The Feng–Rao bounds

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KIAS International Conference on Coding Theory and Applications 2012

Linear code = a subspace.

Operations are:

- Vector addition.
- Scalar multiplication.

$[n, k, d]$ the usual parameters.

To deal with d (and k and even n) the compontwise product is useful:

- $(c_1, \ldots, c_n) * (d_1, \ldots, d_n) = (c_1 d_1, \ldots, c_n d_n)$.

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Claim: Code constructions with a supporting algebra:

- algebraic geometric codes,
- Reed–Muller codes and relatives,
- affine variety codes,

are about getting information on the componentwise product.

Multiplication in "higher" algebra	$\Rightarrow$ $(\Leftarrow)$	Multiplication in quotient ring
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Dual code Parity check matrix		Primary code Generator matrix
The usual Feng–Rao bound		The Andersen–G bound
(Feng–Rao bound for dual codes)	$\Leftrightarrow$	(Feng–Rao bound for primary codes)
Order bound		Footprint bound

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This talk:

- connection between the levels of description,
- connection between dual and primary

Results:

- Consequences of the above connections.
- Information derived from medium and low level descriptions.

Important results that are not covered:

- Higher level results such as Beelen bound, Duursma–Kirov–Park bound and list decoding of algebraic geometric codes by Lee–Bras-Amorós–O'Sullivan's method.

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Ideal $J \subseteq \mathbb{F}[\vec{X}]$

The footprint:

$\Delta_{\prec}(J) = \{\vec{X}^{\vec{\alpha}} \mid \vec{X}^{\vec{\alpha}}$ is not a leading monomial of any polynomial in $J\}$

$I \subseteq \mathbb{F}_q[\vec{X}]$, $I_q = I + \langle X_1^q - X_1, \ldots, X_m^q - X_m \rangle$.

The footprint bound in a special case:

$\#\mathbb{V}_{\mathbb{F}_q}(I_q) = \#\Delta_{\prec}(I_q)$.

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$\#\mathbb{V}_{\mathbb{F}_q}(I_q) = \#\Delta_{\prec}(I_q)$.

$I_q = \langle X^q - X, Y^q - Y \rangle$

$\Delta_{\prec}(I_q) = \{X^i Y^j \mid 0 \leq i, j < q\}$

$\#\mathbb{V}_{\mathbb{F}_q}(I_q) = q^2$

$I_{q^2} = \langle X^{q+1} - Y^q - Y, X^{q^2} - X, Y^{q^2} - Y \rangle.$

Choose monomial ordering with $x^{q+1} \prec Y^q$,

$\Delta_{\prec}(I_{q^2}) \subseteq \{X^i Y^j \mid 0 \leq i < q^2, 0 \leq j < q\}$

$\#\mathbb{V}_{\mathbb{F}_{q^2}}(I_{q^2}) \leq q \cdot q^2 = q^3$

Study of norm/trace gives q^3 zeros.

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Gröbner basis for J w.r.t. $\prec$ is a basis for J such that $\Delta_{\prec}(J)$ can easily be read off.

$\mathcal{G} = \{G_1, \ldots, G_s\} \subseteq J$ is Gröbner basis for J w.r.t. $\prec$ iff any monomial in $\mathrm{lm}(J)$ is divisible by some $\mathrm{lm}(G_i)$.

Gröbner basis for I_q gives exact information on $\#\mathbb{V}_{\mathbb{F}_q}(I_q)$.

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$\mathbb{V}_{\mathbb{F}_q}(I_q) = \{P_1, \ldots, P_n\}$.

Codeword $\vec{c} = (F(P_1), \ldots, F(P_n))$.

$w_H(\vec{c}) = n - \#\Delta_{\prec}(I_q + \langle F \rangle)$ (n minus number of commen zeros).

Information on which leading monomials occour in the code construction gives information on minimum distance.

Improved code construction straight forward.

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$\{M + J \mid M \in \Delta_\prec(J)\}$ is a basis for $\mathbb{F}[\vec{X}]/J$
as a vectorspace over $\mathbb{F}$.

$$G = \begin{bmatrix} M_1(P_1) & \cdots & M_1(P_n) \\ M_2(P_1) & \cdots & M_2(P_n) \\ \vdots & \ddots & \vdots \\ M_k(P_1) & \cdots & M_k(P_n) \end{bmatrix}, \quad M_1, \ldots, M_k \in \Delta_\prec(I_q), M_i \neq M_j$$

is a code of dimension k

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Reed–Muller codes:

Let $I_5 = \langle X^5 - X, Y^5 - Y \rangle$ and

$\vec{c} = (F(P_1), \ldots, F(P_{n=25}))$, with $\mathrm{lm}(F) = X^i Y^j$.

We get $w_H(\vec{c}) = n - \#\Delta_{\prec}(I_5 + \langle F \rangle) \geq (5-i)(5-j)$.

Y^4	XY^4	X^2Y^4	X^3Y^4	X^4Y^4
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Y^2	XY^2	X^2Y^2	X^3Y^2	X^4Y^2
Y	XY	X^2Y	X^3Y	X^4Y
1	X	X^2	X^3	X^4

5	4	3	2	1
10	8	6	4	2
15	12	9	6	3
20	16	12	8	4
25	20	15	10	5

$\mathrm{RM}_5(4,2)$ is $[25, 15, 5]$

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Hermitian codes:

$$I = \langle X^{q+1} - Y^q - Y \rangle,\ I_{q^2} = I + \langle X^{q^2} - X, Y^{q^2} - Y \rangle.$$

$$w(X^i Y^j) = iq + j(q+1)$$

$X^s Y^t \prec_w X^u Y^v$

- if $w(X^s Y^t) < w(X^u Y^v)$
- or $w(X^s Y^t) = w(X^u Y^v)$ and $t < v$

Weighted degree lexicographic ordering.

$I_4 = \langle X^3 - Y^2 - Y, X^4 - X, Y^4 - Y \rangle.$

$$\Delta_{\prec_w}(I_4) \quad \boxed{\begin{array}{cccc} Y & XY & X^2Y & X^3Y \\ 1 & X & X^2 & X^3 \end{array}} \quad \boxed{\begin{array}{cccc} 3 & 5 & 7 & 9 \\ 0 & 2 & 4 & 6 \end{array}}$$

$\vec{c} = (F(P_1), \ldots, F(P_8))$

$\mathrm{lm}(F) = Y$

$w_H(\vec{c}) = \#\{M \in \Delta_{\prec_w}(I_4) \mid M \notin \Delta_{\prec_w}(I_4 + \langle F \rangle)\}.$

$YF \ \text{ rem } \ X^3 - Y^2 - Y = Y(Y + \cdots) \ \text{ rem } \ X^3 - Y^2 - Y = X^3 + \cdots$

$w_H(\vec{c}) \geq \# w\left(\Delta_{\prec_w}(I_4)\right) \cap \left(w(Y) + w(\Delta_{\prec_w}(I_4))\right).$

(what we hit is what we get).

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$I_9 = \langle X^4 - Y^3 - Y, X^9 - X, Y^9 - Y \rangle.\quad w(X) = 3,\ w(Y) = 4.$

$$\begin{array}{ccccccccc}
Y^2 & XY^2 & X^2Y^2 & x^3Y^2 & X^4Y^2 & X^5Y^2 & X^6Y^2 & X^7Y^2 & X^8Y^2 \\
Y & XY & X^2Y & x^3Y & X^4Y & X^5Y & X^6Y & X^7Y & X^8Y \\
1 & X & X^2 & x^3 & X^4 & X^5 & X^6 & X^7 & X^8
\end{array}$$

$$\begin{array}{ccccccccc}
8 & \underline{11} & \underline{14} & \underline{17} & \underline{20} & \underline{23} & \underline{26} & \underline{29} & \underline{32} \\
4 & 7 & 10 & 13 & 16 & \underline{19} & \underline{22} & \underline{25} & \underline{28} \\
0 & 3 & 6 & 9 & 12 & \underline{15} & \underline{18} & \underline{21} & \underline{24}
\end{array}$$

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19 & \underline{16} & 13 & 10 & 7 & 4 & 3 & 2 & 1 \\
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\end{array}$$

One–point algebraic geometric codes:

$P_1, \ldots, P_n, Q$ rational places of function field over $\mathbb{F}_q$.

To construct $C_{\mathcal{L}}(D = P_1 + \cdots + P_n, vQ)$ we need basis for: $\cup_{s=0}^{v} \mathcal{L}(sQ) \subseteq \bigcup_{s=0}^{\infty} \mathcal{L}(sQ)$.

Everything, can be translated into affine variety description:

$\cup_{s=0}^{\infty} \mathcal{L}(sQ) = \mathbb{F}_q[X_1, \ldots, X_m]/I \quad \{P_1, \ldots, P_n\} \subseteq \mathbb{V}_{\mathbb{F}_q}(I)$.

Affine variety description includes determination of minimum distance via footprint bound.

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Weierstrass semigroup:
$H(Q) = -\nu_Q\big(\cup_{s=0}^{\infty} \mathcal{L}(sQ)\big) = \langle w_1, \ldots, w_m \rangle$.

Definition: Given weights $w_1, \ldots, w_m$ define
$w(\vec{X}^{\vec{\alpha}}) = \alpha_1 w_1 + \cdots + \alpha_m w_m$. Define $\prec_w$ by $\vec{X}^{\vec{\alpha}} \prec_w \vec{X}^{\vec{\beta}}$ if

- $w(\vec{X}^{\vec{\alpha}}) < w(\vec{X}^{\vec{\beta}})$
- or $w(\vec{X}^{\vec{\alpha}}) = w(\vec{X}^{\vec{\beta}})$ but $\vec{X}^{\vec{\alpha}} \prec_{\mathcal{M}} \vec{X}^{\vec{\beta}}$

($\prec_{\mathcal{M}}$ can be anything, for instance $\prec_{lex}$)

Example: $w(X) = q, w(Y) = q+1$, $\prec_{\mathcal{M}} = \prec_{lex}$ with $X \prec_{lex} Y$.
$F(X,Y) = X^{q+1} - Y^q - Y$, $w(X^{q+1}) = w(Y^q) = q(q+1)$ and
$\mathrm{Im}(F) = Y^q$.

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$F(X, Y) = X^{q+1} - Y^q - Y$, $w(X^{q+1}) = w(Y^q) = q(q+1)$ and $\mathrm{lm}(F) = Y^q$.

Order domain conditions:

$I = \langle F_1(\vec{X}), \ldots, F_s(\vec{X}) \rangle \subseteq \mathbb{F}[\vec{X}]$ and $w_1, \ldots, w_m$ satisfy ODC if:

1. $\{F_1, \ldots, F_s\}$ is a Gröbner basis w.r.t. $\prec_w$.
2. F_i, $i = 1, \ldots, s$ contains exactly two monomials of highest weight.
3. No two monomials in $\Delta_{\prec_w}(\langle F_1, \ldots, F_s \rangle)$ are of the same weight.

Example: $I = \langle X^{q+1} - Y^q - Y \rangle \subseteq \mathbb{F}_{q^2}[X, Y]$

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Theorem (Miura–1997, Pellikaan–2001):

$\cup_{s=0}^{\infty}\mathcal{L}(sQ) = \mathbb{F}[\vec{X}]/I$ where I and corresponding weights satisfy order domain conditions.

Corollary:

$$\begin{aligned} & C_{\mathcal{L}}(P_1 + \cdots + P_n, vQ) \\ = \; & \mathrm{Span}_{\mathbb{F}_q}\{(M(P_1), \ldots, M(P_n)) \mid M \in \Delta_{\prec_w}(I_q), w(M) \leq v\}. \end{aligned}$$

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Weierstrass semigroup $\Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$.

Corollary: (G–Matsumoto 2009)
A function field having Λ as a Weierstrass semigroup can at most have

$$\#\left(\Lambda \backslash \cup_{i=1}^{m} (q\lambda_i + \Lambda) \right) + 1$$

rational places.

- Term "$q\lambda_i$" comes from $X_i^q - X_i$.
- Term "$+1$" corresponds to the place with Weierstrass semigroup Λ.
- Better than Serre–bound for small q.
- Gives a way for excluding possible Weierstrass semigroups when genus and number of zeros are known.

- Order domains are generalizations of $\cup_{s=0}^{\infty} \mathcal{L}(sQ)$.
- For transcendence degree r, weights are in $\mathbb{N}_0^r$ (when finitely generated) G–Pellikaan 2002.
- Gives a way of generalizing algebraic geometric codes to higher transcendence degree. Think of Reed–Muller code as higher transcendence degree version of Reed–Solomon code.
- Order domain conditions and Pellikaan–Miura correspondence also work for higher transcendence degrees G–Pellikaan 2002.
- So does methods for estimating parameters.
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The footprint-method applied to order domain conditions:

$I = \langle F_1(\vec{X}, \ldots, F_s(\vec{X})\rangle$, $\Delta_{\prec_w}(I_q) = \{M_1, \ldots, M_n\}$.

$\vec{c} = \mathrm{ev}(F)$, $\mathrm{lm}(F) = M_i$.

$$\begin{aligned} w_H(\vec{c}) &= \#\big(\Delta_{\prec_w}(I_q)\backslash\Delta_{\prec_w}(I_q + \langle F\rangle)\big) \\ &= \#\{M \in \Delta_{\prec_w}(I_q) \mid M \text{ is a leading monomial} \\ &\qquad\qquad \text{of a polynomial in } I_q + \langle F\rangle\} \\ &\geq \# \text{ monomials in } \Delta_{\prec_w}(I_q) \text{ hit by } M_i \\ &\qquad\qquad (\text{using } F_1, \ldots, F_s) \\ &= \#\Big(w(\Delta_{\prec_w}(I_q)) \cap \big(w(M_i) + w(\Delta_{\prec_w}(I_q))\big)\Big). \end{aligned}$$

Linear code level:

$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ bases for $\mathbb{F}_q^n$.

$\{\vec{0}\} = L_0 \subsetneq L_1 = \mathrm{Span}\{\vec{b}_1\} \subsetneq L_2 = \mathrm{Span}\{\vec{b}_1, \vec{b}_2\} \subsetneq \cdots \subsetneq L_n = \mathbb{F}_q^n$.

$\bar{\rho}_{\mathcal{B}}(\vec{c}) = i$, if $\vec{c} \in L_i \backslash L_{i-1}$.

(i, j) is OWB if $\bar{\rho}_{\mathcal{B}}(\vec{b}_{i'} * \vec{u}_j) < \bar{\rho}_{\mathcal{B}}(\vec{b}_i * \vec{u}_j)$ for $i' = 1, \ldots, i-1$.

If a supporting algebra is given then information can be extracted regarding above.

Think of $\mathcal{B} = \mathcal{U}$ corresponding to $\{1, X, X^2, \ldots, X^{q-1}\}$.

$\mathrm{ev}(X^i) * \mathrm{ev}(X^j) = \mathrm{ev}(X^{i+j})$ applied when $i + j < q$.

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To hit:

$$\bar{\sigma}(i) = \#\{l \mid \exists j \text{ such that } (i,j) \text{ is OWB} \\ \text{and } \bar{\rho}_{\mathcal{B}}(\vec{b}_i * \vec{u}_j) = l\}$$

Theorem:

If $\bar{\rho}_{\mathcal{B}}(\vec{c}) = i$ then $w_H(\vec{c}) \geq \bar{\sigma}(i)$.

Proof: Assume $(i,j_1),(i,j_2),\ldots,(i,j_\sigma)$ hits $l_1,l_2,\ldots,l_\sigma$.

$\{\vec{c} * \vec{u}_{j_1},\cdots,\vec{c} * \vec{u}_{j_\sigma}\}$ is linearly independent.

Hence, $\vec{c} * \mathrm{Span}\{\vec{u}_{j_1},\cdots \vec{u}_{j_\sigma}\}$ is of dimension σ.

But $\{\vec{c} * \vec{d} \mid \vec{d} \in \mathbb{F}_q^n\}$ is of dimension $w_H(\vec{c})$. $\square$

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Let l be such that $\vec{c} \cdot \vec{b}_l \neq 0$ but $\vec{c} \cdot \vec{b}_{l'} = 0$ for all $l' < l$. Then $w_H(\vec{c}) \geq \bar{\mu}(l)$.

Proof: Same type of arguments as before. $\square$

Primary code: minimum distance $\geq$ smallest $\bar{\sigma}(i)$ value among generating vectors.

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Recent results (with Matsumoto and Ruano):

$\mathcal{G} = \{\vec{g}_1, \ldots, \vec{g}_n\}$, $\mathcal{H} = \{\vec{h}_1, \ldots, \vec{h}_n\}$ and $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_n\}$ with

$$\begin{bmatrix} \vec{g}_1^{\,T} \\ \vdots \\ \vec{g}_n^{\,T} \end{bmatrix} \begin{bmatrix} \vec{h}_n \cdots \vec{h}_1 \end{bmatrix} = I$$

then very nice translation between $\bar{\rho}_{\mathcal{G}}$, $\bar{\sigma}$ with respect to $(\mathcal{G},\mathcal{U})$ on the one side and $\bar{\rho}_{\mathcal{H}}$, $\bar{\mu}$ with respect to $(\mathcal{H},\mathcal{U})$ on the other side. Primary code description $\Leftrightarrow$ dual code description.

- Feng–Rao majority decoding algorithm for dual codes (usually described by means of algebra) can be formulated in linear code set-up (Matsumoto–Miura 2000). Works for WB.
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$R = \mathbb{F}_5[X, Y]$.

$\{P_1 = (1,1), P_2 = (1,2), P_3 = (1,3), P_4 = (2,1), \ldots, P_9 = (3,3)\} \subsetneq \mathbb{F}_5^2$

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Information from

- function field theory,
- Gröbner basis theory,
- algebra,
- order domain theory

translates easily to information on $\bar{\rho}$ and OWB, WWB or WB.

Multiplication corresponds to componentwise product.

Recent list decoding algorithms for algebraic geometric codes decode beoyond the bound for primary codes. (Lee–Bras-Amorós–O'Sullivan 2011, G–Matsumoto–Ruano 2012, Lee–Bras-Amorós–O'Sullivan 2012).

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Everything said so far regarding minimum distance can be lifted to generalized Hamming weights.

$\mathrm{Supp}(D) = \{i \in \{1, \ldots, n\} \mid c_i \neq 0 \text{ for some } \vec{c} \in D\}$.

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Give information about behaviour of

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Generalized Reed–Muller codes:

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Y	XY	X^2Y	X^3Y	X^4Y
1	X	X^2	X^3	X^4

5	4	3	2	1
$\underline{10}$	8	6	4	2
15	$\underline{12}$	9	6	3
20	16	$\underline{12}$	8	4
25	20	15	$\underline{10}$	5

- Minimum distance corresponds to value on border (can be realized as product of linear factors).
- Second smallest weight: What happens if leading monomial is on the border, but minimal value is not realized?
- Use Buchberger's algorithm at a theoretical level. Second smallest weigth IS second smallest number above for degrees up to q^{m-1}. G-2008
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Conclusion:

- The variety of levels can sometimes help in realizing what is “really” going on.
- Lower level descriptions often captures what is going on, but might appear technical.