ and $r > 1$ such that*

$$\|w_{n_j}\|_\infty \geq Cr^j$$

for some subsequence $\{n_j\} \subset \mathbb{N}$.

Proof. Let C and r be the constants from Theorem 2.39. We pick from each set $\{2^j, 2^j + 1, \ldots, 2^{j+1} - 1\}$ an index n_j such that $\|\hat{w}_{n_j}\|_1 \geq Cr^j$ (possible since that average of the $L^1(\mathbb{R})$-norms grows like Cr^j within scale j). Note that $\hat{w}_{n_j}$ is just the modulation of a nonnegative function. Hence, $\|w_{n_j}\|_\infty = (2\pi)^{-1}\|\hat{w}_{n_j}\|_1$ and we are done. ■

2.4 Failure of Some Wavelet Packet Systems to be a Basis for $L^p(\mathbb{R})$

We have proved that the Walsh and Shannon wavelet packets do constitute Schauder bases for $L^p(\mathbb{R})$, $1 < p < \infty$, so one might conjecture that such results hold for any reasonable wavelet packet system. However, it turns out that the assertion is not true for many nice finite filters such as the Daubechies, least asymmetric Daubechies, and Coiflet filters. They all fail because of the following result:

Lemma 2.41. *If $\{w_n(x-k)\}_{k,n}$ is a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, then there exists a finite constant C_p such that*

$$\|w_n\|_p \|w_n\|_{p'} \leq C_p, \qquad n = 0, 1, \ldots. \tag{2.16}$$

Proof. It is a well known result that a Schauder basis $\{e_n\}$ in a Banach space $\mathcal{B}$ with associated coefficient functionals $\{f_n\}$ satisfies

$$\sup_n \|e_n\|_{\mathcal{B}} \|f_n\|_{\mathcal{B}^*} < +\infty.$$

So it suffices to show that $w_n \in L^{p'}(\mathbb{R})$ is the coefficient functional of $w_n \in L^p(\mathbb{R})$. Also, since $\{w_n(x-k)\}_{n,k}$ is an orthonormal system in $L^2(\mathbb{R})$, we just have to verify that $\{w_n(x-k)\}_{n,k}$ is dense in $L^p(\mathbb{R})$. Now, the wavelet system $\{\psi_{j,k}\}$ is dense in $L^p(\mathbb{R})$, but each $\psi_{j,k}$, $j \geq 0$, is a finite linear combination of the functions $\{w_n(x-k)\}_{n,k}$ since the filters m_0 and m_1 are both finite, and we are done. ■

The idea is to find a subsequence of a given wavelet packet system for which (2.41) fails. We have the following useful result.

Corollary 2.42. *If*

$$\tilde{\sigma}_1[S]\tilde{\sigma}_\infty[S] = \alpha > 1,$$

then the associated wavelet packet system $\{w_n(\cdot - k)\}_{n,k}$ *fails to be a Schauder basis for* $L^p(\mathbb{R})$ *for* $p > p_0$*, where* $p_0 = 1/\log_2(\alpha)$.

Proof. Since the functions $\{w_n\}$ all have support contained in some fixed finite interval, we have $\|w_n\|_1 \le C_p\|w_n\|_p$. Thus, for $p > 2$,

$$\begin{aligned}\|w_{2^n-1}\|_{p'}\|w_{2^n-1}\|_p &\ge C_p\|w_{2^n-1}\|_1\|w_{2^n-1}\|_p \\ &\ge \tilde{C}_p 2^{-n/p}\|w_{2^n-1}\|_1\|w_{2^n-1}\|_\infty,\end{aligned}$$

where we have used Lemma 2.33. Note that

$$2^{-n/p}\|w_{2^n-1}\|_1\|w_{2^n-1}\|_\infty \xrightarrow{n\to\infty} \infty$$

for $p > p_0$, so Lemma 2.41 shows that $\{w_n(x-k)\}_{n,k}$ fails to be a Schauder basis for such $L^p(\mathbb{R})$. ■

We already have estimates of $\sigma_\infty[S]$. The following result takes care of $\sigma_1[S]$,

Lemma 2.43. *Let* $m_1(\xi)$ *be a finite high-pass filter with real coefficients associated with a multiresolution analysis. Then*

$$\sigma_1[S] \ge |m_1(2\pi/3)|.$$

Proof. Note that the set $\{-\frac{2\pi}{3}, \frac{2\pi}{3}\}$ is invariant under the transformation $\xi \to 2\xi \pmod{2\pi}$. Also, $|m_1(\frac{2\pi}{3})| = |m_1(-\frac{2\pi}{3})|$ since m_1 has real coefficients. Thus,

$$|m_1(\tfrac{2\pi}{3})\cdots m_1(2^{n-1}\tfrac{2\pi}{3})| = |m_1(\tfrac{2\pi}{3})|^n.$$

Let

$$\sum_{k\in\mathbb{Z}} c_k^n e^{ik\xi} = m_1(\xi)m_1(2\xi)\cdots m_1(2^{n-1}\,\xi).$$

Then

$$|m_1(\tfrac{2\pi}{3})|^n \le \|m_1(\xi)\cdots m_1(2^{n-1}\xi)\|_{L^\infty[0,2\pi)} \le \sum_{k\in\mathbb{Z}}|c_k^n|,$$

and the results follows from Lemma 2.31. ■

Example 2.44. *Let us apply Lemma 2.43 to the Daubechies filter of length* $2N$. *We have*

$$
\begin{aligned}
|m_1^N(\tfrac{2\pi}{3})|^2 &= 1 - |m_0(\tfrac{2\pi}{3})|^2 \\
&= 1 - \cos^{2N}(\tfrac{2\pi}{3}) \sum_{j=0}^{N-1} \binom{N-1+j}{j} \sin^{2j}(\tfrac{2\pi}{3}).
\end{aligned}
$$

Thus,

$$
\sqrt{1 - 2^{-2N} \sum_{j=0}^{N-1} \binom{N-1+j}{j} (\tfrac{\sqrt{3}}{2})^{2j}} \leq \sigma_1[S] \leq 1.
$$

Note that $\sigma_1 \leq 1$ *since the subsequence* $\|w_{2^n-1}\|_1$ *is bounded because the wavelet packets associated with* m_1 *are compactly supported with support contained in some fixed interval independent of* n. *Moreover,*

$$
\begin{aligned}
\sum_{j=0}^{N-1} \binom{N-1+j}{j} (\tfrac{\sqrt{3}}{2})^{2j} &\leq \left(\tfrac{3}{2}\right)^{N-1} \sum_{j=0}^{N-1} \binom{N-1+j}{j} 2^{-j} \\
&= \left(\tfrac{3}{2}\right)^{N-1} 2^{N-1} \\
&= 3^{N-1}.
\end{aligned}
$$

Hence,

$$
\sigma_1 \geq \sqrt{1 - 2^{-2N} \sum_{j=0}^{N-1} \binom{N-1+j}{j} (\tfrac{\sqrt{3}}{2})^{2j}} \geq \sqrt{1 - 4^{-N} 3^{N-1}} \longrightarrow 1
$$

as $N \to \infty$. ■

We have the following unfortunate result about the basic wavelet packets associated with one of the filters listed in Tables 2.1, 2.2, and 2.3.

Theorem 2.45. *For each wavelet packet system* $\{w_n\}$ *associated with one of the filters listed in Tables 2.1, 2.2, and 2.3 there exists a (finite)* $p_0 > 2$ *such that for* $p > p_0$, *the system* $\{w_n(\cdot - k)\}_{n,k}$ *(in any ordering) fails to be a Schauder basis for* $L^p(\mathbb{R})$.

2.4.1 Some Wavelet Packets Generated Using Infinite Filters

Suppose m_0 is a finite low-pass filter for some multiresolution analysis. Then it is easy to check that $|m_0|$ is also a low-pass filter for some multiresolution analysis (see Cohen's condition; Theorem 1.3), and one would expect that the wavelet packets associated with $|m_0|$ are "worse" than those associated with m_0, since $|m_0|$ is not a finite filter. Indeed, we have the following result.

Theorem 2.46. *Let m_0 be a finite low-pass filter with associated wavelet packet system $\{w_n\}$. Let $m_0^{abs} = |m_0|$ with associated wavelet packet system $\{w_n^{abs}\}$. If $\{w_n\}$ fails to be a Schauder basis because*

$$\|\hat{w}_{n_k}\|_\infty \|w_{n_k}\|_\infty \geq Cr^k$$

for some subsequence $\{w_{n_k}\}$ and $r > 1$ then $\{w_n^{abs}\}$ also fails to be a Schauder basis for some $L^p(\mathbb{R})$ spaces since

$$\|w_{n_k}^{abs}\|_p \|w_{n_k}^{abs}\|_{p'} \longrightarrow \infty,$$

for p large.

Proof. We have $\|w_{n_k}\|_\infty \leq \frac{1}{2\pi}\|\hat{w}_{n_k}\|_1$, and

$$\|\hat{w}_{n_k}\|_1 = \|\widehat{w_{n_k}^{\text{abs}}}\|_1 = 2\pi\|w_{n_k}^{\text{abs}}\|_\infty$$

since $\widehat{w_{n_k}^{\text{abs}}}$ is just a modulation of a nonnegative function. By Bernstein's inequality, applied to $\hat{w}_{n_k}$, we have

$$\|\widehat{w_{n_k}^{\text{abs}}}\|_p = \|\hat{w}_{n_k}\|_p \geq C\|\hat{w}_{n_k}\|_\infty,$$

since the Fourier transform of $\hat{w}_{n_k}$ has support contained in some fixed interval given by the filter length of m_0. Moreover, by the Hausdorff-Young inequality, for $p \in [1,2)$,

$$\|\widehat{w_{n_k}^{\text{abs}}}\|_{p'} \leq C\|w_{n_k}^{\text{abs}}\|_p.$$

Hence, $\|w_{n_k}^{\text{abs}}\|_p \geq C\|\hat{w}_{n_k}\|_\infty$ for $p \in [1,2)$ and $\|w_{n_k}^{\text{abs}}\|_\infty \geq \|w_{n_k}\|_\infty$. The result then follows from Lemma 2.41. ■

We now apply the above result the filters listed in Tables 2.1, 2.2, and 2.3.

Corollary 2.47. *Theorem 2.46 applies to the filters listed in Tables 2.1, 2.2, and 2.3.*

Proof. We know that the subsequence $\{w_{2^n-1}\}$ grows exponentially in $L^p(\mathbb{R})$-norm for p large. Furthermore, with ϕ the associated scaling function,

$$\|\hat{w}_{2^n-1}\|_\infty \geq |\hat{w}_{2^n-1}(2^n \tfrac{2\pi}{3})| = |m_1(\tfrac{2\pi}{3})|^n |\hat{\phi}(\tfrac{2\pi}{3})| \geq C|m_1(\tfrac{2\pi}{3})|^n,$$

since $|\hat{\phi}(\xi)| \geq C$ on $[-\pi, \pi]$. The result then follows from the values listed in Tables 2.1, 2.2, and 2.3. ■

2.4.2 Fourier Transforms of Wavelet Packets

The definition of the Shannon wavelet packets shows that the Fourier transforms of these functions are well behaved in $L^p(\mathbb{R})$, $1 < p < \infty$. In fact, we have the following easy theorem.

Theorem 2.48. *The Fourier transforms of the Shannon wavelet packets in frequency order form a Schauder basis for* $L^p(\mathbb{R})$, $1 < p < \infty$.

Proof. We have $\mathcal{F}[\omega_n(\cdot - k)](\xi) = \chi_{[n\pi,(n+1)\pi)}(|\xi|)e^{-ik\xi}$ so the statement follows from the fact that $\{e^{2\pi ik\xi}\}_{k\in\mathbb{Z}}$ form a Schauder basis for $L^p[0,1)$, $1 < p < \infty$. ∎

So, one might conjecture that the Fourier transforms of any reasonable wavelet packet system are equally well behaved. However, we have the following Corollary to Lemma 2.41.

Corollary 2.49. *Let* $\{w_n\}$ *be a wavelet packet system associated with a finite filter* m_0. *If, for some* $p \in (2, \infty)$

$$\|\hat{w}_{n_k}\|_\infty \|w_{n_k}\|_p \longrightarrow \infty$$

then the Fourier transforms of the wavelet packets fail to be a Schauder basis for $L^p(\mathbb{R})$.

Proof. By the Hausdorff Young inequality,

$$\|w_{n_k}\|_p \leq C\|\hat{w}_{n_k}\|_{p'},$$

and Bernstein's inequality shows that $\|\hat{w}_{n_k}\|_p \geq C\|\hat{w}_{n_k}\|_\infty$ so

$$\|\hat{w}_{n_k}\|_p\|\hat{w}_{n_k}\|_{p'} \geq C\|w_{n_k}\|_p\|\hat{w}_{n_k}\|_\infty,$$

and the result follows. ∎

Corollary 2.50. *Theorem 2.49 applies to the wavelet packets associated with one of the filters listed in Tables 2.1, 2.2, and 2.3.*

Proof. We know that the subsequence $\{w_{2^n-1}\}$ grows exponentially in $L^p(\mathbb{R})$-norm for p large. Furthermore, with ϕ the associated scaling function,

$$\|\hat{w}_{2^n-1}\|_\infty \geq |\hat{w}_{2^n-1}(2^n \tfrac{2\pi}{3})| = |m_1(\tfrac{2\pi}{3})|^n|\hat{\phi}(\tfrac{2\pi}{3})| \geq C|m_1(\tfrac{2\pi}{3})|^n,$$

since $|\hat{\phi}(\xi)| \geq C$ on $[-\pi, \pi]$. The result then follows from the values listed in Tables 2.1, 2.2, and 2.3. ∎

2.5 A Formula for σ_p for p Even

In this section we generalize the results expressing σ_∞ as a joint spectral radius of two finite dimensional matrices. First we need the following generalization of the joint spectral radius (introduced by Jia in [23]).

Definition 2.51. *Let A_0 and A_1 be $n \times n$ matrices. We define the p-norm joint spectral radius for $0 < p < \infty$ by*

$$\rho_p(A_0, A_1) = \limsup_{n\to\infty} \left(\sum_{\varepsilon\in\{0,1\}^n} \|A_{\varepsilon_1} A_{\varepsilon_2} \cdots A_{\varepsilon_n}\|^p \right)^{1/pn}.$$

In what follows we let δ_0 denote the sequence $\{\delta_{0,k}\}_{k\in\mathbb{Z}}$, and let S be the subdivision operator introduced in Definition 2.29. We define the operators A_0 and A_1 by their matrix representations

$$(A_\varepsilon)_{ij} = g_{\varepsilon+2i-j},$$

where $m_1(\xi) = \sum_{k=-1}^{N} g_k e^{ik\xi}$ is a finite high-pass filter. Note that A_0, A_1 are essentially the transpose of the matrices appearing in Theorem 2.36. The reason is that the new A_0 and A_1 have nice invariant subspaces of finite dimension, which was first observed by L. F. Villemoes in [42]. The following lemma is well known and can be found in, e.g., [24].

Lemma 2.52. *For $i = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n k$ we have*

$$(S^n\delta_0)_{i-j} = (A_{\varepsilon_n} A_{\varepsilon_{n-1}} \cdots A_{\varepsilon_1})_{kj}.$$

Proof. We use induction on n. For $n = 1$ and $i = \varepsilon_1 + 2k$ we have

$$(S\delta_0)_{i-j} = \sum_{\ell\in\mathbb{Z}} g_{i-j-2\ell}\delta_{0,l} = g_{\varepsilon_1+2k-j} = (A_{\varepsilon_1})_{kj}.$$

Suppose the claim is true for $n-1$. For $i = \varepsilon_1 + 2k_1$ with $k_1 = \varepsilon_2 + \cdots \varepsilon_n 2^n + 2^{n-1}k$ we have

$$\begin{aligned}
(S^n\delta_0)_{i-j} &= \sum_{\ell\in\mathbb{Z}} g_{i-j-2\ell}(S^{n-1}\delta_0)_\ell \\
&= \sum_{\ell\in\mathbb{Z}} g_{\varepsilon_1+2(k_1-\ell)-j}(S^{n-1}\delta_0)_\ell \\
&= \sum_{\ell\in\mathbb{Z}} g_{\varepsilon_1+2\ell-j}(S^{n-1}\delta_0)_{k_1-\ell} \\
&= \sum_{\ell\in\mathbb{Z}} g_{\varepsilon_1+2\ell-j}(A_{\varepsilon_n} \cdots A_{\varepsilon_2})_{k_1\ell} \\
&= \sum_{\ell\in\mathbb{Z}} (A_{\varepsilon_1})_{\ell j}(A_{\varepsilon_n} \cdots A_{\varepsilon_2})_{k_1\ell} \\
&= (A_{\varepsilon_n} A_{\varepsilon_{n-1}} \cdots A_{\varepsilon_1})_{kj}.
\end{aligned}$$

■

It is easy to verify that $\ell^p([-1, N-1])$ is invariant under both A_0 and A_1 so it follows that V_0, the minimal invariant subspace under A_0 and A_1 containing δ_0, is finite dimensional. We let $\mathcal{P} = \{A_0, A_1\}$ and define

$$\mathcal{P}^n \delta_0 = \sum_{\varepsilon \in \{0,1\}^n} A_{\varepsilon_1} \cdots A_{\varepsilon_n} \delta_0,$$

and

$$\|\mathcal{P}^n \delta_0\|_p = \left(\sum_{\varepsilon \in \{0,1\}^n} \|A_{\varepsilon_1} \cdots A_{\varepsilon_n} \delta_0\|^p \right)^{1/p}.$$

The following lemma gives us what we want, an expression for σ_p in terms of two finite dimensional matrices. The first equality in the lemma is a new observation, the second equality was proved in [24].

Lemma 2.53.

$$\sigma_p[S] = \rho_p(A_0|_{V_0}, A_1|_{V_0}) = \lim_{n\to\infty} \|\mathcal{P}^n \delta_0\|_p^{1/n}.$$

Proof. For $i = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n k$ we have

$$(S^n \delta_0)_i = (A_{\varepsilon_n} \cdots A_{\varepsilon_1})_{k0} = (A_{\varepsilon_n} \cdots A_{\varepsilon_1} \delta_0)_k,$$

so

$$\begin{aligned} \sum_{i\in\mathbb{Z}} |(S^n \delta_0)_i|^p &= \sum_{\varepsilon \in \{0,1\}^n} \sum_{k\in\mathbb{Z}} |(A_{\varepsilon_n} \cdots A_{\varepsilon_1} \delta_0)_k|^p \\ &= \|\mathcal{P}^n \delta_0\|_p^p. \end{aligned}$$

Hence,

$$\sigma_p[S] = \lim_{n\to\infty} \|\mathcal{P}^n \delta_0\|_p^{1/n}.$$

To get the second equality, we note that V_0 is finite dimensional so it has a basis of the form

$$\mathcal{B} = \{A_{\varepsilon_1} \cdots A_{\varepsilon_j} \delta_0, j = 0, \ldots, d\}.$$

It follows that there is a constant C such that

$$\|\mathcal{P}^n b\|_p \le C \|\mathcal{P}^n \delta_0\|_p, \quad \text{for } b \in \mathcal{B},$$

and for which

$$\|\mathcal{P}^n|_{V_0}\|_p \le C \max_{b\in\mathcal{B}} \|\mathcal{P}^n b\|_p,$$

so

$$\|\mathcal{P}^n|_{V_0}\|_p \leq C\|\mathcal{P}^n\delta_0\|_p.$$

Clearly,

$$\|\mathcal{P}^n\delta_0\|_p \leq \|\mathcal{P}^n|_{V_0}\|_p\|\delta_0\|_p.$$

Thus,

$$\rho_p(A_0|_{V_0}, A_1|_{V_0}) = \lim_{n\to\infty} \|\mathcal{P}^n|_{V_0}\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{P}^n\delta_0\|_p^{1/n}.$$

■

We have the following amazing result by D.-X. Zhou ([50][3]) that provides a finite dimensional method to calculate the spectral radius of our subdivision operator on $\ell^{2k}(\mathbb{Z})$, $k = 1, 2, \ldots$

Theorem 2.54 ([50]). *For $k \in \mathbb{N}$ and $\mathcal{P} = \{M_1, M_2, \cdots, M_d\}$ a finite collection of real valued $n \times n$-matrices, we have*

$$\rho_{2k}(\mathcal{P}) = \rho\Big(\sum_{M\in\mathcal{P}} (M \otimes M)^{[k]}\Big)^{1/2k},$$

where $A^{[1]} = A$, $A^{[j+1]} = A \otimes A^{[j]}$, and $\otimes$ denotes the Kronecker product.

Unfortunately, the size of the matrix appearing in Theorem 2.54 grows exponentially in k so the method is only feasible for $k < 4$ and $n < 10$ unless one has access to a very powerful computer. We have applied the method to the Daubechies filters of length 4, 6, and 8 to calculate $\tilde{\sigma}_4 = 2^{3/4}\sigma_4[S]$. The results are listed in the following Table:

Daub$_N$	$\tilde{\sigma}_4$
4	1.07197
6	1.03306
8	1.02014

Table 2.4: $\tilde{\sigma}_4$ associated with the Daubechies filters of length 4, 6, and 8.

2.6 Uniformly Bounded Nonstationary Wavelet Packets

We have seen that the basic stationary wavelet packets associated with some of the most widely used filters are not uniformly bounded functions. In this section we prove that using the nonstationary construction of wavelet packets one can obtain uniformly bounded basic wavelet packets. The price we have to pay is that we have to use a sequence of filters with an increasing

[3] I would like to thank Lars Villemoes for bringing Zhou's paper to my attention.

number of nonzero coefficients. A consequence is that the diameter of the support of the basic wavelet packets grows with the frequency. We propose a new construction of wavelet packets in the next section to avoid such support problems.

The following two lemmas give us some basic information on the geometry of the Daubechies filters.

Lemma 2.55. *Let m_0^N be the Daubechies filter of length $2N$. Then*

$$|m_0^N(\xi)| \le |\sin(\xi)|^{N-1}, \qquad \text{for } \pi/2 \le |\xi| \le \pi.$$

Moreover,

$$S(\xi) = |m_0^N(\xi)| + |m_0^N(\xi+\pi)| \le 1 + |\sin(\xi)|^{N-1}, \qquad \xi \in \mathbb{R},$$

and

$$\|S\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})} = 1 + O(1/\sqrt{N}).$$

Proof. We have, for $\pi/2 \le |\xi| \le \pi$,

$$|m_0^N(\xi)|^2 = \cos^{2N}(\xi/2)|P_N(\xi)|^2,$$

where

$$\begin{aligned}
|P_N(\xi)|^2 &= \sum_{j=0}^{N-1} \binom{N-1+j}{j} \sin^{2j}(\xi/2) \\
&= \sum_{j=0}^{N-1} \binom{N-1+j}{j} [2\sin^2(\xi/2)]^j 2^{-j} \\
&\le [2\sin^2(\xi/2)]^{N-1} \sum_{j=0}^{N-1} \binom{N-1+j}{j} 2^{-j} \\
&= [2\sin^2(\xi/2)]^{N-1} |P_N(\pi/2)|^2 \\
&= [4\sin^2(\xi/2)]^{N-1},
\end{aligned}$$

so

$$|m_0^N(\xi)|^2 \le \cos^{2N}(\xi/2)[4\sin^2(\xi/2)]^{N-1} \le [4\cos^2(\xi/2)\sin^2(\xi/2)]^{N-1} = |\sin(\xi)|^{2(N-1)}.$$

To get the second part, we just notice that for $\pi/2 \le |\xi| \le \pi$:

$$|m_0^N(\xi)| \le |\sin(\xi)|^{N-1}, \quad \text{and} \quad |m_0^N(\xi+\pi)| \le 1.$$

For $|\xi| \le \pi/2$ we have, using $|\sin(\xi \pm \pi)| = |\sin(\xi)|$,

$$|m_0^N(\xi)| \le 1, \quad \text{and} \quad |m_0^N(\xi+\pi)| \le |\sin(\xi)|^{N-1}.$$

Finally,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S(\xi)^2\,dx \le 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}[|\sin(\xi)|^{2N-2} + 2|\sin(\xi)|^{N-1}]\,d\xi.$$

Assume N is odd (the case N even is similar). We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{(2N-2)}(\xi)\,d\xi = \frac{1\cdot 3\cdot 5\cdots(2N-3)}{2\cdot 4\cdot 6\cdots(2N-2)} \le \frac{1}{\sqrt{(N-1)\pi}},$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{(N-1)}(\xi)\,d\xi = \frac{1\cdot 3\cdot 5\cdots(N-2)}{2\cdot 4\cdot 6\cdots(N-1)} \le \frac{1}{\sqrt{\pi(N-1)/2}}$$

so, using $\sqrt{1+\alpha^2} \le 1 + \alpha^2/2$ we get estimate we want. ■

Moreover,

Lemma 2.56. *Let $\{m_0^{(p)}\}_{p=1}^{\infty}$ be a family of Daubechies low-pass filters. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that $d_p \equiv \deg(m_0^{(p)}) \ge Cp^{2+\varepsilon}$. Then there exists a constant $B < \infty$ such that*

$$\int_{-\pi}^{\pi}|m_{\varepsilon_1}^{(1)}(\xi)m_{\varepsilon_2}^{(2)}(2\xi)\cdots m_{\varepsilon_j}^{(j)}(2^{j-1}\xi)|\,d\xi \le B2^{-j}, \qquad j = 1, 2, , \dots ,$$

for any choice of $(\varepsilon_k) \in \{0,1\}^{\mathbb{N}}$.

Proof. Fix $\varepsilon \in \{0,1\}^{\mathbb{N}}$, and define $I_{J,K} = I^{\varepsilon}_{J,K}$, $J > K$, by

$$I_{J,K}(\xi) \equiv 2^{K+1}|m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|.$$

It suffices to find a constant A such that $\int_{-\pi}^{\pi} I_{J,J-1}(\xi)\,d\xi \le A$, independent of J and the choice of ε. Let $S_K(\xi) = |m_{\varepsilon_{J-K}}^{(J-K)}(\xi)| + |m_{\varepsilon_{J-K}}^{(J-K)}(\xi+\pi)|$ (note S_K does not depend on ε_{J-K} which follows from the CQF conditions). Then

$$\begin{aligned}
\int_{-\pi}^{\pi} I_{J,K}(\xi)\,d\xi &= 2^{K+1}\int_{-\pi}^{\pi}|m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&= 2^{K+1}\int_{-\pi}^{0}|m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&\qquad + 2^{K+1}\int_{0}^{\pi}|m_{\varepsilon_{J-K}}^{(J-K)}(\xi)m_{\varepsilon_{J-K+1}}^{(J-K+1)}(2\xi)\cdots m_{\varepsilon_J}^{(J)}(2^K\xi)|\,d\xi \\
&= 2^{K}\int_{-\pi}^{\pi}S_K(\xi/2)|m_{\varepsilon_{J-K+1}}^{(J-K+1)}(\xi)m_{\varepsilon_{J-K+2}}^{(J-K+2)}(\xi)\cdots m_{\varepsilon_J}^{(J)}(2^{K-1}\xi)|\,d\xi \\
&= \int_{-\pi}^{\pi}S_K(\xi/2)I_{I,K-1}(\xi)\,d\xi
\end{aligned} \tag{2.17}$$

We have

$$2\pi \le I_{J,0} \le I_{J,1} \le \cdots \le I_{J,K},$$

which follows from (2.17) and the fact that $S_K(\xi) \ge |m^{(J-K)}_{\varepsilon_{J-K}}(\xi)|^2 + |m^{(J-K)}_{\varepsilon_{J-K}}(\xi+\pi)|^2 = 1$ for $K = 1, 2, \dots$. Thus, using Lemma 2.55 and Hölder's inequality,

$$\begin{aligned}
\|I_{J,K}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} &= \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \\
&\le \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|_{L^{4/3}([-\pi,\pi],\frac{dx}{2\pi})} \\
&\le \|I_{J,K-1}(\cdot)S_K(\tfrac{\cdot}{2})\|^2_{L^{4/3}([-\pi,\pi],\frac{dx}{2\pi})} \\
&\le \|I_{J,K-1}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})}\|S_K(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}.
\end{aligned}$$

Hence,

$$\|I_{J,J-1}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \le \|I_{J,0}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \cdot \prod_{j=1}^{J-1} \|S_j(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}.$$

Clearly $\|I_{J,0}\|_{L^1([-\pi,\pi],\frac{dx}{2\pi})} \le 2$, so it suffices to prove that $\prod_{j=1}^{J-1} \|S_j(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})}$ is uniformly bounded in J. By Lemma 2.55,

$$\|S_K(\tfrac{\cdot}{2})\|_{L^2([-\pi,\pi],\frac{dx}{2\pi})} = 1 + O(1/\sqrt{d_{J-K}})),$$

and by assumption

$$\sum_{j=1}^{J-1} \frac{1}{\sqrt{d_{J-j}}} \le \sum_{j=1}^{\infty} \frac{1}{\sqrt{d_j}} \le C \sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon/2}} < \infty.$$

The claim now follows from the Weierstrass product test. ∎

We use the above Lemma to obtain the following result.

Theorem 2.57. *Let $\{h^{(p)}\}_{p=0}^{\infty}$ be a family of Daubechies CQF's with associated transfer functions $\{m_0^{(p)}\}$. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that length$(h^{(p)}) \ge Cp^{2+\varepsilon}$. If $|\hat{w}_0(\xi)| \le B(1+|\xi|)^{-1-\varepsilon}$ for some constant B then the Fourier transforms of associated nonstationary wavelet packets are uniformly bounded in L^1-norm and the wavelet packets are consequently uniformly bounded.*

Proof. Take $n : 2^{J+1} \le n < 2^{J+2}$. Then

$$\hat{w}_n(\xi) = m^{(J)}_{\varepsilon_1}(\xi/2) m^{(J-1)}_{\varepsilon_2}(\xi/4) \cdots m^{(0)}_{\varepsilon_{J+1}}(\xi/2^{J+1})\hat{\phi}(\xi/2^{J+1}).$$

Also, since $|\hat{\phi}(\xi)| \leq B(1+|\xi|)^{-1-\varepsilon}$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} |\hat{w}_n(\xi)|\, d\xi &= \sum_{k\in\mathbb{Z}} \int_{-2^{J+1}\pi+k2^{J+2}\pi}^{2^{J+1}\pi+k2^{J+2}\pi} |\hat{w}_n(\xi)|\, d\xi \\ &\leq \int_{-2^{J+1}\pi}^{2^{J+1}\pi} |m_{\varepsilon_1}^{(J)}(\xi/2) m_{\varepsilon_2}^{(J-1)}(\xi/4)\cdots m_{\varepsilon_{J+1}}^{(0)}(\xi/2^{J+1})|\, d\xi \sum_{k\in\mathbb{Z}} C(1+2\pi|k|)^{-1-\varepsilon}. \\ &\leq B2^{J+1} \int_{-\pi}^{\pi} |m_{\varepsilon_{J+1}}^{(0)}(\xi) m_{\varepsilon_J}^{(1)}(2\xi)\cdots m_{\varepsilon_1}^{(J)}(2^J\xi)|\, d\xi,\end{aligned}$$

and the claim follows from Lemma 2.56. ■

Remark. It is an unfortunate consequence of the above nonstationary construction that the diameter of support for the nonstationary wavelet packets grows just as fast as the filterlength. This problem will be eliminated in the next section using a generalized construction of wavelet packets.

2.7 Highly Nonstationary Wavelet Packets

This section contains a generalization of stationary and nonstationary wavelet packets. The new definition induces more flexibility into the construction, and thus allows for construction of functions with better properties than the corresponding nonstationary construction. We have named the new functions highly nonstationary wavelet packets (HNWPs) and the definition is the following

Definition 2.58 (Highly Nonstationary Wavelet Packets). *Let (ϕ, ψ) be a scaling function/wavelet associated with an MRA, and let $\{m_0^{p,q}\}_{p\in\mathbb{N},1\leq q\leq p}$ be a family of CQFs. Let $w_0 = \phi$ and $w_1 = \psi$ and define the functions w_n, $n \geq 2$, $2^J \leq n < 2^{J+1}$, by*

$$\hat{w}_n(\xi) = m_{\varepsilon_1}^{J,1}(\xi/2) m_{\varepsilon_2}^{J,2}(\xi/4)\cdots m_{\varepsilon_J}^{J,J}(\xi/2^J)\hat{\psi}(\xi/2^J),$$

where $n = \sum_{j=1}^{J+1} \varepsilon_j 2^{j-1}$ is the binary expansion of n. We call $\{w_n\}_{n=0}^{\infty}$ a family of basic highly nonstationary wavelet packets (HNWPs).

Remark. It is obvious that the definition of highly nonstationary wavelet packets includes the stationary and nonstationary wavelet packets as special cases.

The following result shows that the integer translates of the basic HNWPs do give us an orthonormal basis for $L^2(\mathbb{R})$, just like the basic nonstationary wavelet packets.

Theorem 2.59. *Let $\{w_n\}_{n=0}^{\infty}$ be a family of highly nonstationary wavelet packets. Then $\{w_n(\cdot - k)\}_{n\geq 0,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. Recall that

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{n=j}^{\infty} W_j \right),$$

and by definition $w_n \in W_J$ for $2^J \leq n < 2^{J+1}$ so it suffices to show that

$$\{w_n(\cdot - k)\}_{2^J \leq n \leq 2^{J+1}, k \in \mathbb{Z}}$$

is an orthonormal basis for W_J. However, this follows at once from the first J steps of the induction argument in the proof of Theorem 2.3 using the filters $m_n^{(p)} = m_0^{J,J-p+1}$, for $p = 1, \cdots, J$.■

The following corollary to Lemma 2.56 shows the added flexibility in the definition highly nonstationary wavelet packets allows one to get better joint time-frequency localization.

Corollary 2.60. *Let $\{h^{(p)}\}_{p=0}^{\infty}$ be a family of Daubechies CQFs with associated transfer functions $\{m_0^{(p)}\}$. Suppose there are constants $\varepsilon > 0$ and $C > 0$ such that*

$$C^{-1} p^{2+\varepsilon} \leq \mathit{length}(h^{(p)}) \leq C p^{-1-\varepsilon} 2^p.$$

Let $\{w_n\}_n$ be the highly nonstationary wavelet packets associated with $m_0^{p,q} = m_0^{(q)}$ for $p \geq 1$, $q \leq p$, and some pair (ϕ, ψ). If $|\hat{w}_0(\xi)| \leq B(1 + |\xi|)^{-1-\varepsilon}$ for some constant B then the Fourier transforms of associated nonstationary wavelet packets are uniformly bounded in L^1-norm and the wavelet packets are consequently uniformly bounded. Moreover, if w_1 has compact support then there is a $K < \infty$ such that $\mathit{supp}(w_n) \subset [-K, K]$ for all $n \geq 1$.

Proof. The first statement follows directly from the proof of Theorem 2.56. The second follows from the fact that the distribution defined as the inverse Fourier transform of the product $\prod_{j=1}^{J} m_{\varepsilon_j}^{(j)}(\xi/2)\hat{\psi}(\xi/2^J)$ has support contained in

$$\alpha\big[-\textstyle\sum_{j=1}^{J} \mathrm{length}(m_{\varepsilon_j}^{(j)})2^{-j}, \sum_{j=1}^{J} \mathrm{length}(m_{\varepsilon_j}^{(j)})2^{-j}\big] \subset [-\tilde{K}, \tilde{K}],$$

whenever $w_1 = \psi$ has compact support ($\alpha < \infty$ depends on the diameter of the support of w_1). ■

Chapter 3

Periodic Wavelet Packets

Wavelet packets have been introduced to provide a flexible method for time-frequency analysis combining the advantages of wavelet analysis and windowed Fourier analysis. Similarly periodic wavelet packets provide an alternative to Fourier series. Although there are a number of similarities between periodic wavelet packets and the trigonometric system, we show in this chapter that the similarities end when the systems are considered in $L^p[0,1)$ for $p \neq 2$. We show that, unlike the trigonometric system, periodic wavelet packets are not, in general, uniformly bounded in every $L^p[0,1)$-space, and they may fail to be a Schauder basis for such spaces.

We also prove that certain periodic HNWPs do form Schauder bases for $L^p[0,1)$, $1 < p < \infty$.

3.1 Periodic Wavelet Packets

It was proved in [29] that by periodizing any (reasonable) orthonormal wavelet basis associated with a multiresolution analysis one obtain a multiresolution analysis for $L^2[0,1)$.

The same procedure works equally well with wavelet packets,

Definition 3.1. *Let $\{w_n\}_{n=0}^{\infty}$ be a family of (possibly) nonstationary basic wavelet packets satisfying $|w_n(x)| \leq C_n(1+|x|)^{-1-\varepsilon_n}$ for some $\varepsilon_n > 0$, $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$ we define the corresponding periodic wavelet packets $\widetilde{w_n}$ by*

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} w_n(x-k).$$

Note that the hypothesis about the pointwise decay of the wavelet packets w_n ensures that the associated periodic wavelet packets are well defined functions contained in $L^p[0,1)$ for every $p \in [1,\infty]$.

The following result shows that the above definition is useful.

Theorem 3.2 ([22]). *The family* $\{\widetilde{w_n}\}_{n=0}^{\infty}$ *is an orthonormal basis for* $L^2[0,1)$.

Proof. Note that $\widetilde{w_n} \in \widetilde{W}_j$ for $2^{j-1} \leq n < 2^j$ ($\widetilde{W}_j$ is the periodized version of the wavelet space W_j) and that $\widetilde{W}_j$ is 2^{j-1} dimensional (see [29] for details), so it suffices to show that $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal system. We have, using Fubini's Theorem,

$$
\begin{aligned}
\int_0^1 \widetilde{w_n}(x)\overline{\widetilde{w_m}(x)}\,dx &= \int_0^1 \sum_{q\in\mathbb{Z}} w_n(x-q)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \sum_{q\in\mathbb{Z}} \int_0^1 w_n(x-q)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \int_{-\infty}^{\infty} w_n(x)\overline{\sum_{r\in\mathbb{Z}} w_m(x-r)}\,dx \\
&= \sum_{r\in\mathbb{Z}} \int_{-\infty}^{\infty} w_n(x)\overline{w_m(x-r)}\,dx \\
&= \delta_{m,n}.
\end{aligned}
$$

■

3.1.1 Periodic Wavelet Packets and the Trigonometric System

It was observed by Hess-Nielsen in [22] that there are some important similarities between periodic wavelet packets and the trigonometric system. Most strikingly is the fact that the periodic wavelet packets share the translation property of the trigonometric system, although the property is more complicated for the periodic wavelet packets.

Let $G : \mathbb{N}_0 \to \mathbb{N}_0$ be the Gray-code permutation. For $n \in \mathbb{N}$ we write

$$2n = \sum_{\ell=k(n)}^{\infty} \varepsilon_\ell\, 2^\ell, \qquad 1 \leq k(n) < \infty,$$

with $\varepsilon_{k(n)} = 1$ and $\varepsilon_l \in \{0,1\}$ otherwise. Define the number η_n by

$$\eta_n = 1 - 2^{-(k(n)+1)} + \varepsilon_{k(n)+1}(n) \cdot (2^{-k(n)} - 1), \qquad n \in \mathbb{N}.$$

Readers interested in the origin of η_n should consult [22]. The result by Hess-Nielsen is then:

Theorem 3.3 ([22]). *For* $n \in \mathbb{N}$: $\tilde{w}_{G(2n)}(x) = \tilde{w}_{G(2n-1)}(x - \eta_n)$.

This theorem shows that the periodic wavelet packets (in frequency order) resemble the trigonometric system in the sense that each $f \in L^2[0,1)$ has the convergent expansions

$$f = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(2\pi n x) + b_n \sin(2\pi n x)\},$$

and

$$f = \tilde{a}_0 + \sum_{n=1}^{\infty} \{\tilde{a}_n \tilde{w}_{G(2n)} + \tilde{b}_n \tilde{w}_{G(2n-1)}\}, \tag{3.1}$$

where $\tilde{w}_{G(2n)}$ is a translate of $\tilde{w}_{G(2n-1)}$.

We would like to know if the expansion given by (3.1) converges in other spaces that $L^2[0,1)$. In the following section we show that the expansion works just fine in $L^p[0,1)$, $1 < p < \infty$, for the periodic wavelet packets associated with the Walsh type wavelet packets introduced in the previous chapter. However, the convergence property may fail for the $L^p[0,1)$-spaces for $p \neq 2$ even for "nice" periodic wavelet packets. This will be proved in section 3.4.

3.2 Periodic Walsh Type Wavelet Packets as a Basis for $L^p[0,1)$

The periodic version of the Walsh system is the Walsh system itself due to the fact that the support of each Walsh function is contained in $[0,1)$, so the periodic Walsh system is indeed a Schauder basis for $L^p[0,1)$ (R. Paley's original result) for $1 < p < \infty$. The next theorem generalizes Paley's result to the periodic Walsh type wavelet packets.

Theorem 3.4. *Let $\{w_n\}_n$ be a wavelet packet system satisfying the hypothesis of Theorem 2.12. Then the associated periodic system $\{\widetilde{w_n}\}_n$ is a Schauder basis for $L^p[0,1)$ for $1 < p < \infty$.*

Proof. We claim that the periodized system $\{\widetilde{w_n}\}_n$ is dense in $L^p[0,1)$. To verify the claim we let P_N be the projection onto the closed linear span of $\{\widetilde{w_n}\}_{n=0}^{N}$. By the construction of the periodic wavelet packets we have $P_{2^N-1} = P_{\widetilde{V_N}}$, where $\widetilde{V_N}$ is the periodized version of the multiresolution space V_N. But $P_{\widetilde{V_N}} f \to f$ for $f \in L^p[0,1)$ and the claim follows. So it suffices to prove that $\sup_N \|P_N\|_{L^p[0,1)\to L^p[0,1)} < \infty$. Suppose not. Note that each P_n is bounded on $L^p[0,1)$ since its kernel is bounded on $[0,1)^2$, so by the Banach-Steinhaus Theorem there exists $f \in L^p[0,1)$ such that

$$\sup_N \|P_N f\|_{L^p[0,1)} = \infty. \tag{3.2}$$

According to the proof of Theorem 2.12 there exists a constant C_p such that

$$\left\| \sum_{n=0}^{N} \langle g, w_n(\cdot - k)\rangle w_n(\cdot - k) \right\|_p \leq C_p \|g\|_p \tag{3.3}$$

for every $N \geq 1$, $k \in \mathbb{Z}$, and every $g \in L^p(\mathbb{R})$. Fix K such that $\text{supp}(w_n) \subset [-K, K]$ for $n \geq 0$. Then, for $x \in [0,1)$

$$\widetilde{w_n}(x) = \sum_{k=-K}^{K+1} w_n(x-k). \tag{3.4}$$

Choose N such that

$$\Big\| \sum_{n=0}^{N} \langle f, \widetilde{w_n} \rangle \widetilde{w_n} \Big\|_{L^p[0,1)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)}, \tag{3.5}$$

which is possible by (3.2). We insert (3.4) into (3.5);

$$\Big\| \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\{ \sum_{n=0}^{N} \int_0^1 f(x) \overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\} \Big\|_{L^p([0,1),\, dy\,)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)}.$$

By Minkowski's inequality

$$\Big\| \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\{ \sum_{n=0}^{N} \int_0^1 f(x) \overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\} \Big\|_{L^p([0,1),\, dy\,)}$$
$$\leq \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \Big\| \sum_{n=0}^{N} \int_0^1 f(x) \overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p([0,1),\, dy\,)}$$

so we can find k_1 and k_2 such that

$$\begin{aligned} C_p \|f\|_{L^p[0,1)} &= C_p \|\chi_{[0,1)} f\|_{L^p(\mathbb{R})} \\ &< \Big\| \sum_{n=0}^{N} \int_0^1 f(x) \overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p([0,1),\, dy\,)} \\ &\leq \Big\| \sum_{n=0}^{N} \int_0^1 f(x) \overline{w_n(x-k_1)}\, dx\, w_n(y-k_2) \Big\|_{L^p(\mathbb{R},\, dy\,)} \\ &= \Big\| \sum_{n=0}^{N} \int_{\mathbb{R}} \{\chi_{[0,1)}(x) f(x)\} \overline{w_n(x-k_1)}\, dx\, w_n(y-k_1) \Big\|_{L^p(\mathbb{R},\, dy\,)}, \end{aligned}$$

which contradicts (3.3). Hence, our assumption that $\sup_N \|P_N\|_{L^p[0,1) \to L^p[0,1)} = \infty$ is false and we are done. ■

3.2.1 A Counterexample in $L^1[0,1)$

This section contains the analog to the counterexample of Theorem 2.14, the expansion in the periodic Walsh type wavelet packets fails in $L^1[0,1)$.

Theorem 3.5. *Let $\{w_n\}_n$ be a family of smooth Walsh type wavelet packets and let J be defined as in Definition 2.9. Choose $L \in \mathbb{N}$ such that supp$(w_{2^J+1}) \subset [-L+1, L-1]$ and choose $M \in \mathbb{N}$ such that $2^M > 2L$. Let $N(k) = k^3 + M + 1$, and define $K : \mathbb{N} \to \mathbb{N}$ recursively by letting $K(1) = 2^J + 1$, $K(2n) = 2K(n)$, and $K(2n+1) = 2K(n) + 1$. Define f by*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \Bigg(\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \widetilde{w_{K(n)}}(x) \Bigg).$$

Then $f \in L^1[0,1)$, but the periodic wavelet packet expansion of f diverges in $L^1[0,1)$-norm.

Proof. Let $1 \le 2^j \le n < 2^{j+1}$. It is clear from the construction of K and the recursive definition of the wavelet packets that $w_{K(n)} \in \operatorname{span}\{w_{K(1)}(2^j x - k)\}_k$ and the expansion coefficients are given by the expansion coefficients of W_n in the Haar wavelets $h(2^j x - k)$. From this observation and the well known fact that the Walsh functions and the Haar wavelets are related by the Hadamard Transform we get (see Appendix A)

$$w_{K(n)}(x) = \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) w_{K(1)}(2^j x - s)$$

so

$$\begin{aligned}
\widetilde{w_{K(n)}}(x) &= \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) \sum_{r\in\mathbb{Z}} w_{K(1)}(2^j x - 2^j r - s) \\
&\equiv \sum_{s=0}^{2^j-1} W_{n-2^j}(s2^{-j}) g_{j,s}(x)
\end{aligned}$$

We now use the fact that for $x \in [0,1)$ (see Appendix A)

$$\sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} W_{n-2^{N(k)}k}(x) = \sum_{n=2^{k^3}}^{2^{k^3+1}-1} W_n(x) = W_{2^{k^3}}(x) 2^{k^3} \chi_{[0,2^{-k^3})}(x)$$

to get

$$\begin{aligned}
\int_0^1 \left| \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \widetilde{w_{K(n)}}(x) \right| dx &= \int_0^1 \left| \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3+1}-1} \left\{ \sum_{s=0}^{2^{N(k)}-1} W_{n-2^{N(k)}}(s2^{-N(k)}) g_{N(k),s}(x) \right\} \right| dx \\
&= 2^{k^3} \int_0^1 \left| \sum_{s=0}^{2^{N(k)-k^3}-1} W_{2^{k^3}}(s2^{-N(k)}) g_{N(k),s}(x) \right| dx \\
&\le 2^{k^3} \sum_{s=0}^{2^{N(k)-k^3}-1} \int_0^1 \left| W_{2^{k^3}}(s2^{-N(k)}) g_{N(k),s}(x) \right| dx \\
&\le 2^{k^3} 2^{N(k)-k^3} 2^{-N(k)} \|w_{K(1)}\|_{L^1(\mathbb{R})} \\
&= \|w_{K(1)}\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Thus, $f \in L^1[0,1)$. Define the sequence a_n by

$$a_n = \begin{cases} \dfrac{1}{k^2} & \text{for } 2^{N(k)} + 2^{k^3} \le n < 2^{N(k)} + 2^{k^3+1}, \quad k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{n=0}^{\infty} a_n \widetilde{w_{K(n)}}$. Define n_j by

$$n_{2j} = \sum_{i=0}^{j} 2^{2i} \text{ and } n_{2j+1} = \sum_{i=0}^{j} 2^{2i+1},$$

and note that $n_{k^3-2} < 2^{k^3}$ for $k \in \mathbb{N}$. Let us estimate the difference between the following two partial sums of f

$$\begin{aligned}
\int_0^1 \Bigg| & \sum_{n=0}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} a_n \widetilde{w_{K(n)}}(x) - \sum_{n=0}^{2^{N(k)}+2^{k^3}-1} a_n \widetilde{w_{K(n)}}(x) \Bigg| \, dx \\
&= \frac{1}{k^2} \int_0^1 \Bigg| \sum_{s=0}^{2^{N(k)}-1} \sum_{n=2^{N(k)}+2^{k^3}}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} W_{n-2^{N(k)}}(s2^{-N(k)}) g_{N(k),s}(x) \Bigg| \, dx \\
&= \frac{1}{k^2} \int_0^1 \Bigg| \sum_{s=0}^{2^{N(k)}-1} W_{2^{k^3}}(s2^{-N(k)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(s2^{-N(k)}) g_{N(k),s}(x) \Bigg| \, dx,
\end{aligned}$$

and using the fact that W_n is constant on $[p2^{-(k^3+1)}, (p+1)2^{-(k^3+1)})$ for $n \le 2^{k^3}$,

$$\begin{aligned}
= \frac{1}{k^2} \int_0^1 \Bigg| & \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \times \\
& \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x) \Bigg| \, dx
\end{aligned}$$

Define $\{I_p\}_{p=0}^{2^{k^3+1}-1} \subset [0,1)$ by

$$\begin{aligned}
I_p &= \{x | 2^{N(k)}x \in [p2^{((N(k)-k^3-1))} + L, (p+1)2^{(N(k)-k^3-1)} - L]\} \\
&= [p2^{-(k^3+1)} + 2^{-N(k)}L, (p+1)2^{-(k^3+1)} - 2^{-N(k)}L]
\end{aligned}$$

Suppose $x \in I_l$. Consider

$$\sum_{r \in \mathbb{Z}} \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} w_{K(1)}(2^{N(k)}x - 2^{N(k)}r - p2^{(N(k)-k^3-1)} - s). \tag{3.6}$$

Note that

$$\begin{aligned}
2^{N(k)}&x - 2^{N(k)}r - p2^{(N(k)-k^3-1)} - s \\
&\in [(l-p)2^{(N(k)-k^3-1)} + L - r2^{N(k)} - s, (l+1-p)2^{(N(k)-k^3-1)} - L - r2^{N(k)} - s].
\end{aligned}$$

Using that $p \in [0, 2^{k^3+1}-1]$ and $s \in [0, 2^{N(k)-k^3-1}-1]$ we get the bounds

$$(l-p)2^{(N(k)-k^3-1)} + L - r2^{N(k)} - s \geq -2^{N(k)} - r2^{N(k)} + L + 1$$
$$(l+1-p)2^{(N(k)-k^3-1)} - L - r2^{N(k)} - s \leq 2^{N(k)} - L - r2^{N(k)}$$

from which we deduce that it is only the terms with $r = 0$ that contribute to (3.6) since $\mathrm{supp}(w_{K(1)}) \subset [-L+1, L-1]$. A similar argument using the definition of I_l, the fact that $2^{(N(k)-k^3-1)} = 2^M > 2L$, and the compact support of $w_{K(1)}$, shows that for $x \in I_l$

$$\sum_{r\in\mathbb{Z}} \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} w_{K(1)}(2^{N(k)}x - 2^{N(k)}r - p2^{N(k)-k^3-1} - s) = \begin{cases} \sum_{r\in\mathbb{Z}} w_{K(1)}(2^{N(k)}x - r) & \text{for } p = l \\ 0 & \text{for } p \neq l. \end{cases}$$

Hence,

$$\begin{aligned}
&\frac{1}{k^2}\int_{I_l} \Big| \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x) \Big|\, dx \\
&\qquad = \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Big| \int_{I_l} \Big| \sum_{r\in\mathbb{Z}} w_{K(1)}(2^{N(k)}x - r) \Big|\, dx \\
&\qquad = \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Big| 2^{-N(k)} \int_{2^{N(k)} I_l} \big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \big|\, dx \\
&\qquad = \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Big| 2^{-N(k)} \int_{l2^M+L}^{(l+1)2^M-L} \big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \big|\, dx \\
&\qquad = \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Big| 2^{-N(k)}(2^M - 2L) \int_0^1 \big| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \big|\, dx
\end{aligned}$$

Finally we use the following fact about the Lebesgue constants for the Walsh system (see Theorem A.2)

$$\int_0^1 \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(x) \Big|\, dx = 2^{-(k^3+1)} \sum_{l=0}^{2^{k^3+1}-1} \Big| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \Big| > \frac{1}{2}\Big(\frac{k^3-2}{2} + 1 \Big)$$

to get the estimate we want

$$\int_0^1 \left| \sum_{n=0}^{2^{N(k)}+2^{k^3}+n_{k^3-2}-1} a_n \widetilde{w_{K(n)}}(x) - \sum_{n=0}^{2^{N(k)}+2^{k^3}-1} a_n \widetilde{w_{K(n)}}(x) \right| dx$$

$$\geq \frac{1}{k^2} \sum_{l=0}^{2^{k^3+1}-1} \int_{I_l} \left| \sum_{p=0}^{2^{k^3+1}-1} W_{2^{k^3}}(p2^{-(k^3+1)}) \sum_{n=0}^{n_{k^3-2}-1} W_n(p2^{-(k^3+1)}) \times \right.$$

$$\left. \sum_{s=0}^{2^{(N(k)-k^3-1)}-1} g_{N(k),p2^{(N(k)-k^3-1)}+s}(x) \right| dx$$

$$= \frac{1}{k^2} \sum_{l=0}^{2^{k^3+1}-1} \left| \sum_{n=0}^{n_{k^3-2}-1} W_n(l2^{-(k^3+1)}) \right| 2^{-N(k)}(2^M - 2L) \times$$

$$\int_0^1 \left| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \right| dx$$

$$= \frac{1}{k^2}(2^M - 2L) \int_0^1 \left| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \right| dx \, 2^{-M} \int_0^1 \left| \sum_{n=0}^{n_{k^3-2}-1} W_n(x) \right| dx$$

$$> \frac{1}{k^2}(2^M - 2L) \int_0^1 \left| \sum_{r\in\mathbb{Z}} w_{K(1)}(x-r) \right| dx \, 2^{-M} \frac{1}{2}\left(\frac{k^3-2}{2} + 1 \right)$$

$$\geq Ck$$

for some $C > 0$. We conclude that the partial sums

$$P_{K(2^{N(k)}+2^{k^3}+n_{k^3-2}-1)}f - P_{K(2^{N(k)}+2^{k^3}-1)}f$$

diverge in $L^1[0,1)$ as $k \to \infty$. This proves the Theorem. ■

3.2.2 Pointwise Convergence for Periodic Walsh Type Wavelet Packet Expansions

We have the following corollary to Theorem 2.16.

Corollary 3.6. *Let $\{w_n\}_n$ be a wavelet packet system satisfying the hypothesis of Theorem 2.12. Then the expansion of each $f \in L^p[0,1)$, $1 < p < \infty$, in associated periodic system $\{\widetilde{w_n}\}_n$ converges a.e.*

Proof. Let $f \in L^p[0,1)$, and define N as in the proof of Theorem 2.16. Note that

$$\sum_{n=0}^{m} \langle f, \widetilde{w_n} \rangle \widetilde{w_n}(x) = \sum_{k_1=-N}^{N+1} \sum_{k_2=-N}^{N+1} \left\{ \sum_{n=0}^{m} \int_0^1 f(y) \overline{w_n(y-k_1)} \, dy w_n(x-k_2) \right\},$$

so it follows at once from the proof of Theorem 2.16 that the Carleson operator associated with the periodic Walsh type wavelet packets is of strong type (p,p) for $1 < p < \infty$. $\blacksquare$

3.3 Growth in L^p-norm of Periodic Wavelet Packets

We have seen that wavelet packets generated by the Daubechies or Coiflet filters (other that the Haar filter) are not uniformly bounded in $L^p(\mathbb{R})$-norm for p large, and this growth in norm prevents the wavelet packets from being Schauder bases for the $L^p(\mathbb{R})$ spaces. One could hope that periodizing the wavelet packets would somehow collapse their $L^p(\mathbb{R})$-norm and hopefully give us some new nice bases for $L^p[0,1)$. However, this is not so as will be show in this section. But first we need a technical result about multipliers for Fourier series. The result can be deduced, with some work, from a result by de Leeuw (see [12, Corollary 4.6]). We present a more direct proof here.

Lemma 3.7. *Let $\{m_k\}_{k\in\mathbb{Z}}$ be a 2^N-periodic sequence with $\alpha = \inf_k |m_k| > 0$. Then the operator T, defined on $L^2[0,1)$ by*

$$T\{\sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}\} = \sum_{k\in\mathbb{Z}} m_k\, a_k e^{2\pi ikx},$$

extends to an isomorphism on $L^p[0,1)$, $1 < p < \infty$.

Proof. Fix $1 < p < \infty$. First step is to prove that $\{\chi_{2^N\mathbb{Z}+j}(k)\}_{k\in\mathbb{Z}}$ is a p-multiplier. Since the shift operator (i.e. multiplication by $e^{2\pi ijx}$) is an isometric isomorphism on $L^p[0,1)$ it suffices to prove that $\{\chi_{2^N\mathbb{Z}}(k)\}_{k\in\mathbb{Z}}$ is a p-multiplier. We prove this by induction on N. Suppose $N=1$ and let T_2 be the operator induced by the sequence $\{\chi_{2\mathbb{Z}}(k)\}_{k\in\mathbb{Z}}$. Let $p(x) = \sum a_k e^{2\pi ikx}$ be a trigonometric polynomial. Then, for $l = 1, 2, \dots,$

$$\int_0^1 |p(x)|^{2l}\, dx = \sum_{j_1} \cdots \sum_{j_{2l}} a_{j_1}\bar{a}_{j_2} \cdots a_{j_{2l-1}}\bar{a}_{j_{2l}} \int_0^1 e^{2\pi i(j_1 - j_2 + \cdots + j_{2l-1} - j_{2l})x}\, dx. \qquad (3.7)$$

However,

$$\int_0^1 e^{2\pi i(j_1-j_2+\cdots+j_{2l-1}-j_{2l})x}\,dx \neq 0 \Leftrightarrow j_1-j_2+\cdots+j_{2l-1}-j_{2l}=0,$$

and $j_1-j_2+\cdots+j_{2l-1}-j_{2l}=0$ only if the number of odd indices in the set $\{j_1,j_2,\ldots,j_{2l}\}$ is even. Let $\tilde{p}(x)=\sum c(k)a_k e^{2\pi ikx}$, where $c(k)=1-2(k \mod 2)$. The above argument shows that if $j_1-j_2\cdots+j_{2l-1}-j_{2l}=0$ then

$$a_{j_1}\bar{a}_{j_2}\cdots a_{2l-1}\bar{a}_{j_{2l}} = c(j_1)a_{j_1}c(j_2)\bar{a}_{j_2}\cdots c(j_{2l-1})a_{j_{2l-1}}c(j_{2l})\bar{a}_{j_{2l}}.$$

Thus, it follows from (3.7) that $\tilde{p}(x)$ has the same $L^{2l}[0,1)$-norm as $p(x)$. So

$$\|T_2 p\|_{L^{2l}[0,1)} = \frac{1}{2}\|p(x)+\tilde{p}(x)\|_{L^{2l}[0,1)} \leq \|p\|_{L^{2l}[0,1)},$$

and T_2 thus extends to a bounded operator on $L^{2l}[0,1)$ for $l=1,2,\ldots$ The Riesz-Thorin interpolation theorem and a duality argument (T_2 is obviously self-adjoint) shows that T_2 extends to bounded operator on every $L^p[0,1)$ for $1<p<\infty$.

Suppose the result holds for $N-1$, $N\geq 2$. Let T_j be the operator induced by $\{\chi_{2^j\mathbb{Z}}(k)\}_{k\in\mathbb{Z}}$. Note that

$$T_N f(x) = T_2[(T_{N-1}f)(2^{1-N}\cdot)](2^{N-1}x)$$

so T_N is bounded on $L^p[0,1)$.

We now apply the above 2^N times to obtain

$$\sum_{j=0}^{2^N-1}\|\sum_{k\in 2^N\mathbb{Z}+j} c_k e^{2\pi ikx}\|_{L^p[0,1)} \leq C_p 2^N \|\sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}\|_{L^p[0,1)}.$$

Hence, using Minkowski's inequality,

$$\|\sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}\|_{L^p[0,1)} \simeq \sum_{j=0}^{2^N-1}\|\sum_{k\in 2^N\mathbb{Z}+j} c_k e^{2\pi ikx}\|_{L^p[0,1)}.$$

Returning to the operator T we have, using $\{m_k\}$ is 2^N-periodic and

$$0<\inf_k |m_k| \leq \sup_k |m_k| < \infty,$$

$$
\begin{aligned}
\Big\|\sum_{k\in\mathbb{Z}} m_k c_k e^{2\pi ikx}\Big\|_{L^p[0,1)} &\simeq \sum_{j=0}^{2^N-1}\Big\|\sum_{k\in 2^N\mathbb{Z}+j} m_k c_k e^{2\pi ikx}\Big\|_{L^p[0,1)}\\
&= \sum_{j=0}^{2^N-1}|m_j|\Big\|\sum_{k\in 2^N\mathbb{Z}+j} c_k e^{2\pi ikx}\Big\|_{L^p[0,1)}\\
&\simeq \sum_{j=0}^{2^N-1}\Big\|\sum_{k\in 2^N\mathbb{Z}+j} c_k e^{2\pi ikx}\Big\|_{L^p[0,1)}\\
&\simeq \Big\|\sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}\Big\|_{L^p[0,1)},
\end{aligned}
$$

which shows that T is an isomorphism on $L^p[0,1)$ and we are done. ■

The following result shows that periodizing most compactly supported wavelet packets will not collapse their L^p-norm significantly.

Theorem 3.8. *Let $\{w_n\}_n$ be a wavelet packet basis associated with the finite filters (m_0, m_1). Choose N such that $\operatorname{diam}\operatorname{supp}(w_n) \le 2^N$. Fix $L \in 2\mathbb{Z}^+ + 1$. If*

$$
(m_0^{-1}(0) \cup m_1^{-1}(0)) \cap \Big(\bigcup_{k=1}^{N} 2^{-k}(2\mathbb{Z}+1)\pi\Big) = \emptyset
$$

then there exist finite constants $c_p, C_p > 0$ (depending on L) such that

$$
c_p\|w_{2^n-1}\|_p \le \|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)} \le C_p\|w_{2^n-1}\|_p, \qquad \text{for } n \ni 2^{n+N} - L \ge 1.
$$

Proof. We have, using that $m_1(k\pi) = -(k \mod 2)$,

$$
\begin{aligned}
\widetilde{w_{2^{n+N}-L}}(x) &= \sum_{k\in\mathbb{Z}} \hat{w}_{2^{N+n}-L}(2\pi k)e^{2\pi ikx}\\
&= \sum_{k\in\mathbb{Z}} m_1(\pi k)m_{\varepsilon_2}(\tfrac{\pi k}{2})\cdots m_{\varepsilon_J}(\tfrac{\pi k}{2^N})\hat{w}_{2^n-1}(\tfrac{\pi k}{2^N})e^{2\pi ikx}\\
&= -\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}\big(\tfrac{(2\ell+1)\pi}{2}\big)\cdots m_{\varepsilon_J}\big(\tfrac{(2\ell+1)\pi}{2^N}\big)\hat{w}_{2^n-1}\big(\tfrac{(2\ell+1)\pi}{2^N}\big)e^{2\pi i2\ell x}e^{2\pi ix},
\end{aligned}
$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_J$ are the first J bits of the binary expansion of $2^{n+N} - L$. Note that $\varepsilon_1 = 1$ since L is odd and $\varepsilon_2, \dots, \varepsilon_J$ do not depend on n, only on L. Thus,

$$
\begin{aligned}
\|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)} &= \Big\|\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}(\tfrac{(2\ell+1)\pi}{2})\cdots m_{\varepsilon_J}(\tfrac{(2\ell+1)\pi}{2^N})\hat{w}_{2^n-1}(\tfrac{(2\ell+1)\pi}{2^N})e^{2\pi i2\ell x}e^{2\pi ix}\Big\|_{L^p[0,1)}\\
&= \Big\|\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}(\tfrac{(2\ell+1)\pi}{2})\cdots m_{\varepsilon_J}(\tfrac{(2\ell+1)\pi}{2^N})\hat{w}_{2^n-1}(\tfrac{(2\ell+1)\pi}{2^N})e^{2\pi i2\ell x}\Big\|_{L^p[0,1)}\\
&= \Big\|\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}(\tfrac{(2\ell+1)\pi}{2})\cdots m_{\varepsilon_J}(\tfrac{(2\ell+1)\pi}{2^N})\hat{w}_{2^n-1}(\tfrac{(2\ell+1)\pi}{2^N})e^{2\pi i\ell x}\Big\|_{L^p[0,1)}.
\end{aligned}
$$

Note that

$$\{m_{\varepsilon_2}(\tfrac{(2\ell+1)\pi}{2})\cdots m_{\varepsilon_J}(\tfrac{(2\ell+1)\pi}{2^N})\}_{\ell\in\mathbb{Z}}$$

is a 2^N-periodic sequence. Moreover, the sequence is non-vanishing (by assumption). Hence, by Lemma 3.7 for $1 < p < \infty$,

$$\begin{aligned}\|\widetilde{w_{2^{n+N}-L}}\|^p_{L^p[0,1)} &\simeq \Big\|\sum_{\ell\in\mathbb{Z}} \hat{w}_{2^n-1}(\tfrac{(2\ell+1)\pi}{2^N})e^{2\pi i\ell x}\Big\|^p_{L^p[0,1)}\\ &= 2^{-N}\Big\|\sum_{\ell\in\mathbb{Z}} \hat{w}_{2^n-1}(\tfrac{2\ell\pi}{2^N}+\tfrac{\pi}{2^N})e^{2\pi i2^{-N}\ell x}\Big\|^p_{L^p[0,2^N)}.\end{aligned}$$

However,

$$2^{-N}\sum_{\ell\in\mathbb{Z}} \hat{w}_{2^n-1}(\tfrac{2\ell\pi}{2^N}+\tfrac{\pi}{2^N})e^{2\pi i2^{-N}\ell x}$$

is just the Fourier series on $[0,2^N)$ of the function

$$g(x) = \sum_{k\in\mathbb{Z}} f(x-2^Nk),$$

where $f(x) = w_{2^n-1}(x)e^{-i2^{-N}\pi x}$. Also, $\|g\|_{L^p[0,2^N)} = \|w_{2^n-1}\|_{L^p(\mathbb{R})}$ since $\operatorname{diam}\operatorname{supp}(w_{2^n-1}) \leq 2^N$. So we conclude that for $1 < p < \infty$

$$\|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)} \simeq \|w_{2^n-1}\|_{L^p(\mathbb{R})},$$

for n sufficiently large. ■

Corollary 3.9. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables 2.1, 2.2, and 2.3. Fix $L \in 2\mathbb{Z}^+ + 1$. Then there is a $p_0 > 2$ such that for $p \geq p_0$ there is a constant $r_p > 1$ (depending on L) such that*

$$\|\widetilde{w_{2^n-L}}\|_{L^p[0,1)} \geq C_p r_p^n,$$

for n large.

Proof. Follows at once from Corollary 2.37 and Theorem 3.8, since the combined zero-set of the filters m_0 and m_1 is $\pi\mathbb{Z}$, and $(2\ell+1)/2^j \notin \mathbb{Z}$ for $j \geq 1$. ■

Corollary 3.9 can be used to generalize Theorem 2.37. The following result emphasizes that it is the high-pass filter (m_1) that causes the growth in L^p-norm of the wavelet packets.

Corollary 3.10. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables 2.1, 2.2, and 2.3. Fix $L \in 2\mathbb{Z}^+ + 1$. Then there is a $p_0 > 2$ such that for $p \geq p_0$ there is a constant $r_p > 1$ (depending on L) such that*

$$\|w_{2^n-L}\|_{L^p(\mathbb{R})} \geq C_p r_p^n,$$

for n large.

Proof. Follows at once from Corollary 3.9, Minkowski's inequality, and the fact that the wavelet packets all have support contained in some fixed interval. ■

We have the following result analogous to Theorem 2.38.

Corollary 3.11. *Let m_0 be the Daubechies filter of length 4 and let $\{\widetilde{w_n}\}$ be the associated periodic wavelet packets. Then*

$$\|\widetilde{w_{2^n-1}}\|_{L^p[0,1)} \xrightarrow{n\to\infty} \infty$$

for every $p > 2$.

Proof. We just have to prove that

$$\|\widetilde{w_{2^n-1}}\|_{L^1[0,1)} \xrightarrow{n\to\infty} 0$$

since $\|\widetilde{w_{2^n-1}}\|_{L^2[0,1)} = 1$. But $\|\widetilde{w_{2^n-1}}\|_{L^1[0,1)} \leq \|w_{2^n-1}\|_{L^1(\mathbb{R})}$, and we have

$$\|w_{2^n-1}\|_{L^1(\mathbb{R})} \xrightarrow{n\to\infty} 0$$

by the proof of Theorem 2.38, and we are done. ■

3.4 Failure of Some Periodic Wavelet Packets to be a Basis

We proved in the previous chapter that compactly supported wavelet packets may fail to be Schauder bases for the $L^p(\mathbb{R})$-spaces. We show in this section that a similar (unfortunate) result holds true for periodic wavelet packets. The failure is due to the following analog of Lemma 2.41.

Lemma 3.12. *If $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, then there exists a finite constant C_p such that*

$$\|\widetilde{w_n}\|_{L^p[0,1)}\|\widetilde{w_n}\|_{L^{p'}[0,1)} \leq C_p, \qquad n = 0, 1, \dots . \tag{3.8}$$

Proof. Same as for Lemma 2.41. ■

The main theorem is the following

Theorem 3.13. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables 2.1, 2.2, and 2.3. Then there is a $p_0 > 2$ such that for $p \geq p_0$ the periodic wavelet packet system $\{\widetilde{w_n}\}_n$ (in any ordering) fails to be a Schauder basis for $L^p[0,1)$.*

Proof. Choose p_0 such that

$$\sup_n \|w_{2^n-1}\|_{p'} \|w_{2^n-1}\|_p = \infty,$$

for each $p \geq p_0$. Fix $p \geq p_0$. Then here is a constant $c_p \in (0, \infty)$ and an integer N such that

$$\|\widetilde{w_{2^{n+N}-1}}\|_{L^{p'}[0,1)} \|\widetilde{w_{2^{n+N}-1}}\|_{L^p[0,1)} \geq c_p \|w_{2^n-1}\|_{p'} \|w_{2^n-1}\|_p.$$

Hence,

$$\sup_j \|\widetilde{w_{2^j-1}}\|_{L^{p'}[0,1)} \|\widetilde{w_{2^j-1}}\|_{L^p[0,1)} = \infty.$$

The result then follows from Lemma 3.12. ■

3.4.1 Some Periodic Wavelet Packets Generated Using Infinite Filters

The results in this section are the periodic analog of the results from Section 2.4.1. Again, for certain technical reasons we have to restrict our attention to transfer functions that are given by the absolute value of a transfer function for a FIR filter.

First, we check that the Bernstein type inequality from Lemma 2.33 can be generalized to periodic wavelet packets.

Lemma 3.14. *Let $\{\tilde{V}_j\}$ be the periodization of a multiresolution analysis $\{V_j\}$ with associated scaling function ϕ satisfying $|\phi(x)| \leq C(1 + |x|)^{-1-\varepsilon}$, for some $\varepsilon > 0$. Then there is a constant C_p such that for $f \in \tilde{V}_n$,*

$$\|f\|_{L^\infty[0,1)} \leq 2^{n/p} C_p \|f\|_{L^p[0,1)}.$$

Proof. We have $f = \sum_{k=0}^{2^n-1} c_k \tilde{\phi}_{n,k}$ for some sequence $\{c_k\}$. Thus,

$$\begin{aligned}
\|f\|_{L^\infty[0,1)} &= \Big\| \sum_{k=0}^{2^n-1} c_k \sum_{q \in \mathbb{Z}} 2^{n/2} \phi(2^n(x-q) - k) \Big\|_\infty \\
&\leq 2^{n/2} \|\{c_k\}\|_{\ell^\infty(\mathbb{Z})} \Big\| \sum_{q \in \mathbb{Z}} |\phi(2^n x - q)| \Big\|_\infty \\
&\leq C 2^{n/2} \|\{c_k\}\|_{\ell^\infty(\mathbb{Z})} \\
&\leq C 2^{n/2} \|\{c_k\}\|_{\ell^p(\mathbb{Z})}.
\end{aligned}$$

We also have

$$c_k = \int_0^1 f(x) \overline{\tilde{\phi}_{n,k}^{1/p}(x)\, \tilde{\phi}_{n,k}^{1/p'}(x)}\, dx,$$

so

$$|c_k| \leq \|\tilde{\phi}_{n,k}\|_{L^1[0,1)}^{1/p'} \left(\int_0^1 |f(x)|^p |\tilde{\phi}_{n,k}(x)|\, dx \right)^{1/p}.$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{2^n-1} |c_k|^p &\leq \sum_{k=0}^{2^n-1} \|\tilde{\phi}_{n,k}\|_{L^1[0,1)}^{p/p'} \int_0^1 |f(x)|^p |\tilde{\phi}_{n,k}(x)|\, dx \\
&\leq C(2^{-n/p'} 2^{n/2p'} 2^{n/2p})^p \int_0^1 |f(x)|^p \sum_{q\in\mathbb{Z}} |\phi(2^n x - q)|\, dx \\
&\leq \tilde{C}(2^{-n/p'} 2^{n/2p'} 2^{n/2p})^p \int_0^1 |f(x)|^p\, dx. \\
&= \tilde{C}(2^{n/p} 2^{-n/2})^p \|f\|_{L^p[0,1)}^p.
\end{aligned}$$

Thus,

$$\|f\|_{L^\infty[0,1)} \leq C 2^{n/p} [2^{n/2} 2^{-n/p} \|\{c_k\}\|_{\ell^p(\mathbb{Z})}] \leq \tilde{C} 2^{n/p} \|f\|_{L^p[0,1)},$$

and we are done. ∎

We have the following result

Theorem 3.15. *Let m_0 be a finite low-pass filter with associated wavelet packet system $\{w_n\}$. Let $m_0^{abs} = |m_0|$ with associated periodic wavelet packet system $\{\widetilde{w_n^{abs}}\}$. Suppose m_0 only vanishes on $(2\mathbb{Z}+1)\pi$ and suppose there is an $r > 1$ such that*

$$\|\hat{w}_{2^n-1}\|_\infty \|w_{2^n-1}\|_\infty \geq Cr^n.$$

Then there is a $p_0 > 2$ such that for $p > p_0$, $\{\widetilde{w_n^{abs}}\}$ fails to be a Schauder basis for $L^p[0,1)$ since

$$\|\widetilde{w_{2^n-1}^{abs}}\|_{L^p[0,1)} \|\widetilde{w_{2^n-1}^{abs}}\|_{L^{p'}[0,1)} \xrightarrow{n\to\infty} \infty.$$

Proof. Fix $2 < p < \infty$. By the proof of Theorem 3.8 there is an $N > 1$ such that

$$\|\widetilde{w_{2^{n+N}-1}^{\text{abs}}}\|_{L^{p'}[0,1)} \simeq \Big\| \sum_{k\in\mathbb{Z}} \hat{w}_{2^n-1}^{\text{abs}}(\tfrac{2\pi k}{2^N} + \tfrac{\pi}{2^N}) e^{2\pi i k x} \Big\|_{L^{p'}[0,1)},$$

so by the Hausdorff-Young inequality

$$\|\widetilde{w_{2^n-1}^{\text{abs}}}\|_{L^{p'}[0,1)} \geq C \|\{\hat{w}_{2^n-1}^{\text{abs}}(\tfrac{2\pi k}{2^N} + \tfrac{\pi}{2^N})\}_k\|_{\ell^p(\mathbb{Z})}.$$

By Theorem 2.19 we have

$$\|\{\hat{w}_{2^n-1}^{\text{abs}}(\tfrac{2\pi k}{2^N} + \tfrac{\pi}{2^N})\}_k\|_{\ell^p(\mathbb{Z})} \simeq \|\hat{w}_{2^n-1}\|_p,$$

and using Bernstein's inequality (note, $\hat{w}_n$ is band-limited), we get

$$\|\widetilde{w_{2^n-1}^{\text{abs}}}\|_{L^{p'}[0,1)} \geq C \|\hat{w}_{2^n-1}\|_\infty.$$

Also,

$$\|\widetilde{w_{2^n-1}}\|_\infty \le \|\{\hat{w}_{2^n-1}(2\pi k)\}_k\|_{\ell^1(\mathbb{Z})} = \|\widetilde{w^{\text{abs}}_{2^n-1}}\|_\infty,$$

since $\hat{w}^{\text{abs}}_{2^n-1}$ is a modulation of a nonnegative function. Hence, using Lemma 3.14,

$$\begin{aligned}
\|\widetilde{w^{\text{abs}}_{2^n-1}}\|_{L^p[0,1)}\|\widetilde{w^{\text{abs}}_{2^n-1}}\|_{L^{p'}[0,1)} &\ge C2^{-n/p}\|\hat{w}_{2^n-1}\|_\infty\|\widetilde{w^{\text{abs}}_{2^n-1}}\|_\infty \\
&\ge C2^{-n/p}\|\hat{w}_{2^n-1}\|_\infty\|\widetilde{w_{2^n-1}}\|_\infty \\
&\ge \tilde{C}2^{-n/p}\|\hat{w}_{2^n-1}\|_\infty\|\widetilde{w_{2^n-1}}\|_p \\
&\ge \tilde{\tilde{C}}2^{-n/p}\|\hat{w}_{2^n-1}\|_\infty\|w_{2^{n-N}-1}\|_{L^p(\mathbb{R})},
\end{aligned}$$

but

$$2^{-n/p}\|\hat{w}_{2^n-1}\|_\infty\|w_{2^{n-N}-1}\|_{L^p(\mathbb{R})} \xrightarrow{n\to\infty} \infty$$

for p large and we are done. ■

An immediate corollary of the above result is

Corollary 3.16. *Theorem 3.15 applies to the filters listed in Tables 2.1, 2.2, and 2.3.*

Proof. Same as for Corollary 2.47. ■

3.5 Periodic HNWPs With Near Perfect Frequency Localization

The Shannon wavelet packets are not contained in $L^1(\mathbb{R})$ so one has to be careful trying to periodize the functions. We can avoid this problem by viewing the Shannon filter as the limit of a sequence of Meyer filters. For Meyer filters, N. Hess-Nielsen observed that periodic wavelet packets in frequency ordering are just shifted sine and cosines at the low frequencies. More precisely, for $n \in \mathbb{N}$ we use the binary expansion $2n = \sum_{\ell=0}^{\infty} \varepsilon_\ell 2^\ell$ to define a sequence $\{\kappa_n\}$ by

$$\kappa_n = \sum_{\ell=0}^{\infty} |\varepsilon_\ell - \varepsilon_{\ell+1}|2^{-\ell-1}.$$

Then the result is

Theorem 3.17 ([22]). *Choose ε such that $\pi/6 > \varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $\varepsilon \le 2^{-N}$. For m_0 a Meyer filter with $m_0(\xi) = 1$ for $\xi \in [-\frac{\pi}{2} + \frac{\varepsilon}{\pi-\varepsilon}, \frac{\pi}{2} - \frac{\varepsilon}{\pi-\varepsilon}]$ we consider the periodized wavelet packets $\{\widetilde{w_n}\}_n$ in frequency order generated using m_0 and the associated high-pass filter. They fulfill*

$$\begin{aligned}
\widetilde{w_{2n}}(x) &= \sqrt{2}\cos[2\pi n(x-\kappa_n)] \\
\widetilde{w_{2n-1}}(x) &= \sqrt{2}\sin[2\pi n(x-\kappa_n)],
\end{aligned}$$

for each n, $0 < n < 2^{N-1}$.

The periodized version of the Shannon wavelet packet system should correspond to the limit of the above results as we let $\varepsilon \to 0$. This consideration leads us to the following definition:

Definition 3.18 (Periodic Shannon Wavelet Packets). *We define the periodic Shannon wavelet packets* $\{\widetilde{S_n}\}$ *(in frequency order) by* $\widetilde{S_0} = 1$ *and for* $n \in \mathbb{N}$:

$$\begin{aligned} \widetilde{S_{2n}}(x) &= \sqrt{2}\cos[2\pi n(x - \kappa_n)] \\ \widetilde{S_{2n-1}}(x) &= \sqrt{2}\sin[2\pi n(x - \kappa_n)]. \end{aligned}$$

This system has all the useful properties one can hope for:

Theorem 3.19. *The system* $\{\widetilde{S_n}\}_n$ *is an orthonormal basis for* $L^2[0,1)$ *and a Schauder basis for* $L^p[0,1)$, $1 < p < \infty$.

Proof. The L^2 result follows from the fact that any finite subsystem of $\{\widetilde{S_n}\}_n$ is a subset of the orthonormal basis considered in Theorem 3.17 for sufficiently small ε. To get the L^p result it suffices to notice that for any sequence $(\delta_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$, $\{e^{2\pi i k(x-\delta_k)}\}_k$ is a Schauder basis for $L^p[0,1)$, which follows easily by calculating the associated partial sums

$$\begin{aligned} \sum_{|k|\le N} \langle f, e^{-2\pi i k\delta_k} e^{2\pi i k\cdot}\rangle e^{2\pi i k(x-\delta_k)} &= \sum_{|k|\le N} e^{2\pi i k\delta_k}\langle f, e^{2\pi i k\cdot}\rangle e^{-2\pi i k\delta_k} e^{2\pi i kx} \\ &= \sum_{|k|\le N} \langle f, e^{2\pi i k\cdot}\rangle e^{2\pi i kx}, \end{aligned}$$

where we have used that the coefficient functional of $e^{2\pi i n(x-\delta_n)}$ is just $e^{2\pi i n(x-\delta_n)}$ since $\{e^{2\pi i k(x-\delta_k)}\}_k$ is an orthonormal system in $L^2[0,1)$. ■

3.5.1 Periodic Shannon Wavelets

Our goal in this section is to construct periodic HNWPs that are equivalent in $L^p[0,1)$ to small perturbations of the periodic Shannon wavelet packets. To get such results we need some results on the periodic Shannon wavelets. The Shannon wavelet is not in $L^1(\mathbb{R})$ so it does not really make sense to try to periodize it. However, if we view the Shannon filter as the limit of a sequence of Meyer filters, we obtain the following natural definition of the periodized Shannon system

Definition 3.20 (Periodic Shannon Wavelets). *Let* $\Sigma_0 = 1$. *For* $n = 2^J + k, 0 \le k < 2^J$, $J \ge 0$, *we define* Σ_n *by*

$$\Sigma_n(x) = f_J(x - 2^{-J}k),$$

where

$$f_J(x) = 2^{-J/2} \sum_{\ell=2^{J-1}}^{2^J} b(\ell)\left[e^{2\pi i\ell/2^{J+1}} e^{-2\pi i\ell x} + e^{-2\pi i\ell/2^{J+1}} e^{2\pi i\ell x}\right],$$

and

$$b(\ell) = \begin{cases} 1/\sqrt{2}, & \text{if } \ell \in \{2^j\}_{j\geq 0}, \\ 1, & \text{otherwise.} \end{cases}$$

We call $\{\Sigma_n\}_{n=0}^{\infty}$ *the family of periodic Shannon wavelets.*

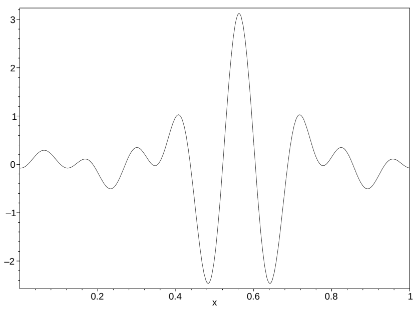

Figure 3.1: The function $f_3(\cdot - 1/2)$.

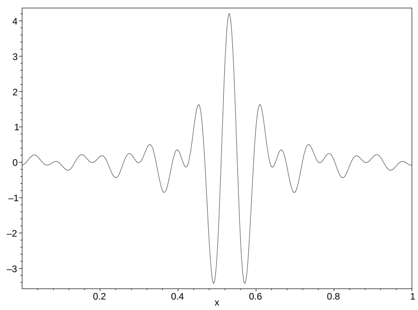

Figure 3.2: The function $f_4(\cdot - 1/2)$.

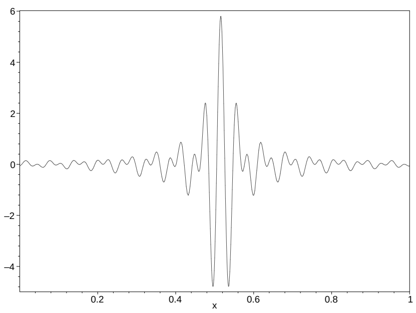

Figure 3.3: The function $f_5(\cdot - 1/2)$.

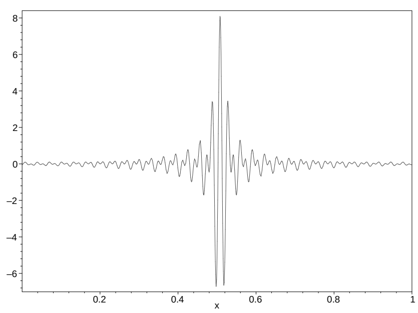

Figure 3.4: The function $f_6(\cdot - 1/2)$.

Since any finite subset of $\{\Sigma_n\}_{n\geq 0}$ is a subsystem of a periodized Meyer wavelet system (the Meyer wavelet needed depends on the subset of $\{\Sigma_n\}_{n\geq 0}$, of course), it follows that the system is indeed an orthonormal basis for $L^2[0,1)$. First, let us show that the periodic Shannon wavelets are equivalent to the Haar system in $L^p[0,1)$. We will need the following lemma by P. Wojtaszczyk,

Lemma 3.21 ([48]). *Let f be a trigonometric polynomial of degree n. Then there exists a constant $C > 0$ such that*

$$Mf(x) \geq C \sup_{|t-x|\leq \pi/n} |f(t)|,$$

where M is the classical Hardy-Littlewood maximal operator,

to get the following Theorem. The proof is in the spirit of Wojtaszczyk's work [48].

Theorem 3.22. *The periodic Shannon wavelets are equivalent to the (periodic) Haar wavelets in $L^p[0,1]$, $1 < p < \infty$.*

Proof. First, we have to introduce and analyze some auxiliary functions. For $n = 2^J + k$, $0 \leq k < 2^J$ we define

$$\Phi_n(x) = 2^{-(J-1)/2} \sum_{s=2^{J-1}}^{2^J-1} \exp\left\{2\pi i s\left(x - \frac{k+1/2}{2^J}\right)\right\}.$$

Note that

$$e^{-2^{J-1}2\pi i x}\Phi_n(x) = e^{-\pi i(k+1/2)}2^{-(J-1)/2} \sum_{s=0}^{2^{J-1}-1} \exp\left\{2\pi i s\left(x - \frac{k+1/2}{2^J}\right)\right\}. \tag{3.9}$$

In particular, $\{\Phi_{2n}\}_{n\geq 0}$ and $\{\Phi_{2n-1}\}_{n\geq 1}$ are both orthonormal systems, since each of the blocks

$$\{\Phi_{2n}\}_{2^J\leq 2n<2^{J+1}} \qquad \text{and} \qquad \{\Phi_{2n-1}\}_{2^J\leq 2n-1<2^{J+1}}$$

is a unitary image of the orthonormal system

$$\left\{2^{-(J-1)/2} \sum_{s=0}^{2^{J-1}-1} e^{2\pi i s(x-k/2^{J-1})}\right\}_{k=0}^{2^{J-1}-1}.$$

Moreover, it is easy to check that

$$\operatorname{span}\{\Phi_{2n}\}_{0\leq 2n<2^J} = \operatorname{span}\{\Phi_{2n-1}\}_{0\leq 2n-1<2^J} = \operatorname{span}\{e^{2\pi i n x}\}_{n=0}^{2^{J-1}-1}.$$

Let $\{a_k\}_{k\geq 0} \subset \mathbb{C}$ and define

$$\begin{aligned} f(x) &= e^{-2^{J-1}2\pi i x} \sum_{2^J\leq 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}(x) \\ &= 2^{-(J-1)/2}e^{-i\pi/2} \sum_{0\leq 2k<2^J} a_{2^J+2k}\left\{\sum_{s=0}^{2^{J-1}-1} \exp\left\{2\pi i s\left(x - \frac{k}{2^{J-1}} - \frac{1}{2^{J+1}}\right)\right\}\right\} \end{aligned} \tag{3.10}$$

In particular,

$$|f(\tfrac{\ell}{2^{J-1}} + \tfrac{1}{2^{J+1}})| = \tfrac{1}{\sqrt{2}}2^{J/2}|a_{2^J+2\ell}|,$$

since

$$\sum_{s=0}^{2^{J-1}-1} e^{2\pi i(\ell-k)s/2^{J-1}} = 2^{J-1}\delta_{\ell,k}.$$

It follows form Lemma 3.21 and (3.10) that

$$M\Big(\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big)(x) \ge C \sum_{2^J\le 2\ell<2^{J+1}} |a_{2\ell}|[|h_{2\ell}(x)| + |h_{2\ell+1}(x)|]. \tag{3.11}$$

Hence, by using the Littlewood-Paley theorem and the Fefferman-Stein inequality for vector valued maximal functions,

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty} a_{2n}\Phi_{2n}\Big\|_p &= \Big\|a_0\Phi_0 + \sum_{J=0}^{\infty}\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big\|_p \\
&\ge \bigg(\int_0^1 \Big(|a_0\Phi_0|^2 + \sum_{J=0}^{\infty}\Big|\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big|^2\Big)^{p/2} dx\bigg)^{1/p} \\
&\ge C_p\bigg(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty}\Big|M\Big(\sum_{2^J\le 2\ell<2^{J+1}} a_{2\ell}\Phi_{2\ell}\Big)\Big|^2\Big)^{p/2} dx\bigg)^{1/p},
\end{aligned}$$

and by (3.11),

$$\begin{aligned}
&\ge C_p\bigg(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty}\Big(\sum_{2^J\le 2\ell<2^{J+1}} |a_{2\ell}|[|h_{2\ell}| + |h_{2\ell+1}|]\Big)^2\Big)^{p/2} dx\bigg)^{1/p} \\
&\ge C_p\Big\|\sum_{l=0}^{\infty} a_{2n}[h_{2n} + h_{2n+1}]\Big\|_p \\
&\ge C_p\Big\|\sum_{l=0}^{\infty} a_{2n}h_{2n}\Big\|_p,
\end{aligned}$$

where we have used the unconditionality of the Haar system (in particular, the projection onto the even numbered Haar functions is bounded on $L^p[0,1)$, $1<p<\infty$). A similar proof shows that

$$\Big\|\sum_{n=1}^{\infty} a_{2n-1}\Phi_{2n-1}\Big\|_p \ge C_p\Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p.$$

Actually, it is the opposite inequalities we really need. However, since $\mathrm{span}\{\Phi_{2n}\}_{n\ge 0}$ is dense in $H^q[0,1)$, for $f=\sum a_{2n}\Phi_{2n}$ and $\varepsilon>0$ there is a $g=\sum b_{2n}\Phi_{2n}$ with $\|g\|_q\le 1+\varepsilon$ such that

$$\begin{aligned}\|f\|_p-\varepsilon &\le |\langle \sum a_{2n}\Phi_{2n}, \sum b_{2n}\Phi_{2n}\rangle| \\ &= |\langle \sum a_{2n}h_{2n}, \sum b_{2n}h_{2n}\rangle| \\ &\le \Big\|\sum a_{2n}h_{2n}\Big\|_p \Big\|\sum b_{2n}h_{2n}\Big\|_q \\ &\le C\Big\|\sum a_{2n}h_{2n}\Big\|_p \Big\|\sum b_{2n}\Phi_{2n}\Big\|_q .\end{aligned}$$

Since ε was arbitrary, we have

$$\Big\|\sum a_{2n}\Phi_{2n}\Big\|_p \le C\Big\|\sum a_{2n}h_{2n}\Big\|_p,$$

and similarly,

$$\Big\|\sum a_{2n+1}\Phi_{2n+1}\Big\|_p \le C\Big\|\sum a_{2n+1}h_{2n+1}\Big\|_p.$$

Finally, we can prove the theorem. Let $\mathcal{R}$ denote the Riesz projection, i.e. the projection onto $\{e^{2\pi inx}\}_{n\ge 0}$. Then for any finite sequence $\{a_k\}_{k\ge 0}\subset\mathbb{C}$ we have

$$\begin{aligned}\Big\|\sum_{n=0}^{\infty} a_n\Sigma_n\Big\|_p &\le \Big\|\sum_{n=0}^{\infty} a_n\mathcal{R}\Sigma_n\Big\|_p + \Big\|\sum_{n=0}^{\infty} a_n(1-\mathcal{R})\Sigma_n\Big\|_p \\ &\le \Big\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\Big\|_p + \Big\|\sum_{n=1}^{\infty} a_{2n-1}\mathcal{R}\Sigma_{2n-1}\Big\|_p \\ &\qquad + \Big\|\sum_{n=0}^{\infty} a_{2n}(1-\mathcal{R})\Sigma_{2n}\Big\|_p + \Big\|\sum_{n=1}^{\infty} a_{2n-1}(1-\mathcal{R})\Sigma_{2n-1}\Big\|_p. \end{aligned} \tag{3.12}$$

Let $P: L^p[0,1)\to L^p[0,1)$ denote the bounded projection[1] onto the frequencies $\{e^{2\pi i 2^j x}\}_{j\ge 0}$. We have, using Khintchine's inequality for lacunary Fourier series (see [45, I.B.8]),

$$\begin{aligned}\Big\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\Big\|_p &\le \Big\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\Big\|_p + \Big\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p \\ &\le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\Big\|_2 + \Big\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p\Big).\end{aligned}$$

[1]The operator P is bounded since for $2\le p<\infty$,

$$\Big\|\sum_{j\ge 0} c_{2^j}e^{i2^j x}\Big\|_{L^p[0,1)} \le C\Big\|\sum_{j\ge 0} c_{2^j}e^{i2^j x}\Big\|_{L^2[0,1)} \le C\Big\|\sum_{k\in\mathbb{Z}} c_k e^{ikx}\Big\|_{L^2[0,1)} \le C\Big\|\sum_{k\in\mathbb{Z}} c_k e^{ikx}\Big\|_{L^p[0,1)},$$

where we have used Khintchine's inequelity for lacunary Fourier series. The case $1<p<2$ follows by duality.

A direct calculation shows that

$$P\Big(\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\mathcal{R}\Sigma_{2^J+2\ell}\Big) = 2^{-(J+1)/2}\Big\{\Big(e^{-i\pi/2}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^{J-1}x} + \Big(e^{-i\pi}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^J x}\Big\},$$

whereas,

$$P\Big(\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Phi_{2^J+2\ell}\Big) = 2^{-(J-1)/2}\Big(e^{-i\pi/2}\sum_{0\le 2\ell<2^J} a_{2^J+2\ell}\Big)e^{2\pi i 2^{J-1}x}.$$

Thus,

$$\Big\|\sum_{n=0}^{\infty} a_{2n}P\mathcal{R}\Sigma_{2n}\Big\|_2 \le \Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n}\Big\|_2,$$

and we get

$$\begin{aligned}
\Big\|\sum_{n=0}^{\infty} a_{2n}\mathcal{R}\Sigma_{2n}\Big\|_p &\le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n}\Big\|_2 + \Big\|\sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p\Big)\\
&\le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n} + \sum_{n=0}^{\infty} a_{2n}(1-P)\mathcal{R}\Sigma_{2n}\Big\|_p\Big)\\
&= C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}P\Phi_{2n} + \sum_{n=0}^{\infty} a_{2n}(1-P)\Phi_{2n}\Big\|_p\Big)\\
&= C\Big\|\sum_{n=0}^{\infty} a_{2n}\Phi_{2n}\Big\|_p\\
&\le C\Big\|\sum_{n=0}^{\infty} a_{2n}h_{2n}\Big\|_p.
\end{aligned}$$

Similarly, we obtain

$$\Big\|\sum_{n=1}^{\infty} a_{2n-1}\mathcal{R}\Sigma_{2n-1}\Big\|_p \le C\Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p.$$

The remaining two terms in (3.12) can easily be estimated by assuming (w.l.o.g., of course) that $\{a_k\}\subset\mathbb{R}$ and taking complex conjugates of the above estimates (note, the coefficients at negative frequencies of Σ_n are just the conjugate of the coefficients at positive frequencies). We conclude that

$$\Big\|\sum_{n=0}^{\infty} a_n\Sigma_n\Big\|_p \le C\Big(\Big\|\sum_{n=0}^{\infty} a_{2n}h_{2n}\Big\|_p + \Big\|\sum_{n=1}^{\infty} a_{2n-1}h_{2n-1}\Big\|_p\Big) \le C\Big\|\sum_{n=0}^{\infty} a_nh_n\Big\|_p,$$

where we have used that the projection onto the even numbered Haar functions is bounded on $L^p[0,1)$. To obtain the opposite inequality, we let $\varepsilon > 0$ and let $f = \sum a_n h_n$. The Haar system is dense in $L^q[0,1]$ so there is a function $g = \sum b_n h_n \in \operatorname{span}(h_n)$ with $\|g\|_q \le 1+\varepsilon$ such that

$$\begin{aligned}
\|f\|_p - \varepsilon &\le |\langle \sum a_n h_n, \sum b_n h_n \rangle| \\
&= |\langle \sum a_n \Sigma_n, \sum b_n \Sigma_n \rangle| \\
&\le \Big\| \sum a_n \Sigma_n \Big\|_p \Big\| \sum b_n \Sigma_n \Big\|_q \\
&\le C \Big\| \sum a_n \Sigma_n \Big\|_p \Big\| \sum b_n h_n \Big\|_q \\
&\le C(1+\varepsilon) \Big\| \sum a_n \Sigma_n \Big\|_p ,
\end{aligned}$$

where we have used the orthonormality of the system Σ_n. Since ε was arbitrary we have

$$\Big\| \sum a_n h_n \Big\|_p \le C \Big\| \sum a_n \Sigma_n \Big\|_p ,$$

and we are done. ■

The following Theorem is the periodic analog to Theorem 2.10. The proof is new, but the result is due to Meyer ([28])

Theorem 3.23. *Let $\{\Psi_n\}_n$ be a periodic wavelet system associated with a wavelet ψ satisfying $|\psi(x)| \le C(1+|x|)^{-2-\varepsilon}$. Then $\{\Psi_n\}_n$ is equivalent to the (periodic) Haar wavelets in $L^p[0,1]$.*

Proof. By duality, it suffices to prove that

$$\Big\| \sum_{n=0}^{\infty} a_n \Psi_n \Big\|_p \ge C \Big\| \sum_{n=0}^{\infty} a_n h_n \Big\|_p .$$

We have, by the Fefferman-Stein inequality,

$$\begin{aligned}
\Big\| \sum_{n=0}^{\infty} a_n \Psi_n \Big\|_p &= \Big\| a_0 \Psi_0 + \sum_{J=0}^{\infty} \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big) \Big\|_p \\
&\ge C \Big(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty} \Big| \sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big|^2 \Big)^{p/2} dx \Big)^{1/p} \\
&\ge C \Big(\int_0^1 \Big(|a_0|^2 + \sum_{J=0}^{\infty} \Big| M \Big(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \Big) \Big|^2 \Big)^{p/2} dx \Big)^{1/p}
\end{aligned}$$

It follows form [46, p. 208] that for $n = 2^J + k$,

$$|\Psi_n(x)| \le C2^{J/2}(1 + 2^J|x - k/2^J|)^{-1-\varepsilon}. \tag{3.13}$$

Hence, for $x \in [k2^{-J}, (k+1)2^{-J})$ (see [39, pp. 62-63]),

$$|a_n| = \left| \int_0^1 \left(\sum_{\ell=2^J}^{2^{J+1}-1} a_\ell \Psi_\ell(y) \right) \overline{\Psi_n(y)}\, dy \right| \le C2^{-J/2} M\left(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \right)(x),$$

where we have used the estimate (3.13), which shows that $2^{J/2}|\Psi_n|$ is an approximation of the identity centered at $k2^{-J}$. Thus

$$M\left(\sum_{k=2^J}^{2^{J+1}-1} a_k \Psi_k \right) \ge C \sum_{k=2^J}^{2^{J+1}-1} |a_k||h_k|,$$

and we have

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} a_n \Psi_n \right\|_p &\ge C \left(\int_0^1 \left(|a_0|^2 + \sum_{s=0}^{\infty} \left| \sum_{k=2^J}^{2^{J+1}-1} |a_k| h_k| \right|^2 \right)^{p/2} dx \right)^{1/p} \\ &\ge C \left\| \sum_{n=0}^{\infty} a_n h_n \right\|_p . \end{aligned}$$

■

The following corollary is immediate

Corollary 3.24. *Let $\{\Psi_n\}_n$ be a periodic wavelet packet system associated with a wavelet ψ satisfying $|\psi(x)| \le C(1+|x|)^{-2-\varepsilon}$. Then $\{\Psi_n\}_n$ is equivalent to the periodic Shannon wavelets in $L^p[0,1]$, $1 < p < \infty$.*

We let $\{w_n\}_n$ be a HNWP system for which $|w_1(x)| \le C(1+|x|)^{-2-\varepsilon}$, and let $\{\widetilde{w}_n\}_n$ be the corresponding periodic system. For $2^J \le n \le 2^{J+1}$ write

$$\widetilde{w}_n(x) = \sum_{s=2^J}^{2^{J+1}} c_{n,s} \Psi_s(x),$$

where Ψ_n is the corresponding periodic wavelet. Define a new system $\{\widetilde{w}_n^S\}$ by

$$\widetilde{w}_n^S(x) = \sum_{s=2^J}^{2^{J+1}} c_{n,s} \Sigma_s(x),$$

where Σ_s is the periodic Shannon wavelets. Then we have the following result

Corollary 3.25. *The systems $\{\widetilde{w}_n\}_n$ and $\{\widetilde{w}_n^S\}_n$ are equivalent in $L^p[0,1)$, $1 < p < \infty$, in the sense that there exists an isomorphism Q on $L^p[0,1)$ such that*

$$Q\widetilde{w}_n = \widetilde{w}_n^S.$$

Proof. Take Q to be the isomorphism from Corollary 3.24 defined by $Q\Psi_n = \Sigma_n$. ■

Remark. The significance of the previous Corollary is that when dealing with periodic HNWPs $\{\widetilde{w}_n\}_n$ in $L^p[0,1)$, we may assume that the wavelet $\psi = w_1$ is a Meyer wavelet $\psi^{M,\delta}$ with arbitrarily good frequency localization, i.e. $\psi(\xi) = 1$ for $|\xi| \in (\pi+\delta, 2\pi-\delta)$ for a small number δ. To see this, let $\{\widetilde{w}_n^{M,\delta}\}_n$ be the periodic HNWP system obtained using the same filters that generated $\{\widetilde{w}_n\}_n$ but with $\psi^{M,\delta}$ as the wavelet. From the previous discussion of the periodic Meyer wavelets we see that by periodizing $\psi_{j,k}^{M,\delta}$ we get exactly $\widetilde{S_n}$ for $n \le N(\delta)$, where $N(\delta) \to \infty$ as $\delta \to 0$. Hence, $\widetilde{w}_n^S = \widetilde{w}_n^{M,\delta}$ for $j \le N$, and $\widetilde{w}_n^S$ can be mapped onto $\widetilde{w}_n$ by the isomorphism of Corollary 3.25.

3.5.2 Perturbation of Periodic Shannon Wavelet Packets

We need the following perturbation theorem by Krein and Liusternik (see [49])

Theorem 3.26. *Let $\{x_n\}$ be a Schauder basis for a Banach space X and let $\{f_n\}$ be the associated sequence of coefficient functionals. If $\{y_n\}$ is a sequence of vectors in X with dense linear span and if*

$$\sum_{n=1}^{\infty} \|x_n - y_n\|_X \cdot \|f_n\|_{X^*} < \infty$$

then $\{y_n\}$ is a Schauder basis for X equivalent to $\{x_n\}$.

to prove our main theorem on periodic HNWPs;

Theorem 3.27. *Let $\{d_n\}_{n=0}^{\infty} \subset 2\mathbb{N}$ be such that $d_n \ge Cn4^n \log(n+1)$ for some constant $C > 0$. Let $\{\widetilde{w}_n\}_n$ be a periodic HNWP system (in frequency order) given by the filters $\{m_0^{n,q}\}_{n\ge1, 1\le q\le n}$, where*

$$m_0^{n,q}(\xi) = m_0^{(d_n)}(\xi), \quad q = 1, 2, \dots, n,$$

is the Daubechies filter of length d_n. Suppose $|w_1(x)| \le C(1+|x|)^{-2-\varepsilon}$ for some $\varepsilon > 0$. Then $\{\widetilde{w}_n\}_n$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$.

Proof. By the remark at the end of the previous section, we may assume that w_1 is a Meyer wavelet with arbitrarily good frequency localization. We note that since $\{\widetilde{w}_n\}_n$ is orthonormal in $L^2[0,1)$, a simple duality argument will give us the result for $2 < p < \infty$ if we can prove it for $1 < p < 2$. Fix $1 < p < 2$. Define the phase functions $\eta_n : \mathbb{R} \to [0, 2\pi)$ by

$$|m_0^{(d_n)}(\xi)| = e^{-i\eta_n(\xi)}\, m_0^{(d_n)}(\xi).$$

Define a family of low-pass filters by

$$m_0^{n,q}(\xi) = e^{i\eta_n(\xi)} m_0^{M,\delta},$$

where $m_0^{M,\delta}$ is a Meyer filter with localization δ. Take $\psi^{M,\delta}$ as the wavelet and consider the corresponding periodic HNWPs $\{\widetilde{w}_n^{M,\delta}\}_n$. For fixed n, there is a $\delta_n > 0$ such that $0 < \delta, \tilde{\delta} < \delta_n$ implies that $\widetilde{w}_n^{M,\delta} = \widetilde{w}_n^{M,\tilde{\delta}}$. Let $\widetilde{w}_n^{M}$ denote this limit function. It follows from Theorem 3.17 and the proof of Theorem 3.19 that $\{\widetilde{w}_n^{M}\}_{n=0}^{\infty}$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, consisting of shifted sines and cosines (more precisely, $\widetilde{w}_n^{M}$ is a shifted version of $\widetilde{S}_n$). The property of this new basis we need, is that the Fourier coefficients of $\widetilde{w}_n^{M}$ have the same phase (but not the same length) as the the Fourier coefficients of $\widetilde{w}_n$. We want to apply the perturbation result (Theorem 3.26), so we need to show that

$$\sum_{n=0}^{\infty} \|\widetilde{w}_n - \widetilde{w}_n^M\|_p \cdot \|\widetilde{w}_n^M\|_q \simeq \sum_{n=0}^{\infty} \|\widetilde{w}_n - \widetilde{w}_n^M\|_p < \infty.$$

However, by Hölder's inequality

$$\sum_{n=0}^{\infty} \|\widetilde{w}_n - \widetilde{w}_n^M\|_p \leq \sum_{n=0}^{\infty} \|\widetilde{w}_n - \widetilde{w}_n^M\|_2,$$

so it suffices to estimate $\|\widetilde{w}_n - \widetilde{w}_n^M\|_2$. To ensure that

$$\sum_{n=0}^{\infty} \|\widetilde{w}_n - \widetilde{w}_n^M\|_2 < \infty \tag{3.14}$$

we will show that for $2^J \leq n < 2^{J+1}$,

$$\|\widetilde{w}_n - \widetilde{w}_n^M\|_2 \leq C 2^{-J} J^{-1} \log(J)^{-2},$$

with C a constant independent of J.

The Fourier series for $\widetilde{w}_n^M$ is particularly simple and contains only two non-zero terms, equal to $e^{\pm i\alpha} 2^{-1/2}$, where $\alpha \in \mathbb{R}$ depends on the phase of the Daubechies filters used to generate $\{\widetilde{w}_n\}_n$. We want to estimate the corresponding two most significant coefficients in the Fourier series for $\widetilde{w}_n$. We have, for $2^J \leq n < 2^{J+1}$,

$$\widetilde{w}_n(x) = \sum_{k \in \mathbb{Z}} \hat{w}_n(2\pi k) e^{2\pi i k x},$$

and we can choose $w_1 = \psi^{M,\delta}$ to be a Meyer wavelet with sufficiently good frequency localization so this reduces to the following trigonometric polynomial

$$\widetilde{w}_n(x) = \sum_{2^{J-1} \leq |k| < 2^J} \hat{w}_n(2\pi k) e^{2\pi i k x}.$$

Recall that

$$\hat{w}_n(\xi) = m_{\varepsilon_1}^{(d_J)}(\xi/2) m_{\varepsilon_2}^{(d_J)}(\xi/4) \cdots m_{\varepsilon_J}^{(d_J)}(\xi/2^J) \hat{\psi}^{M,\delta}(\xi/2^J),$$

where $G(n) = \sum_{j=1}^{J+1} \varepsilon_j 2^{j-1}$ is the binary expansion of of the Gray-code permutation of n. Let k_n be the positive index corresponding to the significant coefficient of $\widetilde{w}_n^M$. We deduce from Theorem 3.17 that

$$m_{\varepsilon_1}^{(d_J)}(2\pi k_n/2) m_{\varepsilon_2}^{(d_J)}(2\pi k_n/4) \cdots m_{\varepsilon_J}^{(d_J)}(2\pi k_n/2^J) \hat{\psi}^{M,\delta}(2\pi k_n/2^J)$$

has exactly one factor equal to $2^{-1/2}$ in absolute value, namely the factor with argument $2^{-s} 2\pi k_n$ satisfying

$$\frac{2\pi k_n}{2^s} \in \frac{\pi}{2} + 2\pi\mathbb{Z}.$$

The arguments of the remaining factors are at least a distance of $2^{1-J}\pi$ from the set $\pi/2 + 2\pi\mathbb{Z}$. Moreover, Theorem 3.17 shows that the arguments of the remaining J factors are situated where the respective m_ε's are "big", i.e. in the set $[-\pi/2, \pi/2]$ for the low-pass filters appearing in the product and in the set $[-\pi, -\pi/2] \cup [\pi/2, \pi]$ for the high-pass filters appearing in the product.

Notice that, by construction, the Fourier coefficients of $\widetilde{w}_n$ and $\widetilde{w}_n^M$ have the same phase. Also, the Fourier series of $\widetilde{w}_n^M$ contains only two non-zero terms and $\widetilde{w}_n$ is normalized in $L^2[0,1)$. From this we see that to show $\|\widetilde{w}_n - \widetilde{w}_n^M\|_2 \leq C 2^{-J} J^{-1} \log(J)^{-2}$, for $2^J \leq n < 2^{J+1}$, it suffices to verify that

$$|m_{\varepsilon_1}^{(d_J)}(2\pi k_n/2) m_{\varepsilon_2}^{(d_J)}(2\pi k_n/4) \cdots m_{\varepsilon_J}^{(d_J)}(2\pi k_n/2^J) \hat{\psi}^{M,\delta}(2\pi k_n/2^J)| \geq 2^{-1/2}\left(1 - \frac{C}{2^J J \log^2(J)}\right), \tag{3.15}$$

for some constant C independent of J. The d_J's have already been chosen, so we will work our way back to see that everything works out. We now consider (3.15) as an inequality in $d_J = N(J)$. Hence, (3.15) will be satisfied if

$$|m_0^{(N(J))}(\pi/2 - 2^{1-J}\pi)| \geq \left(1 - \frac{C}{2^J J \log^2(J)}\right)^{1/J}. \tag{3.16}$$

By the CQF conditions, (3.16) is equivalent to

$$|m_0^{(N(J))}(\pi/2 + 2^{1-J}\pi)|^2 \leq 1 - \left(1 - \frac{C}{2^J J \log^2(J)}\right)^{2/J}.$$

From lemma 2.55 we have

$$|m_0^{(N(J))}(\pi/2+2^{1-J}\pi)| \leq |\cos(2^{1-J}\pi)|^{N(J)-1},$$

which gives us an explicit way to pick a sequence $N(J)$ that works. We put

$$\cos(2^{1-J}\pi)^{2(N(J)-1)} \leq 1-\left(1-\frac{C}{2^J J\log^2(J)}\right)^{2/J}$$

A simple estimate shows that

$$1-\left(1-\frac{C}{2^J J\log^2(J)}\right)^{2/J} \leq \frac{\tilde{C}}{2^J J\log^2(J)}.$$

Hence,

$$2(N(J)-1)\log\cos(2^{1-J}\pi) \leq C-(J+\log(J)+2\log\log(J)) \tag{3.17}$$

Using

$$\log\cos(x) = -\frac{1}{2}x^2+O(x^4), \quad \text{as } x\to 0,$$

in (3.17), we see that choosing

$$N(J) \geq CJ2^{2J}\log(J)$$

for any $C>0$ will work. This is exactly our hypothesis about the d_J's. ■

Remark. It follows from the above estimates that the factor $\log(n+1)$ in the hypothesis about the sequence $\{d_n\}$ can be replaced by α_n with $\{\alpha_n\}$ any positive increasing sequence with $\alpha_n\to\infty$.

Chapter 4

Modified Hilbert Transforms

The classical Hilbert transform defined on $L^2[0,1)$ is given by

$$H(e^{2\pi inx}) = -i\,\mathrm{sgn}(n)\,e^{2\pi inx}, \tag{4.1}$$

and it is well known that H extends to a bounded operator on $L^p[0,1)$. Note that an equivalent definition of H is given by requiring that $H(1) = 0$, $H(\cos(2\pi nx)) = \sin(2\pi nx)$, and $H(\sin(2\pi nx)) = -\cos(2\pi nx)$ for $n \geq 1$. We emulate this last definition, using certain Walsh type wavelet packets in place of the trigonometric system, to get a family of transforms bounded on $L^p[0,1)$ for each $1 < p < \infty$.

4.1 A Modified Hilbert Transform for the Walsh System

Guided by (4.1), we want to define a modified Hilbert transform for the Walsh system $\{W_n\}_{n=0}^{\infty}$. First, we define the binary operator $\dot{+} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by

$$m \dot{+} n = \sum_{i=0}^{\infty} |m_i - n_i| 2^i,$$

where $m = \sum_{i=0}^{\infty} m_i 2^i$ and $n = \sum_{i=0}^{\infty} n_i 2^i$, and define the notion of $\mathbb{Z}_2$-linearity

Definition 4.1. *Let $T : \mathbb{N}_0 \to \mathbb{N}_0$ be a permutation. T is called $\mathbb{Z}_2$-linear if $T(m \dot{+} n) = T(m) \dot{+} T(n)$.*

The most important example if such an operator is given by the Gray-code permutation:

Example 4.2. *One important example of such a map is the Gray-code permutation $G : \mathbb{N}_0 \to$*

$\mathbb{N}_0$, *defined by* $G(n)_i = n_i \dot{+} n_{i+1}$. *To see that it is* $\mathbb{Z}_2$*-linear note that*

$$\begin{aligned} G(n \dot{+} m)_i &= (n_i \dot{+} m_i) \dot{+} (n_{i+1} \dot{+} m_{i+1}) \\ &= (n_i \dot{+} n_{i+1}) \dot{+} (m_i \dot{+} m_{i+1}) \\ &= G(n)_i \dot{+} G(m)_i. \end{aligned}$$

We can now define the transformation.

Definition 4.3. *Let* T *be any* $\mathbb{Z}_2$*-linear permutation of* $\mathbb{N}_0$. *We define the modified Hilbert transform* $\mathcal{H}_T$ *by*

$$\mathcal{H}_T(W_{T(n)}) = \begin{cases} 0, & \text{if } n = 0, \\ W_{T(n-1)}, & \text{if } n \text{ is even,} \\ -W_{T(n+1)}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we let $\cos(2\pi nx)$ correspond to $W_{T(2n)}(x)$ and $\sin(2\pi nx)$ correspond to $W_{T(2n-1)}$. It is obvious that $\mathcal{H}_T$ is bounded on $L^2[0,1)$ (of norm 1). The Walsh system and the trigonometric system share a number of properties, however, they are not equivalent bases in $L^p[0,1)$ for $p \neq 2$ (see [35]) so $\mathcal{H}_T$ is not trivially bounded on $L^p[0,1)$. The main result of the next section is that $\mathcal{H}_T$ can be extended to a bounded operator on $L^p[0,1)$ for $1 < p < \infty$.

4.1.1 Boundedness of the Operator $\mathcal{H}_T$

The following Lemma shows that there is an important metric relationship between the Walsh systems $\{W_n(x)\}_{n=0}^{\infty}$ and $\{W_{T(n)}(x)\}_{n=0}^{\infty}$ for T any $\mathbb{Z}_2$-linear map.

Lemma 4.4 ([35]). *Let* $T : \mathbb{N}_0 \to \mathbb{N}_0$ *be a* $\mathbb{Z}_2$*-linear permutation. Then there exist measure preserving mappings* $\mathcal{M}$, $\tilde{\mathcal{M}} : [0,1) \to [0,1)$ *such that*

$$\begin{aligned} W_{T(n)}(x) &= W_n(\mathcal{M}(x)), \\ W_n(x) &= W_{T(n)}(\tilde{\mathcal{M}}(x)). \end{aligned}$$

We use Lemma 4.4 to prove the following result.

Lemma 4.5. *Let* $T : \mathbb{N}_0 \to \mathbb{N}_0$ *be a* $\mathbb{Z}_2$*-linear permutation. Then* $\{W_n\}_{n=0}^{\infty}$ *and* $\{W_{T(n)}\}_{n=0}^{\infty}$ *are equivalent Schauder bases for* $L^p[0,1)$, $1 < p < \infty$.

Proof. Let $f = \sum_{n=0}^{N} a_n W_n$, and $\tilde{f} = \sum_{n=0}^{N} a_n W_{T(n)}$. Then, for $1 < p < \infty$,

$$\begin{aligned}
\|f\|_p^p &= \int_0^1 \Big| \sum_{n=0}^{N} a_n W_n(x) \Big|^p \, dx \\
&= \int_0^1 \Big| \sum_{n=0}^{N} a_n W_n(\mathcal{M}(x)) \Big|^p \, dx \\
&= \int_0^1 \Big| \sum_{n=0}^{N} a_n W_{T(n)}(x) \Big|^p \, dx \\
&= \|\tilde{f}\|_p^p.
\end{aligned}$$

It follows that $\{W_n\}_{n=0}^{\infty}$ and $\{W_{T(n)}\}_{n=0}^{\infty}$ are equivalent systems on $L^p[0,1)$. Since $\{W_n\}_{n=0}^{\infty}$ is a Schauder basis for $L^p[0,1)$, the Lemma follows. ■

We also need the following result:

Lemma 4.6 ([35]). *The operators $\mathcal{W}^{(\pm)} : L^2[0,1) \to L^2[0,1)$, defined by*

$$\mathcal{W}^{(\pm)} W_n = W_{n \pm 1},$$

where we let $W_{-1} = 0$, extend to bounded operators on $L^p[0,1)$ for $1 < p < \infty$.

Lemma 4.5 shows that there is an isomorphism on $L^p[0,1)$, $1 < p < \infty$, mapping W_n onto $W_{T(n)}$. Using this result, Lemma 4.6, and the next Lemma, we can decompose $\mathcal{H}_T$ into a sum of bounded operators.

Lemma 4.7. *Let $P : L^2[0,1) \to L^2[0,1)$ be the operator defined by*

$$P\Big\{ \sum_{n=0}^{2N} a_n W_{T(n)} \Big\} = \sum_{n=0}^{N} a_{2n} W_{T(2n)}.$$

Then P extends to a bounded operator on $L^p[0,1)$, $1 < p < \infty$.

Proof. First we prove that P extends to a bounded operator on $L^{2k}[0,1)$ for $k = 1, 2, \ldots$. Let $f = \sum_{n=0}^{N} a_n W_{T(n)}$ be a Walsh polynomial with real coefficients. Then

$$\begin{aligned}
f^{2k} &= \sum_{i_1=0}^{N} \sum_{i_2=0}^{N} \cdots \sum_{i_{2k}=0}^{N} a_{i_1} a_{i_2} \cdots a_{i_{2k}} W_{T(i_1)} W_{T(i_2)} \cdots W_{T(i_{2k})} \\
&= \sum_{i_1=0}^{N} \sum_{i_2=0}^{N} \cdots \sum_{i_{2k}=0}^{N} a_{i_1} a_{i_2} \cdots a_{i_{2k}} W_{T(i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k})}.
\end{aligned}$$

Hence,

$$\|f\|_{2k}^{2k} = \sum_{i_1=0}^{N} \sum_{i_2=0}^{N} \cdots \sum_{i_{2k}=0}^{N} a_{i_1} a_{i_2} \cdots a_{i_{2k}} \int_0^1 W_{T(i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k})}(x)\, dx.$$

Note that

$$\int_0^1 W_{T(i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k})}(x)\, dx \neq 0 \Longleftrightarrow T(i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k}) = 0,$$

and that $T(i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k}) = 0$ if and only if $i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k} = 0$. But $i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k} = 0$ only if the number of odd indices in the set $\{i_1, i_2, \dots i_{2k}\}$ is even. Let

$$c(n) = \begin{cases} 1 & \text{if n is even} \\ -1 & \text{if n is odd} \end{cases},$$

and let $\tilde{f} = \sum_{n=0}^{N} c(n) a_n W_{T(n)}$. The above argument shows that if $i_1 \dotplus i_2 \dotplus \cdots \dotplus i_{2k} = 0$ then

$$a_{i_1} a_{i_2} \cdots a_{i_{2k}} = c(i_1) a_{i_1} c(i_2) a_{i_2} \cdots c(i_{2k}) a_{i_{2k}}.$$

We conclude that f and $\tilde{f}$ have the same $L^{2k}[0,1)$-norm. Hence,

$$\|Pf\|_{2k} = \frac{1}{2}\|f + \tilde{f}\|_{2k} \leq \|f\|_{2k}.$$

The family $\{W_{T(n)}\}_{n=0}^{\infty}$ is a Schauder basis for $L^{2k}[0,1)$ so it follows that P extends to a bounded operator on $L^{2k}[0,1)$. Using the Riesz-Thorin interpolation theorem, we conclude that P extends to a bounded operator on $L^p[0,1)$ for $2 \leq p < \infty$. A simple duality argument shows that P is also bounded on $L^p[0,1)$ for $1 < p < 2$. ■

We are ready to prove the main result.

Theorem 4.8. *Let $T : \mathbb{N}_0 \to \mathbb{N}_0$ be a $\mathbb{Z}_2$-linear permutation. Then the operator $\mathcal{H}_T : L^2[0,1) \to L^2[0,1)$ extends to a bounded operator on $L^p[0,1)$ for $1 < p < \infty$.*

Proof. It follows from Lemma 4.6 that there exists an isomorphism, K, on $L^p[0,1)$ such that $KW_n = W_{T(n)}$. We have,

$$\mathcal{H}_T = K\mathcal{W}^{(-)}K^{-1}P_T - K\mathcal{W}^{(+)}K^{-1}(I - P_T),$$

a composition of bounded operators on $L^p[0,1)$ for $1 < p < \infty$. ■

4.1.2 Walsh Exponentials

An important fact about the trigonometric system is that it is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, and we have

$$\lim_{M,N\to\infty} \sum_{-M\le n\le N} \hat{f}(n)e^{2\pi inx} = f,$$

for every $f \in L^p[0,1)$, with convergence in $L^p[0,1)$-norm. A closely related property is that the Riesz projection, given by

$$R\Big\{ \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}\Big\} = \sum_{n=0}^{\infty} \hat{f}(n)e^{2\pi inx},$$

is bounded on $L^p[0,1)$. We want to emulate the exponential system using Walsh functions. One way to do that is to define a set of Walsh exponentials by the following.

Definition 4.9. *Let T be any $\mathbb{Z}_2$-linear permutation of $\mathbb{N}_0$. Then we let $e_0(x) = 1$ and define*

$$e_n(x) = \frac{1}{\sqrt{2}}\left\{(W_{T(2|n|)}(x) + i\, sgn(n)\, W_{T(2|n|-1)}(x)\right\}, \qquad n \in \mathbb{Z} - \{0\}.$$

It is immediate that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1)$. Moreover, the Walsh exponentials and $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ share the following property.

Theorem 4.10. *The system $\{e_n\}_{n\in\mathbb{Z}}$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, in the sense that*

$$\lim_{M,N\to\infty} \sum_{-M\le n\le N} (f, e_n)e_n = f \tag{4.2}$$

for all $f \in L^p[0,1)$, $1 < p < \infty$, with convergence in $L^p[0,1)$-norm.

Proof. Clearly, $\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^p[0,1)$ so it suffices to prove that the partial sum operators $S_{-M,N}(f)$, defined by

$$S_{-M,N}(f) = \sum_{-M\le n\le N} (f, e_n)e_n,$$

are uniformly bounded on each $L^p[0,1)$. Let $f = \sum_{n=0}^{\infty} a_n W_{T(n)} \in L^p[0,1)$. Note that $(f,1) = a_0$, and

$$(f, e_n) = \frac{1}{\sqrt{2}}\left\{a_{|2n|} - i\,\mathrm{sgn}(n)\, a_{2|n|-1}\right\}, \qquad n \in \mathbb{Z} - \{0\}. \tag{4.3}$$

We have

$$\begin{aligned} S_{-M,N}(f) &= \sum_{-M\le n\le N} (f, e_n)e_n \\ &= \sum_{-M\le n<0} (f, e_n)e_n + (f,1)1 + \sum_{0<n\le N} (f, e_n)e_n \\ &= I + II + III. \end{aligned}$$

Clearly, $\|II\|_p \le \|f\|_p$. Let $S_N(f) = \sum_{0<n\le N} a_n W_{T(n)}$ and let K be defined as in Theorem 4.8, P be defined as in Lemma 4.7, and $\mathcal{W}^{(\pm)}$ be defined as in Lemma 4.6. Then, using (4.3) and Definition 4.9,

$$\begin{aligned}\sqrt{2}I = \sum_{-M\le n<0} & \Big\{a_{2|n|}W_{T(2|n|)} - ia_{2|n|}W_{T(2|n|-1)} + ia_{2|n|-1}W_{T(2|n|)} \\ & + a_{2|n|-1}W_{T(2|n|-1)}\Big\} \\ = [PS_M & - iK\mathcal{W}^{(-)}K^{-1}PS_M + iK\mathcal{W}^{(+)}K^{-1}(I-P)S_M + (I-P)S_M](f),\end{aligned}$$

where all the operators inside the brackets are bounded on $L^p[0,1)$. The same argument can be used to estimate II, and (4.2) follows. ■

An easy corollary of the proof of Theorem 4.10 is the following.

Corollary 4.11. *The Riesz-projection, defined for $f = \sum_{n=-M}^{N}(f,e_n)e_n$ by*

$$\mathcal{R}\Big\{\sum_{n=-M}^{N}(f,e_n)e_n\Big\} = \sum_{n=0}^{N}(f,e_n)e_n,$$

extends to a bounded operator on $L^p[0,1)$, $1<p<\infty$.

Proof. Follows from the estimate of II and III in the proof of Theorem 4.10. ■

4.2 Periodic Walsh Type Systems

We now want to extend all of the results to the periodic Walsh type wavelet packets. First we deal with the shift operator and the projection operator onto the even numbered periodic Walsh type wavelet packets.

4.2.1 The Shift and Projection Operator

The next lemma shows that the shift operator is bounded for the periodic Walsh type wavelet packets. We show later in this chapter the such a result may fail for more general periodic wavelet packets.

Lemma 4.12. *Let $\{w_n\}_n$ be a wavelet packet system satisfying the hypothesis of Theorem 2.12, and let $\{\widetilde{w_n}\}_n$ be the associated periodic system. Then the shift operators $\mathcal{S}^{\pm}$, defined by*

$$\mathcal{S}^{\pm}\widetilde{w_n} = \widetilde{w_{n\pm 1}},$$

with $\widetilde{w_{-1}} \equiv 0$, are bounded on $L^p[0,1)$ for $1<p<\infty$.

Proof. Since $\{\widetilde{w_n}\}_n$ is a Schauder basis for $L^p[0,1)$ it suffices to prove that the family $\{\mathcal{S}^{\pm}P_n\}_{n=0}^{\infty}$, with P_n as in Theorem 3.4, is a uniformly bounded family of operators on $L^p[0,1)$. Suppose not. Note that each $\mathcal{S}^{\pm}P_n$ is bounded on $L^p[0,1)$ (since its kernel is bounded on $[0,1)^2$). Hence, by the Banach-Steinhaus Theorem, there exists $f \in L^p[0,1)$ such that

$$\sup_n \|\mathcal{S}^{\pm}P_n f\|_{L^p[0,1)} = \infty. \tag{4.4}$$

Let $\mathcal{W}_N^{\pm}$ be the operator defined on $L^p(\mathbb{R})$ by

$$\mathcal{W}_N^{\pm} g = \sum_{n=0}^{N} \langle g, w_n(\cdot - k)\rangle w_{n\pm 1}(\cdot - k).$$

It is easy to check that $\{\mathcal{W}_N^{\pm}\}_{N=0}^{\infty}$ is a uniformly bounded family of operators on $L^p(\mathbb{R})$ since the shift operator is bounded for the Walsh system and we have the equivalence given by (2.2). Let $C_p = \sup_n \|\mathcal{W}_n^{\pm}\|_{L^p[0,1)\to L^p[0,1)}$. By (4.4) we can find $N \geq 1$ such that

$$\left\| \sum_{n=0}^{N} \langle f, \widetilde{w_n}\rangle \widetilde{w_{n\pm 1}} \right\|_{L^p[0,1)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)}, \tag{4.5}$$

with K defined as in Theorem 2.2. We have

$$\left\| \sum_{k_1=-K}^{K+1} \sum_{k_2=-K}^{K+1} \left\{ \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\,dx\, w_{n\pm1}(y-k_2) \right\} \right\|_{L^p([0,1),\,dy)} > (2K+2)^2 C_p \|f\|_{L^p[0,1)},$$

and proceeding as in the proof of Theorem 3.4 we can find k_1 and k_2 such that

$$\begin{aligned}
C_p\|f\|_{L^p[0,1)} &= C_p \|\chi_{[0,1)} f\|_{L^p(\mathbb{R})} \\
&< \left\| \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\,dx\, w_{n\pm1}(y-k_2) \right\|_{L^p([0,1),\,dy)} \\
&\leq \left\| \sum_{n=0}^{N} \int_0^1 f(x)\overline{w_n(x-k_1)}\,dx\, w_{n\pm1}(y-k_2) \right\|_{L^p(\mathbb{R},\,dy)} \\
&= \left\| \sum_{n=0}^{N} \int_{\mathbb{R}} \{\chi_{[0,1)}(x) f(x)\}\overline{w_n(x-k_1)}\,dx\, w_{n\pm1}(y-k_1) \right\|_{L^p(\mathbb{R},\,dy)},
\end{aligned}$$

which contradicts the definition of C_p. Hence, our assumption that

$$\sup_n \|\mathcal{S}^{\pm}P_n\|_{L^p[0,1)\to L^p[0,1)} = \infty$$

is wrong and we are done. ■

Next we show the the projection onto the even numbered Walsh type wavelet packets is bounded on $L^p[0,1)$ for $1 < p < \infty$. It is interesting to note that the proof works for any type of periodic wavelet packets not just the Walsh type.

Lemma 4.13. *Let $\{w_n\}_n$ be a wavelet packet system satisfying the hypothesis of Theorem 2.12, and let $\{\widetilde{w_n}\}_{n=0}^{\infty}$ be the associated periodic system. Then the projection P onto $\{\widetilde{w_{2n}}\}_{n=0}^{\infty}$ is bounded on $L^p[0,1)$ for $1 < p < \infty$.*

Proof. Note that for $\ell \geq 0$ we have

$$\begin{aligned}\widetilde{w_{2\ell}}(x) &= \sum_{k\in\mathbb{Z}} \hat{w}_{2\ell}(2\pi k)e^{2\pi ikx} \\ &= \sum_{k\in\mathbb{Z}} m_0(\pi k)\hat{w}_\ell(\pi k)e^{2\pi ikx} \\ &= \sum_{s\in\mathbb{Z}} \hat{w}_\ell(2\pi s)e^{2\pi i(2\,s)x}\end{aligned}$$

and

$$\begin{aligned}\widetilde{w_{2\ell+1}}(x) &= \sum_{k\in\mathbb{Z}} \hat{w}_{2\ell+1}(2\pi k)e^{2\pi ikx} \\ &= \sum_{k\in\mathbb{Z}} m_1(\pi k)\hat{w}_\ell(\pi k)e^{2\pi ikx} \\ &= -\sum_{s\in\mathbb{Z}} \hat{w}_\ell((2s+1)\pi)e^{2\pi i(2\,s+1)x}.\end{aligned}$$

Thus, $\widetilde{w_{2\ell}}$ is made up entirely of even frequencies and $\widetilde{w_{2\ell+1}}$ is made up entirely of odd frequencies. The claim therefore follows from the fact that the projection onto $\{e^{2\pi ikx}\}_{k\in 2\mathbb{Z}}$ is bounded on $L^p[0,1)$ (see the proof of Lemma 3.7). ■

4.2.2 Boundedness of the Hilbert Transform

We can now generalize all of the results obtained for the Walsh functions to the periodic Walsh type wavelet packets. The proof of the following is a trivial consequence of the results from the previous section.

Corollary 4.14. *Substitute $W_{T(n)}$ by $\widetilde{w_{T(n)}}$ in Definitions 4.3 and 4.9. Then Theorem 4.8, Theorem 4.10, and Corollary 4.11 hold for the system $\{\widetilde{w_{T(n)}}\}_{n=0}^{\infty}$.*

4.3 Failure of the Hilbert Transform

One of the important results needed to prove boundedness of the modified Hilbert Transform for periodic Walsh type wavelet packets is that the shift operator $\mathcal{S}^{(\pm)}$ is bounded on $L^p[0,1)$, $1 < p < \infty$. However, as we shall see in this section, the shift operator is not bounded for general periodic wavelet packet systems. The reason is that the periodic wavelet packets are not, in general, uniformly bounded in $L^p[0,1)$.

First we prove the following result about the lacunary subsequence $\{\widetilde{w_{2^n}}\}$ of the periodic wavelet packets. The result shows the the subsequence essentially agrees with the Rademacher functions.

Theorem 4.15. *Let $\{\widetilde{w_n}\}_{n=0}^{\infty}$ be any periodized wavelet packet system for which $w_1 \in C^1(\mathbb{R})$ has compact support, and let $\{W_n\}_{n=0}^{\infty}$ be the Walsh System. Then the following subsystem are equivalent in $L^p[0,1)$, $1 < p < \infty$,*

$$\{W_{2^n}\}_{n=0}^{\infty} \leftrightarrow \{\widetilde{w_{2^n}}\}_{n=0}^{\infty}$$
$$\{W_{3\cdot 2^n}\}_{n=0}^{\infty} \leftrightarrow \{\widetilde{w_{3\cdot 2^n}}\}_{n=0}^{\infty},$$

in the sense that there exists an isomorphism Q on $L^p[0,1)$ mapping one subsystem to the other.

Proof. Let $\widetilde{W_n^s}$ be the periodized smooth Walsh type wavelet packets defined in the proof of Lemma 2.11 generated using the wavelet w_1. It suffices to prove that $\widetilde{w_n} \equiv \widetilde{W_n^s}$ whenever $n \in \{2^n\}_{n=0}^{\infty} \cup \{3\cdot 2^n\}_{n=0}^{\infty}$ (see Corollary 3.24). Note that the Fourier expansion of $\widetilde{w_n}$ is is given by

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} \hat{w}_n(2\pi k)e^{2\pi ikx}.$$

In particular,

$$\begin{aligned}
\widetilde{w_{2n}}(x) &= \sum_{k\in\mathbb{Z}} \hat{w}_{2n}(2\pi k)e^{2\pi ikx} \\
&= \sum_{k\in\mathbb{Z}} m_0(k\pi)\hat{w}_n(k\pi)e^{2\pi ikx} \\
&= \sum_{k\in 2\mathbb{Z}} \hat{w}_n(k\pi)e^{2\pi ikx} \\
&= \sum_{l\in\mathbb{Z}} \hat{w}_n(2\pi l)e^{2\pi il(2x)} \\
&= \widetilde{w_n}(2x). \qquad (4.6)
\end{aligned}$$

So, it suffices to prove that $\widetilde{w_1} \equiv \widetilde{W_1^s}$, $\widetilde{w_2} \equiv \widetilde{W_2^s}$, and $\widetilde{w_3} \equiv \widetilde{W_3^s}$. Surely, $\widetilde{w_1} \equiv \widetilde{W_1^s}$ since they are the periodized version of the same wavelet. It follows from (4.6) that $\widetilde{w_2} \equiv \widetilde{W_2^s}$ since they have identical Fourier series. Also, a calculation similar to (4.6) shows that

$$\widetilde{w_3}(x) = -\sum_{k\in\mathbb{Z}} \hat{\psi}((2k+1)\pi)e^{2(2k+1)\pi ix}.$$

Hence, $\widetilde{w_3} \equiv \widetilde{W_3^s}$, and we are done. ∎

The above theorem is bad news for shift operator on $L^p[0,1)$.

Theorem 4.16. *There is a periodized wavelet packet system* $\{\widetilde{w_n}\}_{n=0}^{\infty}$ *for which the shift operator* $\mathcal{W}^{(\pm)}$*, defined by* $\mathcal{W}^{(\pm)}\widetilde{w_n} \equiv \widetilde{w_{n\pm1}}$*, fails to be bounded on* $L^p[0,1)$*,* $p \neq 2$*.*

Proof. By duality, it suffices to prove the result for $\mathcal{W}^{(-)}$. Note that $\mathcal{W}^{(-)}$ maps $\widetilde{w_{2^n}}$ onto $\widetilde{w_{2^n-1}}$. Theorem 4.15 shows that the system $\{\widetilde{w_{2^n}}\}$ is uniformly bounded in $L^p[0,1)$, $1 < p < \infty$. Hence, any system for which $\sup_n \|\widetilde{w_{2^n-1}}\|_p = \infty$, $p > 2$, provides a counterexample since a bounded operator cannot map a bounded sequence onto an unbounded one. See Theorem 2.38 for such a counterexample. ■

We can also use each of the periodic wavelet packet systems associated with one of the filters from Tables 2.1, 2.2, and 2.3 to get more counterexamples if we limit Theorem 4.16 to large p.

The same type of argument used in Theorem 4.16 can also be used to obtain the following results, which shows that general periodic wavelet packet systems behave nothing like the Walsh system in $L^p[0,1)$.

Corollary 4.17. *The the systems* $\{\widetilde{w_n}\}$ *and* $\{\widetilde{w_{G(n)}}\}$ *are, in general, not equivalent in* $L^p[0,1)$*,* $p \neq 2$*.*

Proof. Note that $G(2^n - 1) = 2^n$ and use the same argument as in Theorem 4.16. ■

Corollary 4.18. *The modified Hilbert Transform is, in general, not bounded on* $L^p[0,1)$*,* $p \neq 2$*, for the periodic wavelet packet system in Paley order.*

Proof. The modified Hilbert Transform restricted to $\{\widetilde{w_{2^n}}\}$ is just the shift operator. ■

Appendix A

The Walsh System

This appendix contains some results on the Walsh system that are used in the previous chapters. All the results are taken from the two monographs [14, 35] that both contain **much more** about the Walsh systems than we have included here.

A.1 Definitions and Properties

We need two equivalent definitions of the Walsh system on $[0,1)$. The first one fit into the wavelet packet scheme

Definition A.1. *The Walsh system* $\{W_n\}_{n=0}^{\infty}$ *is defined recursively on* $[0,1)$ *by letting* $W_0 = \chi_{[0,1)}$ *and*

$$W_{2n}(x) = W_n(2x) + W_n(2x+1)$$
$$W_{2n+1}(x) = W_n(2x) - W_n(2x+1).$$

It is not hard to see that the Walsh system is the basic wavelet packets associated with the Haar multiresolution analysis. It turns out that the Walsh system is closed under pointwise multiplication, but this is hard to verify using Definition A.1. An alternative definition of the Walsh system can be given in terms of the Rademacher functions. Consider the function

$$r_0(x) = \begin{cases} 1 & \text{for } x \in [0,1/2), \\ -1 & \text{for } x \in [1/2,1). \end{cases}$$

Extend r_0 to the real line and define $r_n(x) = r_0(2^n x)$. Then the Walsh system can be obtained by taking all possible finite products of the Rademacher functions. More precisely, for $n =$

$\sum_{i=0}^{\infty} n_i 2^i \in \mathbb{N}_0$, we define

$$w_n(x) = \prod_{i=0}^{\infty} (r_i(x))^{n_i} \chi_{[0,1)}(x).$$

To see that the definitions agree, we just have to note that $w_0 = \chi_{[0,1)}$ and, using the properties of the Rademacher functions,

$$\begin{aligned} w_{2n}(x) &= w_n(2x) + w_n(2x+1) \\ w_{2n+1}(x) &= w_n(2x) - w_n(2x+1), \end{aligned}$$

i.e $W_n \equiv w_n$ for $n \in \mathbb{N}_0$. Using the multiplicative definition, it follows easily that the Walsh system is closed under pointwise multiplication. In fact, define the binary operator $\dot{+} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by

$$m \dot{+} n = \sum_{i=0}^{\infty} |m_i - n_i| 2^i,$$

where $m = \sum_{i=0}^{\infty} m_i 2^i$ and $n = \sum_{i=0}^{\infty} n_i 2^i$. Then

$$W_m(x) W_n(x) = W_{m \dot{+} n}(x). \tag{A.1}$$

Moreover, (A.1) shows that the Walsh functions are characters for the group of all binary sequences (indexed by $\mathbb{N}_0$) under bitwise addition.

Theorem A.2. *Let $\{L_n\}_{n \in \mathbb{N}}$ be the Lebesgue constants for the Walsh system defined by*

$$L_n = \int_0^1 \left| \sum_{k=0}^{n-1} W_k(x) \right| dx.$$

Define n_k, $k \in \mathbb{N}$, by $n_{2s} = \sum_{i=0}^{s} 2^{2i}$, and $n_{2s+1} = \sum_{i=0}^{s} 2^{2i+1}$, then for all $k \in \mathbb{N}$

$$L_{n_k} > \frac{1}{2}\left(\frac{k}{2} + 1\right).$$

Proof. See [14, Chapter 2]. ■

Theorem A.3. *Let*

$$D_n(x) = \sum_{k=0}^{n-1} W_k(x).$$

Then

$$D_{2^k}(x) = 2^k \chi_{[0,2^{-k})}(x).$$

Proof. We prove the result by induction on k. If $k=0$ the the result is trivial ($w_0 \equiv \chi_{[0,1)}$). Suppose the result holds for some $k \geq 0$. Note that, using (A.1),

$$\begin{aligned} D_{2^{k+1}}(x) &= D_{2^k}(x) + \sum_{n=2^k}^{2^{k+1}-1} W_n(x) \\ &= D_{2^k}(x) + W_{2^k}(x) \sum_{n=0}^{2^k-1} W_n(x) \\ &= (1+r_k(x)) D_{2^k}(x), \end{aligned}$$

and

$$(1+r_k(x))\big|_{x\in[0,2^k)} = 2\chi_{[0,2^{-(k+1)})}(x).$$

Thus by induction hypothesis

$$\begin{aligned} D_{2^{k+1}}(x) &= (1+r_k(x)) D_{2^k}(x) \\ &= 2\chi_{[0,2^{-(k+1)})}(x) 2^k \chi_{[0,2^{-k})}(x) \\ &= 2^{k+1}\chi_{[0,2^{-(k+1)})}(x). \end{aligned}$$

■

Lemma A.4. *Let $f_1 \in L^2(\mathbb{R})$, and define $\{f_n\}_n \geq 2$ recursively by*

$$f_{2n+\varepsilon}(x) = f_n(2x) + (-1)^\varepsilon f_n(2x-1).$$

Then for $n, J \in \mathbb{N}$, $2^J \leq n < 2^{J+1}$, we have

$$f_n(x) = \sum_{s=0}^{2^J-1} W_{n-2^J}(s2^{-J}) f_1(2^J - s).$$

Proof. Proof is by induction on n. First, note that for $n = 2, 3$,

$$\begin{aligned} f_2(x) &= f_1(2x) + f_1(2x-1) = W_0(0) f_1(2x) + W_0(1/2) f_1(2x-1), \\ f_3(x) &= f_1(2x) - f_1(2x-1) = W_1(0) f_1(2x) + W_1(1/2) f_1(2x-1), \end{aligned}$$

and for the inductive step observe that

$$\begin{aligned} f_{2[1\varepsilon_{J-1}\cdots\varepsilon_1]_2+\varepsilon}(x) &= f_{[1\varepsilon_{J-1}\cdots\varepsilon_1]_2}(2x) + (-1)^\varepsilon f_{[1\varepsilon_{J-1}\cdots\varepsilon_1]_2}(2x-1) \\ &= \sum_{s=0}^{2^{J-1}-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1]}(s2^{-(J-1)}) f_1(2^J x - s) \\ &\qquad + (-1)^\varepsilon \sum_{s=0}^{2^{J-1}-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1]}(s2^{-(J-1)}) f_1(2^J x - 2^{J-1} - s), \end{aligned}$$

and using (A.1),

$$= \sum_{s=0}^{2^J-1} W_{[\varepsilon_{J-1}\cdots\varepsilon_1\varepsilon]}(s2^{-J}) f_1(2^J x - s).$$

■

Remark. The matrix $H \in \mathbb{R}^{2^J \times 2^J}$ defined by

$$H_{i,j} = 2^{-J/2} W_i(j2^{-J}), \qquad i,j = 0,1,\ldots 2^J - 1,$$

is called the Hadamard Transform, and it follows from the previous lemma that the expansion coefficients of Wavelet packets generated by Haar filters can be expressed in terms of this transform.

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