On Weierstrass semigroups and rational places

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Terminology

$\mathbb{F}/\mathbb{F}_q$ $\mathbb{F}_q$ full constant field

$N(\mathbb{F})$ the number of rational places

$g(\mathbb{F})$ the genus

$N_q(g) = \max\{N(\mathbb{F}) \mid \mathbb{F}$ is a function field over $\mathbb{F}_q, g(\mathbb{F}) = g\}$

Let Λ be a Weierstrass semigroup for a rational place $\mathcal{P}$

$\Lambda = -\nu_{\mathcal{P}}(\cup_{m=0}^{\infty}(m\mathcal{P})\backslash\{0\})$

We have $\#\mathbb{N}_0\backslash\Lambda = g$

Terminology

Λ sometimes holds more information about $N(\mathbb{F})$ than does g alone.

If no function field over $\mathbb{F}_q$ has a rational place having Λ as Weierstrass semigroup then $N_q(\Lambda) = 0$

else

$$N_q(\Lambda) = \max\{N(\mathbb{F}) \mid \mathbb{F} \text{ is a function field over } \mathbb{F}_q \text{ having a rational place with Weierstrass semigroup equal to } \Lambda\}$$

Lewittes' bound

$\Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$, $\lambda_1 < \cdots < \lambda_m$.

λ_1 is called the multiplicity.

Lewittes showed
$N_q(\Lambda) \leq q\lambda_1 + 1$

Example:
Hermitian function field $X^{q+1} - Y^q - Y$ over $\mathbb{F}_{q^2}$.
$\mathcal{P}_\infty$ has Weierstrass semigroup $\Lambda = \langle q, q+1 \rangle$.
$q^2\lambda_1 + 1 = q^3 + 1$ which is attained.

An improvement to Lewittes' bound

$$
\begin{aligned}
N_q(\Lambda) &\leq \#(\Lambda \backslash (\cup_{i=0}^{m} q\lambda_i + \Lambda)) + 1 \\
&\leq \#(\Lambda \backslash (q\lambda_1 + \Lambda)) + 1 \\
&= q\lambda_1 + 1
\end{aligned}
$$

where $\eta + \Lambda = \{\eta + \lambda \mid \lambda \in \Lambda\}$

A first example

$\Lambda = \langle 3, 5 \rangle$, $\lambda_1 = 3$, $\lambda_2 = 5$, $q = 2$

$\Lambda = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 13, \dots\}$

$q\lambda_1 + \Lambda = \{6, 9, 11, 12, 14, 15, 16, 17, \dots\}$

$q\lambda_2 + \Lambda = \{10, 13, 15, 16, 18, 19, 20, \dots\}$

$\Lambda \backslash (q\lambda_1 + \Lambda) = \{0, 3, 5, 8, 10, 13\}$

$\Lambda \backslash ((q\lambda_1 + \Lambda) \cup (q\lambda_2 + \Lambda)) = \{0, 3, 5, 8\}$

Lewittes' bound: $N_2(\Lambda) \leq 7$

New bound: $N_2(\Lambda) \leq 5$

Proposition
If $\lambda_1 < q$ then let j be the largest index such that $\lambda_j < q$ holds else let $j = 0$.
We have

$$q\lambda_1 - g \leq \#(\Lambda \backslash (\cup_{i=1}^m (q\lambda_i + \Lambda))) \leq (\prod_{i=1}^{j} \lambda_i) q^{m-j} \leq q^m + 1$$

Remark The conductor of a semigroup $\Lambda \subseteq \mathbb{N}_0$ with finitely many gaps is the smallest number c such that there are no gaps greater or equal to c.
If $q\lambda_1 + c \leq q\lambda_2$ the new bound and Lewittes' bound are the same. In particular if $q\lambda_1 + 2g \leq q\lambda_2$. Happens for Garcia-Stichtenoth's second tower.

More examples

$\Lambda = \langle 8, 9, 20 \rangle \qquad g = 20$						
q	2	3	4	8	9	16
bounds	$17/9$	$25/16$	$33/25$	$65/65$	$73/73$	$129/129$
$N_q(g)$	$19 - 21$	$30 - 34$	$40 - 45$	$76 - 83$	$70 - 91$	$127 - 139$

$\Lambda = \langle 13, 15, 24, 31 \rangle \qquad g = 38$						
q	2	3	4	8	9	16
bounds	$27/13$	$40/28$	$53/40$	$105/97$	$118/112$	$209/207$
$N_q(g)$	$30 - 33$	$*$	$64 - 74$	$129 - 135$	$105 - 149$	$193 - 233$

... and still more examples

$\Lambda = \langle 13, 14, 20 \rangle \qquad g = 42$						
q	2	3	4	8	9	16
bounds	$27/9$	$40/17$	$53/33$	$105/95$	$118/102$	$209/195$
$N_q(g)$	$33 - 35$	$52 - 59$	$75 - 80$	$129 - 147$	$122 - 161$	$209 - 254$

$\Lambda = \langle 10, 11, 20, 22 \rangle \qquad g = 45$						
q	2	3	4	8	9	16
bounds	$21/5$	$31/10$	$41/17$	$81/65$	$91/82$	$161/141$
$N_q(g)$	$33 - 37$	$54 - 62$	$80 - 84$	$144 - 156$	$136 - 170$	$242 - 268$

All semigroups of genus 8

$N_2(8) = 11$: Excluded $13/33$
$N_3(8) \in \{17, 18\}$: Assuming 18, excluded $26/31$
$N_4(8) \in \{21, 22, 23, 24\}$: Assuming 24, excluded $26/31$

Semigroup	$q = 2$	$q = 3$	$q = 4$
$\langle 2, 17\rangle$	$5/5$	$7/7$	$9/9$
$\langle 3, 10, 17\rangle$	$7/6$	$10/10$	$13/13$
$\langle 3, 11, 16\rangle$	$7/7$	$10/10$	$13/13$
$\langle 3, 13, 14\rangle$	$7/7$	$10/10$	$13/13$
$\langle 4, 6, 13\rangle$	$9/9$	$13/13$	$17/17$
$\langle 4, 6, 15, 17\rangle$	$9/9$	$13/13$	$17/17$
$\langle 4, 7, 17\rangle$	$9/6$	$13/11$	$17/17$
$\langle 4, 9, 10\rangle$	$9/9$	$13/12$	$17/17$
$\langle 4, 9, 11\rangle$	$9/7$	$13/13$	$17/17$
$\langle 4, 9, 14, 15\rangle$	$9/8$	$9/9$	$17/17$
$\langle 4, 10, 11, 17\rangle$	$9/9$	$13/13$	$17/17$
$\langle 4, 10, 13, 15\rangle$	$9/9$	$13/13$	$17/17$
$\langle 4, 11, 13, 14\rangle$	$9/9$	$13/13$	$17/17$

All semigroups of genus 8 - cont.

$N_2(8) = 11$: Excluded $13/33$
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$N_4(8) \in \{21, 22, 23, 24\}$: Assuming 24, excluded $26/31$

Semigroup	$q = 2$	$q = 3$	$q = 4$
$\langle 5, 6, 13 \rangle$	$11/7$	$16/12$	$21/18$
$\langle 5, 6, 14 \rangle$	$11/7$	$16/12$	$21/19$
$\langle 5, 7, 9 \rangle$	$11/7$	$16/13$	$21/19$
$\langle 5, 7, 11 \rangle$	$11/9$	$16/14$	$21/19$
$\langle 5, 7, 13, 16 \rangle$	$11/8$	$16/14$	$21/20$
$\langle 5, 8, 9 \rangle$	$11/9$	$16/15$	$21/20$
$\langle 5, 8, 11, 12 \rangle$	$11/9$	$16/14$	$21/21$
$\langle 5, 8, 11, 14, 17 \rangle$	$11/9$	$16/15$	$21/20$
$\langle 5, 8, 12, 14 \rangle$	$11/9$	$16/15$	$21/21$
$\langle 5, 9, 11, 12 \rangle$	$11/9$	$16/16$	$21/21$
$\langle 5, 9, 11, 13, 17 \rangle$	$11/9$	$16/15$	$21/21$
$\langle 5, 9, 12, 13, 16 \rangle$	$11/10$	$16/16$	$21/21$
$\langle 5, 11, 12, 13, 14 \rangle$	$11/11$	$16/16$	$21/21$

All semigroups of genus 8 - cont.

$N_2(8) = 11$: Excluded $13/33$

$N_3(8) \in \{17, 18\}$: Assuming 18, excluded $26/31$

$N_4(8) \in \{21, 22, 23, 24\}$: Assuming 24, excluded $26/31$

Semigroup	$q = 2$	$q = 3$	$q = 4$
$\langle 6, 7, 8, 17 \rangle$	$13/8$	$19/15$	$25/22$
$\langle 6, 7, 9, 17 \rangle$	$13/10$	$19/17$	$25/22$
$\langle 6, 7, 10, 11 \rangle$	$13/11$	$19/16$	$25/21$
$\langle 6, 7, 10, 15 \rangle$	$13/10$	$19/17$	$25/23$
$\langle 6, 7, 11, 15, 16 \rangle$	$13/9$	$19/16$	$25/23$
$\langle 6, 8, 11, 13, 15 \rangle$	$13/11$	$19/19$	$25/25$
$\langle 6, 8, 10, 13, 15, 17 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 6, 8, 10, 11, 15 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 6, 8, 10, 11, 13 \rangle$	$13/11$	$19/18$	$25/25$
$\langle 6, 8, 9, 10 \rangle$	$13/11$	$19/19$	$25/25$
$\langle 6, 8, 9, 11 \rangle$	$13/10$	$19/19$	$25/25$
$\langle 6, 8, 9, 13 \rangle$	$13/11$	$19/19$	$25/25$
$\langle 6, 9, 10, 11, 14 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 6, 9, 10, 11, 13 \rangle$	$13/11$	$19/19$	$25/25$

All semigroups of genus 8 - cont.

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$N_4(8) \in \{21, 22, 23, 24\}$: Assuming 24, excluded $26/31$

Semigroup	$q = 2$	$q = 3$	$q = 4$
$\langle 6, 9, 10, 13, 14, 17 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 6, 9, 11, 13, 14, 16 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 6, 10, 11, 13, 14, 15 \rangle$	$13/12$	$19/19$	$25/25$
$\langle 7, 8, 9, 10, 11 \rangle$	$15/10$	$22/18$	$29/26$
$\langle 7, 8, 9, 10, 12 \rangle$	$15/10$	$22/18$	$29/26$
$\langle 7, 8, 9, 10, 13 \rangle$	$15/10$	$22/18$	$29/26$
$\langle 7, 8, 9, 11, 12 \rangle$	$15/11$	$22/18$	$29/27$
$\langle 7, 8, 9, 11, 13 \rangle$	$15/11$	$22/18$	$28/27$
$\langle 7, 8, 9, 12, 13 \rangle$	$15/11$	$22/18$	$29/27$
$\langle 7, 8, 10, 12, 13 \rangle$	$15/12$	$22/19$	$29/27$
$\langle 7, 8, 10, 11, 12 \rangle$	$15/11$	$22/19$	$29/29$
$\langle 7, 8, 10, 11, 13 \rangle$	$15/11$	$22/19$	$29/27$
$\langle 7, 8, 11, 12, 13, 17 \rangle$	$15/12$	$22/20$	$29/28$

All semigroups of genus 8 - cont.

$N_2(8) = 11$: Excluded $13/33$
$N_3(8) \in \{17, 18\}$: Assuming 18, excluded $26/31$
$N_4(8) \in \{21, 22, 23, 24\}$: Assuming 24, excluded $26/31$

Semigroup	$q = 2$	$q = 3$	$q = 4$
$\langle 7, 9, 10, 11, 12, 13 \rangle$	$15/11$	$22/20$	$29/27$
$\langle 7, 9, 10, 11, 13, 15 \rangle$	$15/11$	$22/20$	$29/28$
$\langle 7, 9, 10, 12, 13, 15 \rangle$	$15/12$	$22/21$	$29/28$
$\langle 7, 9, 11, 12, 13, 15, 17 \rangle$	$15/12$	$22/21$	$29/28$
$\langle 7, 10, 11, 12, 13, 15, 16 \rangle$	$15/13$	$22/21$	$29/29$
$\langle 8, 9, 10, 11, 12, 13, 14 \rangle$	$17/13$	$25/22$	$33/31$
$\langle 8, 9, 10, 11, 12, 13, 15 \rangle$	$17/13$	$25/22$	$33/31$
$\langle 8, 9, 10, 11, 12, 14, 15 \rangle$	$17/13$	$25/22$	$33/31$
$\langle 8, 9, 10, 11, 13, 14, 15 \rangle$	$17/13$	$25/22$	$33/31$
$\langle 8, 9, 10, 12, 13, 14, 15 \rangle$	$17/14$	$25/22$	$33/32$
$\langle 8, 9, 11, 12, 13, 14, 15 \rangle$	$17/14$	$25/24$	$33/32$
$\langle 8, 10, 11, 12, 13, 14, 15, 17 \rangle$	$17/15$	$25/23$	$33/33$
$\langle 9, 10, 11, 12, 13, 14, 15, 16, 17 \rangle$	$19/15$	$28/26$	$37/35$

An estimation of the new upper bound

Corollary

Define

$$t = \#\{\lambda \in \Lambda \mid \lambda \in [\lambda_1 + 1, \lambda_1 + \lceil \lambda_1/q \rceil - 1]\}.$$

We have

$$N(\mathbb{F}) \leq q\lambda_1 - t + 1.$$

Example

Consider the extreme case $\lambda_1 = g + 1$, $\Lambda = \{0, g+1, g+2, \ldots\}$. We get $t = \lceil (g+1)/q \rceil - 1$, and therefore

$$N(\mathbb{F}) \leq q(g+1) + 2 - \lceil (g+1)/q \rceil$$

For this particular semigroup:

$$\#(\Lambda \backslash (\cup_{i=1}^{m} (\lambda_i + \Lambda))) = q\lambda_1 - t + 1.$$

Bounds on $N_q(g)$

Proposition

$$N_q(g) \leq (q - \frac{1}{q})g + q + 2 - \frac{1}{q}.$$

Consequently,

$$\begin{aligned} N_2(g) &\leq 1\frac{1}{2}g + 3\frac{1}{2} \\ N_3(g) &\leq 2\frac{2}{3}g + 4\frac{2}{3} \\ N_4(g) &\leq 3\frac{3}{4}g + 5\frac{3}{4} \end{aligned}$$

whereas Serre's upper bound states

$$\begin{aligned} N_2(g) &\leq 2g + 3 \\ N_3(g) &\leq 3g + 4 \\ N_4(g) &\leq 4g + 5 \end{aligned}$$

Towers of function fields

Corollary
Assume a tower of function fields is given with $g^{(i)} \to \infty$ for $i \to \infty$ and $\liminf_{i\to\infty}(\frac{N^{(i)}}{g^{(i)}}) = \kappa > 0$. Let $(\mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots)$ be any sequence such that $\mathcal{P}^{(i)}$ is a rational place of $F^{(i)}$ for $i = 1, 2, \ldots$. Let $\lambda_1^{(i)}$ be the multiplicity of the Weierstrass semigroup related to $\mathcal{P}^{(i)}$ and let m_i be the number of generators in some description of $\Lambda^{(i)}$. We have

$$\liminf_{i\to\infty}(\lambda_1^{(i)}/g^{(i)}) \geq \kappa/q \qquad (1)$$

$$m_i \to \infty \ \text{ for } \ i \to \infty$$

Proof: Follows from Lewittes' bound.

Towers of function fields

Garcia-Stichtenoth's second tower:

$$\lim_{i\to\infty}(\lambda_1^{(i)}/g^{(i)}) = 1/q.$$

Above corollary reads

$$\liminf_{i\to\infty}(\lambda_1^{(i)}/g^{(i)}) \geq (q-1)/q^2 = \frac{1-1/q}{q}.$$

Corollary: One cannot construct asymptotically good towers of function fields having telescopic Weierstrass semigroups.

Sketch of a proof

Step 1:
Use order domain theory to describe $\cup_{m=0}^{\infty}\mathcal{L}(m\mathcal{P})$ as a factor ring $\mathbb{F}_q[X_1,\ldots,X_m]/I$ where I is a certain type of ideal.

Step 2:
Observe, that the rational places except $\mathcal{P}$ corresponds to the elements in $\mathbb{V}_{\mathbb{F}_q}(I)$.

Step 3:
Apply Gröbner basis techniques to the problem of estimating the size of $\mathbb{V}_{\mathbb{F}_q}(I)$
NOTE: we do not assume to know I but only assume to know Λ.

Definition Given $w_1, \ldots, w_m \in \mathbb{N}$ define $w(X_1^{a_1} \cdots X_m^{a_m}) = a_1 w_1 + \cdots + a_m w_m$. Given further a fixed monomial ordering $\prec$ on $\mathcal{M}(X_1, \ldots, X_m)$ the weighted graded ordering $\prec_w$ on $\mathcal{M}(X_1, \ldots, X_m)$ is given by $X_1^{\alpha_1} \cdots X_m^{\alpha_m} \prec_w X_1^{\beta_1} \cdots X_m^{\beta_m}$ if one of the following two conditions holds:

$$
\begin{aligned}
&(1) \quad w(X_1^{\alpha_1} \cdots X_m^{\alpha_m}) < w(X_1^{\beta_1} \cdots X_m^{\beta_m}) \\
&(2) \quad w(X_1^{\alpha_1} \cdots X_m^{\alpha_m}) = w(X_1^{\beta_1} \cdots X_m^{\beta_m}) \\
&\qquad \text{and } X_1^{\alpha_1} \cdots X_m^{\alpha_m} \prec X_1^{\beta_1} \cdots X_m^{\beta_m}
\end{aligned}
$$

Definition
Denote by $\mathcal{M}(X_1, \dots, X_m)$ the set of monomials in $X_1, \dots, X_m$. Given a monomial ordering $\prec$ on $\mathcal{M}(X_1, \dots, X_m)$ and an ideal $I \subseteq k[X_1, \dots, X_m]$ the footprint of I is the set

$$\Delta_{\prec}(I) = \{M \in \mathcal{M}(X_1, \dots, X_m) \mid M \text{ is not a leading monomial of any polynomial in } I\}.$$

Main example of numerical weight function

Main example of numerical weight function:

$$\rho : \cup_{m=0}^{\infty}\mathcal{L}(m\mathcal{P}) \to -\Lambda \cup \{-\infty\}$$

given by $\rho(f) = -\nu_{\mathcal{P}}(f)$.

A result from order domain theory

Theorem

Let I be an ideal in $k[X_1, \ldots, X_m]$ and assume

- $\mathcal{G} = \{F_1(X_1, \ldots, X_m), \ldots, F_s(X_1, \ldots, X_m)\}$ is a Gröbner basis for I with respect to a weighted degree ordering $\prec_w$.
- The elements of $\Delta_{\prec_w}(I)$ have mutually distinct weights
- Every element of $\mathcal{G}$ has exactly two monomials of highest weight in its support.

Write $\Lambda = \{w(M) \mid M \in \Delta_{\prec_w}(I)\}$. For $f \in k[X_1, \ldots, X_m]/I$ denote by F the (unique) remainder of any polynomial in f after division with $\mathcal{G}$. Then $R = k[X_1, \ldots, X_m]/I$ is an order domain with a numerical weight function $\rho : R \rightarrow \Lambda \cup \{-\infty\}$ defined by $\rho(0) = -\infty$ and $\rho(f) = \max\{w(M) \mid M \in \mathrm{Supp}(F)\}$ for $f \neq 0$. On the other hand let R be an order domain with a numerical weight function $\rho : R \rightarrow \Lambda \cup \{-\infty\}$ where $\Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$. Then there exists a description as above with $w(X_1) = \lambda_1, \ldots, w(X_m) = \lambda_m$.

Lemma

Let $\cup_{m=0}^{\infty}\mathcal{L}(m\mathcal{P})$ *be described as a factor ring as in previous Theorem. The number of rational places of* $\mathbb{F}$ *equals* $\#\mathbb{V}_{\mathbb{F}_q}(I)+1=\#\mathbb{V}_{\mathbb{F}_q}(I_q)+1$ *where*

$$I_q=\langle X_1^q-X_1,\ldots,X_m^q-X_m,F_1(X_1,\ldots,X_m),\ldots,F_s(X_1,\ldots,X_m)\rangle.$$

How to count

Proposition

Let $J \subseteq \mathbb{F}_q[X_1, \ldots, X_m]$ be an ideal such that $X_1^q - X_1, \ldots, X_m^q - X_m \in J$. Let $\prec$ be any monomial ordering on $\mathcal{M}(X_1, \ldots, X_m)$. The footprint $\Delta_\prec(J)$ is finite and $\#\mathbb{V}_{\mathbb{F}_q}(J) = \#\Delta_\prec(J)$ holds.

Lemma Given description as in above theorem. Assume that $G(X_1, \ldots, X_m)$ has precisely one monomial of highest weight in its support. Denote this highest weight by $w(G(X_1, \ldots, X_m))$. The polynomial

$$G(X_1, \ldots, X_m) \text{ rem } (F_1(X_1, \ldots, X_m), \ldots, F_s(X_1, \ldots, X_m)$$

also has precisely one monomial of highest weight in its support and this weight equals $w(G(X_1, \ldots, X_m))$.

Apply Lemma to $(X_1^{a_1} \cdots X_m^{a_m})(X_i^q - X_i)$ for all choices of $a_1, \ldots, a_m, i$.