On the Feng-Rao Bound for Generalized Hamming Weights

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Outline

A First Description of the Area

Motivating Example - the Reed-Solomon Code

The General Theory in Headlines

Linear Codes:

$B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis for $\mathbb{F}_q^n$. Choose any $G \subseteq B$.

$C^\perp(B, G) = (C(B, G))^\perp$ (parity check matrix)

$C(B, G) = \mathrm{span}\{\mathbf{b}_i \mid \mathbf{b}_i \in G\}$ (generator matrix)

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$$\begin{aligned} B &= \{\mathbf{b}_1 = \text{ev}(1), \mathbf{b}_2 = \text{ev}(X), \ldots, \mathbf{b}_n = \text{ev}(X^{n-1})\} \\ G &= \{\mathbf{b}_1, \ldots, \mathbf{b}_s\} \end{aligned}$$

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Generalized Hamming Weights

$$D = \{(1,1,0,0,1,0)\\ (1,0,1,0,0,0)\\ (0,0,1,0,1,0)\}$$

$$\text{Supp}(D) = \{1,2,3,5\}$$

$$d_t(C) := \min\{\#\text{Supp}(D) \mid D \text{ is a subcode of } C \text{ of dimension } t\}$$

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Non-zero codewords in $C(B, G_k)$ are of the form:

$$\mathbf{c} = \sum_{i=1}^{a} \alpha_i \mathbf{b}_i = \text{ev}\left(\sum_{i=1}^{a} \alpha_i X^{i-1}\right), \quad \alpha_a \neq 0 \text{ and } a \leq k.$$

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$\mathbf{e} * \mathbf{f} = (e_1 f_1, \ldots, e_n f_n) \qquad \text{ev}(F) * \text{ev}(G) = \text{ev}(FG)$

$$\begin{cases} \mathbf{c} * \mathbf{b}_1 & \in C(B, G_a) \backslash C(B, G_{a-1}) \\ \mathbf{c} * \mathbf{b}_2 & \in C(B, G_{a+1}) \backslash C(B, G_a) \\ & \vdots \\ \mathbf{c} * \mathbf{b}_{n-a+1} & \in C(B, G_n) \backslash C(B, G_{n-1}) \end{cases}$$

$\mathbf{c} * \mathbf{b}_1, \mathbf{c} * \mathbf{b}_2, \ldots, \mathbf{c} * \mathbf{b}_{n-i+1}$ are linearly independent.

From this one can deduce that $w_H(\mathbf{c}) \geq n - i + 1$.

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$$C^{\perp}(B,G_{n-k}) = \{\mathbf{c} \in \mathbb{F}_q^n \mid \mathbf{c} \cdot \mathbf{b}_i = 0, i = 1, \ldots, n-k\}$$

If $\mathbf{c} \neq \mathbf{0}$ then $\exists j \in \{n-k+1, \ldots, n\}$ with

$$\mathbf{c} \cdot \mathbf{d} \neq 0, \ \forall \mathbf{d} \in C(B,G_j) \backslash C(B,G_{j-1})$$

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Consider $\mathbf{r}_h = \sum_{i=1}^{h} \alpha_i \mathbf{b}_i$, $h \in \{1, \dots, j\}$, $\alpha_h \neq 0$.

$$\mathbf{r}_h * \mathbf{b}_{j-h} \in C(B, G_j) \backslash C(B, G_{j-1})$$

$$\mathbf{c} \cdot (\mathbf{r}_h * \mathbf{b}_{j-h}) \neq 0$$

$$\Downarrow$$

$$\mathbf{c} * \mathbf{r}_h \neq \mathbf{0}$$

But set of all possible $\mathbf{r}_h$'s, $h = 1, \dots, j$ is a space of dimension j.
From this one can deduce that $w_H(\mathbf{c}) \geq j$.

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General Theory

Define $G_s := \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$.

Let $\mathbf{b}_i * \mathbf{b}_j \in C(B, G_s) \backslash C(B, G_{s-1})$. Then (i,j) is said to be OWB if $\mathbf{b}_u * \mathbf{b}_j \in C(B, G_{s-1})$ for all $u < i$.

Minimum distance of $C^\perp(B, G)$:
For every $\mathbf{b}_l \notin G$ count number of i's such that a j exists with (i,j) OWB and $\mathbf{b}_i * \mathbf{b}_j \in C(B, G_l)$. Minimum distance greater or equal to smallest found value.

tth generalized Hammingweight of $C^\perp(B, G)$:
Consider all possible combinations of t different $\mathbf{b}_l \notin G$.

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Minimum distance of $C(B,G)$:
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tth generalized Hammingweight of $C(B,G)$:
Consider all possible combinations of t different $\mathbf{b}_i \in G$.

Minimum distance of $C(B, G)$:
For every $\mathbf{b}_i \in G$ count number of l's such that a j exists with (i, j) OWB and $\mathbf{b}_i * \mathbf{b}_j \in C(B, G_l)$. Minimum distance greater or equal to smallest found value.

tth generalized Hammingweight of $C(B, G)$:
Consider all possible combinations of t different $\mathbf{b}_i \in G$.

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- improved bounds on duals of one-point geometric Goppa codes
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