One-point AG-codes from an affine-variety point of view

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Monomial orderings

A monomial ordering $\prec$ is a total ordering on $\{\vec{X}^{\vec{\alpha}} = X_1^{\alpha_1} \cdots X_m^{\alpha_m} | \vec{\alpha} \in \mathbb{N}_0^m\}$ such that

- $\vec{X}^{\vec{\alpha}} \prec \vec{X}^{\vec{\beta}} \Rightarrow \vec{X}^{\vec{\alpha}+\vec{\gamma}} \prec \vec{X}^{\vec{\beta}+\vec{\gamma}}$.
- Every subset has a unique smallest element.

Examples: $\prec_{lex}$, $\prec_{glex}$, $\prec_{grlex}$, $\prec_{wdeglex}$.

$X^2Y^3 \prec_{glex} XY^5$ because $5 < 6$.
$X^2Y^3 \prec_{glex} X^3Y^2$ because $5 = 5$ and $2 < 3$.

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Footprint

$I \subseteq \mathbb{F}[\vec{X}]$ an ideal.

$$\Delta_{\prec}(I) = \{\vec{X}^{\vec{\alpha}} \mid \vec{X}^{\vec{\alpha}} \text{ is not leading monomial of any polynomial in } I\}$$

If $I = \langle F(\vec{X}) \rangle$ then
$\Delta_{\prec}(I) = \{\vec{X}^{\vec{\alpha}} \mid \vec{X}^{\vec{\alpha}} \text{ is not divisible with } \mathrm{Im}(F)\}$.

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The main tools

Theorem:
$\{M + I \mid M \in \Delta_{\prec}(I)\}$ constitutes a basis for $\mathbb{F}[\vec{X}]/I$ as a vectorspace.

Definition:
$\mathbb{V}_{\mathbb{F}}(I)$ means set of commen zeros of the polynomials in I.

Corollary:
$|\mathbb{V}_{\mathbb{F}}(I)| \leq |\Delta_{\prec}(I)|$ (whenever latter is finite).

Proof: Consider $\{P_1, \ldots, P_n\} \subseteq \mathbb{V}_{\mathbb{F}}(I)$ and define
$\mathrm{ev} : \mathbb{F}[\vec{X}]/I \rightarrow \mathbb{F}^n$ by $\mathrm{ev}(F + I) = (F(P_1), \ldots, F(P_n))$.
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An importan special case

Corollary: Let $F(\vec{X}) \in \mathbb{F}_q[\vec{X}]$, $\text{lm}(F) = X_1^{i_1} \cdots X_m^{i_m}$. Then F has at most $q^m - \prod_{s=1}^{m}(q - i_s)$ zeros.

Proof:

$$\begin{aligned} \text{number of zeros} &\leq |\Delta_{\prec}(\langle F(\vec{X}) \rangle + \langle X_1^q - X_1, \ldots, X_m^q - X_m \rangle)| \\ &\leq |\{\vec{X}^{\vec{\alpha}} | 0 \leq \alpha_1 < q, \ldots, 0 \leq \alpha_m < q, \vec{X}^{\vec{i}} \not| \vec{X}^{\vec{\alpha}}\}|. \end{aligned}$$

One-point algebraic geometric codes

$P_1, \ldots, P_n, Q$ rational places of function field over $\mathbb{F}_q$.

To construct $C_{\mathcal{L}}(D = P_1 + \cdots + P_n, vQ)$ we need basis for: $\cup_{s=0}^{v} \mathcal{L}(sQ) \subseteq \bigcup_{s=0}^{\infty} \mathcal{L}(sQ)$.

Everything, can be translated into affine variety description:

$\cup_{s=0}^{\infty} \mathcal{L}(sQ) = \mathbb{F}_q[X_1, \ldots, X_m]/I \quad \{P_1, \ldots, P_n\} \subseteq \mathbb{V}_{\mathbb{F}_q}(I)$.

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Weights versus valuation

Weierstrass semigroup:
$H(Q) = -\nu_Q\big(\cup_{s=0}^{\infty} \mathcal{L}(sQ)\big) = \langle w_1, \ldots, w_m\rangle.$

Definition: Given weights $w_1, \ldots, w_m$ define $w(\vec{X}^{\vec{\alpha}}) = \vec{\alpha} \cdot (w_1, \ldots, w_m)$. Define $\prec_w$ by $\vec{X}^{\vec{\alpha}} \prec_w \vec{X}^{\vec{\beta}}$ if

- $w(\vec{X}^{\vec{\alpha}}) < w(\vec{X}^{\vec{\beta}})$
- or $w(\vec{X}^{\vec{\alpha}}) = w(\vec{X}^{\vec{\beta}})$ but $\vec{X}^{\vec{\alpha}} \prec_{\mathcal{M}} \vec{X}^{\vec{\beta}}$

($\prec_{\mathcal{M}}$ can be anything, for instance $\prec_{lex}$)

Example: $w(X) = q, w(Y) = q+1$, $\prec_{\mathcal{M}} = \prec_{lex}$ with $X \prec_{lex} Y$.
$F(X,Y) = X^{q+1} - Y^q - Y$, $w(X^{q+1}) = w(Y^q) = q(q+1)$ and $\mathrm{lm}(F) = Y^q$.

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Order domain conditions

$I = \langle F_1(\vec{X}), \ldots, F_s(\vec{X}) \rangle \subseteq \mathbb{F}[\vec{X}]$ and $w_1, \ldots, w_m$ satisfy ODC if:

1. $\{F_1, \ldots, F_s\}$ is a Gröbner basis w.r.t. $\prec_w$.
2. F_i, $i = 1, \ldots, s$ contains exactly two monomials of highest weight.
3. No two monomials in $\Delta_{\prec_w}(\langle F_1, \ldots, F_s \rangle)$ are of the same weight.

Example: $I = \langle X^{q+1} - Y^q - Y \rangle \subseteq \mathbb{F}_{q^2}[X, Y]$

1. OK
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3. $\Delta_{\prec_w}(I) = \{X^i Y^j \mid 0 \leq j < q, 0 \leq i\}$ OK

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Presentation Theorem

Theorem (Miura, Pellikaan):

$\cup_{s=0}^{\infty}\mathcal{L}(sQ) = \mathbb{F}[\vec{X}]/I$ where I and corresponding weights satisfy order domain conditions.

Corollary:

$$\begin{aligned} & C_{\mathcal{L}}(P_1 + \cdots + P_n, vQ) \\ = \; & \mathrm{Span}_{\mathbb{F}_q}\{(M(P_1), \ldots, M(P_n)) \mid M \in \Delta_{\prec_w}(I), w(M) \leq v\}. \end{aligned}$$

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Dimension and generator matrix

Remember in general $\{M + J \mid M \in \Delta_{\prec}(J)\}$ is a basis for $\mathbb{F}[\vec{X}]$.

Define $I_q = I + \langle X_1^q - X_1, \ldots, X_m^q - X_m \rangle$.

$\text{ev} : \mathbb{F}_q[\vec{X}](I_q \rightarrow \mathbb{F}_q^n$ given by $\text{ev}(F + I_q) = (F(P_1), \ldots, F(P_n))$ is a bijection.

$$\begin{aligned} & C_{\mathcal{L}}(P_1 + \cdots + P_n, vQ) \\ = \; & \text{Span}_{\mathbb{F}_q}\{(M(P_1), \ldots, M(P_n)) \mid M \in \Delta_{\prec_w}(I_q), w(M) \leq v\}. \end{aligned}$$

Dimension can be read off directly. So can generator matrix.

Hermitian function field

$$I_9 = \langle X^4 - Y^3 - Y, X^9 - X, Y^9 - Y \rangle \subseteq \mathbb{F}_9[X, Y].$$

8	11	14	17	20	23	26	29	32	35	38	⋯
4	7	10	13	16	19	22	25	28	31	34	⋯
0	3	6	9	12	15	18	21	24	27	30	⋯

$$H^*(Q) = w(\Delta_{\prec_w}(I_9)) \subseteq w(\Delta_{\prec_w}(I)) = H(Q).$$

What about minimum distance?

Applying the footprint bound

Let $I = \langle F_1(\vec{X}), \ldots, F_s(\vec{X}) \rangle$ and $w_1, \ldots, w_m$ satisfy ODC.

Code word $\vec{c} = \mathrm{ev}(F + I_q)$ where $\mathrm{Supp}(F) \subseteq \Delta_{\prec_w}(I_q)$.

Hamming weight equals
$n - |\Delta_{\prec_w}(\langle F(\vec{X}) \rangle + I_q)|$.

For every monomial M

$\mathrm{Im}\big(MF(\vec{X}) \ \ \mathrm{rem} \ \ \{F_1(\vec{X}), \ldots, F_s(\vec{X})\}\big)$

DOES NOT BELONG TO $\Delta_{\prec_w}(\langle F(\vec{X}) \rangle + I_q)$.

We can easily detect the above leading monomial!

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The weights tell it all...

$w(\mathrm{lm}(MF(\vec{X})) = w(\mathrm{lm}\big(MF(\vec{X}) \ \text{ rem } \ \{F_1(\vec{X}), \ldots, F_s(\vec{X})\}\big))$
because:

- No two monomials in $F(\vec{X})$ are of the same weight (as no two monomials in $\Delta_{\prec_w}(I)$ are of the same weight).
- Every $F_i(\vec{X})$ has exactly two monomials of highest weight.

Hamming weight of $\vec{c}$

In conclusion we can estimate

$$\begin{aligned} & w_H(\vec{c}) \\ = \quad & n - |\Delta_{\prec_w}(\langle F(\vec{X})\rangle + I_q)| \\ = \quad & |\Delta_{\prec_w}(I_q) \backslash \Delta_{\prec_w}(\langle F(\vec{X})\rangle + I_q)| \\ \geq \quad & |w(\Delta_{\prec_w}(I_q)) \cap \{w(M \cdot \mathrm{lm}(F)) | M \text{ a monomial}\}| \\ = \quad & |H^*(Q) \cap \big(w(\mathrm{lm}(F)) + H(Q)\big)| \\ \geq \quad & n - |H(Q) \backslash \big(w(\mathrm{lm}(F)) + H(Q)\big)| \\ = \quad & n - w(\mathrm{lm}(F)). \end{aligned} \tag{1}$$

Last line corresponds to Goppa bound. Last equality comes from semigroup theory.

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Minimum distance of Hermitian codes

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Green=Goppa bound, Blue=Equation 1.

Improved code construction straight forward.

Everything works for general one-point algebraic geometric code.

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- Improved bound and improved code construction translates immediately to the general case of one-point AG codes where affine description is not known but $H^*(Q)$ is known.
- Generalization of one-point construction to higher transcendence degree. We have trdg $= r \Leftrightarrow w(X_i) \in \mathbb{N}_0^r$. Improved bound works (but Goppa bound does not).
- Let $\mathbb{F}/\mathbb{F}_q$ be a function field that possesses a Weierstrass semigroup $\Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$. The number of rational places is at most

$$\#\big(\Lambda \backslash \cup_{i=1}^{m} (q\lambda_i + \Lambda)\big) + 1.$$

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