Generalizations of the Reed-Solomon Codes Via Gröbner Basis Theory

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Tokyo Institute of Technology, August 8, 2006

Thanks to

Professor Tomohiko Uyematsu
and
Professor Ryutaroh Matsumoto

for invitation
and for great hospitality

Outline

Model

$$
\begin{array}{ccccccccc}
\vec{m} & & \vec{c} & & \vec{r} = \vec{c} + \vec{e} & & \vec{m}' \\
\rightarrow & \text{Encoder} & \rightarrow & \text{Channel} & \rightarrow & \text{Decoder} & \rightarrow \\
 & & & \uparrow & & & \\
 & & & \vec{e} & & &
\end{array}
$$

$\vec{m} = (m_1, \dots, m_k) \in \mathbb{F}_q^k$, $k < n$, $\vec{c} = (c_1, \dots, c_n) \in \mathbb{F}_q^n$
$\vec{e} = (e_1, \dots, e_n) \in \mathbb{F}_q^n$, $\vec{m}' \in \mathbb{F}_q^k$

$P_i(e_i = 0) = p$ is large, $P_i(e_i = \alpha) = (1 - P)/(1 - q)$ for $\alpha \neq 0$ and P_i, P_j are independent

Linear code

A (linear) code C is a subspace $C \subseteq \mathbb{F}_q^n$
$k = \dim(C)$, $C \simeq \mathbb{F}_q^k$.

Encoding:
Choose basis $\{\vec{g}_1, \ldots, \vec{g}_k\}$ for C. The generator matrix is

$$G = \begin{bmatrix} \vec{g}_1 \\ \vdots \\ \vec{g}_k \end{bmatrix}$$

Encode by $\vec{c} = \vec{m}G$.

Minimum distance

$$w_H((w_1, \ldots, w_n)) = \#\{i \mid w_i \neq 0\}$$

$$\mathrm{dist}_H(\vec{w}_1, \vec{w}_2) = w_H(\vec{w}_1 - \vec{w}_2)$$

$$d = \min\{\mathrm{dist}_H(\vec{c}_1, \vec{c}_2) \mid \vec{c}_1, \vec{c}_2 \in C, \vec{c}_1 \neq \vec{c}_2\}$$

Within the distance $\left\lfloor \frac{d-1}{2} \right\rfloor$ of a word $\vec{w}$ there can be at most one codeword.

If at most $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors occurs we can correct them by choosing the nearest code word to the received word.

$$d = \min\{w_H(\vec{c}) \mid \vec{c} \in C, \vec{c} \neq \vec{0}\}.$$

The three parameters

The length n, the dimension k and the minimum distance d.
$[n, k, d]$

If $\frac{k}{n}$ is high then fast transmission.
If $\frac{d}{n}$ is high then good protection against noise.

The challenge is to get $\frac{k}{n}$ as well as $\frac{d}{n}$ high simultaneously.

Reed-Solomon code - generator matrix description

$\mathrm{RS}_q(k)$ over $\mathbb{F}_q = \{P_1, P_2, \ldots, P_q\}$.

$$G = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ P_1 & P_2 & \cdots & P_q \\ P_1^2 & P_2^2 & \cdots & P_q^2 \\ \vdots & \vdots & \ddots & \ldots \\ P_1^{k-1} & P_2^{k-1} & \cdots & P_q^{k-1} \end{bmatrix}$$

$\mathrm{RS}_q(k)$ consists of the code words $\vec{c} = \vec{i}G$, where $\vec{i} = (i_0, \ldots, i_{k-1}) \in \mathbb{F}_q^k$.

$$\mathrm{RS}_q(k) = \{(f(P_1), f(P_2), \ldots, f(P_q)) \mid \deg(f) < k\}.$$

Parameters of Reed-Solomon code

Length: $n = q$.

Key observation 1:
$d = n - (k-1) = n - k + 1$ as a polynomial of degree at most $k-1$ can have at most $k-1$ zeros.

Key observation 2:

$$(1, 1, \dots, 1), (P_1, P_2, \dots, P_n), \dots, (P_1^{n-1}, P_2^{n-1}, \dots, P_n^{n-1})$$

are linearly independent. Hence, for $k < q$ we have $\dim(\mathrm{RS}_q(k)) = k$.

$\mathrm{RS}_q(k)$ is $[n = q, k, d = n - k + 1]$.

Some properties of the Reed-Solomon code

Advantages:

- Large minimum distance
- A lot of structure
- More efficient decoding algorithms including list decoding
- Useful as outer code in concatenated codes

Disadvantage:

- ... but very short

The goal

Want to construct long codes with a lot of structure and with high minimum distance.

Strategy:
Reed-Solomon like generator matrices that support generalizations of Key observation 1 and Key observation 2.

Mathematical language (theory)

Traditional: Algebraic geometry or function field theory.
In particular the Riemann-Roch Theorem.

More recent: Gröbner basis theory (much simpler)

Monomials and polynomials

One variable:

$$\mathcal{M}(X) = \{1, X, X^2, X^3, \ldots\} = \{X^i \mid i \in \mathbb{N}_0\}$$

Two variables:

$$\mathcal{M}(X, Y) = \{X^i Y^j \mid i, j \in \mathbb{N}_0\}$$

Example: $X^3 Y^4$

More variables:

$$\mathcal{M}(X_1, X_2, \ldots, X_m) = \{X_1^{i_1} X_2^{i_2} \cdots X_m^{i_m} \mid i_1, i_2, \ldots, i_m \in \mathbb{N}_0\}$$

Polynomial is linear combination of monomials.

Example: $F(X, Y) = X^3 Y + 2XY + 1$

Monomial orderings

One variable:

$$1 < X < X^2 < X^3 < \cdots$$

More variables:
$\prec$ is a monomial ordering on $\mathcal{M}(X_1, X_2, \ldots, X_m)$ if the following hold

1. If $K \prec M$ and $M \prec N$ then also $K \prec N$.
2. If $K \prec M$ then also $KN \prec MN$.
3. Any set of monomials in $\mathcal{M}(X_1, X_2, \ldots, X_m)$ has a (unique) smallest element.

Monomial orderings - continued

The lexicographic ordering:
$X_1^{i_1} \cdots X_m^{i_m} \prec_{lex} X_1^{j_1} \cdots X_m^{j_m}$ if leftmost nonzero entry of $(j_1 - i_1, \ldots, j_m - i_m)$ is positive.

Example:
$X_1 X_2^4 \prec_{lex} X_1^2 X_2$ as $(2,1) - (1,4) = (1,-3)$ has leftmost nonzero entry positive.
If we choose $X_1 = X$ and $X_2 = Y$ then $XY^4 \prec_{lex} X^2Y$.
If we choose $X_1 = Y$ and $X_2 = X$ then $X^2Y \prec_{lex} XY^4$.

Monomial orderings - continued

The graded lexicographic ordering:
$X_1^{i_1} \cdots X_m^{i_m} \prec_{deg} X_1^{j_1} \cdots X_m^{j_m}$ if either (1) or (2) below holds:

(1) $i_1 + \cdots + i_m < j_1 + \cdots + j_m$

(2) $i_1 + \cdots + i_m = j_1 + \cdots + j_m$
and leftmost nonzero entry of $(j_1 - i_1, \ldots, j_m - i_m)$
is positive

Example: $X^2YZ^2 \prec_{deg} X^4YZ$ and $X^3Y^2Z \prec_{deg} X^4YZ$

Monomomial orderings - continued

The weighted graded lexicographic ordering:
Given $w_1, \ldots, w_m \in \mathbb{R}_+$ then $X_1^{i_1} \cdots X_m^{i_m} \prec_w X_1^{j_1} \cdots X_m^{j_m}$ if either (1) or (2) below holds:

(1) $w_1 i_1 + \cdots + w_m i_m < w_1 j_1 + \cdots + w_m j_m$

(2) $w_1 i_1 + \cdots w_m i_m = w_1 j_1 + \cdots + w_m j_m$
and leftmost nonzero entry of $(j_1 - i_1, \ldots, j_m - i_m)$ is positive

Example: $w_1 = w(X) = 2$ and $w_2 = w(Y) = 3$.
$X^4 Y \prec_w Y^4$ as $w(X^4 Y) = 11 < 12 = w(Y^4)$
$X_1 = Y$ and $X_2 = X$ gives $X^3 \prec_w Y^2$ as $w(X^3) = w(Y^2) = 6$ and $(2,0) - (0,3) = (2,-3)$.

Zeros of a polynomial

One variable:
$2 \in \mathbb{Z}_5 = \mathbb{F}_5$ is a zero of $F(X) = X^2 + X + 4$ as $2^2 + 2 + 4 \equiv 0 \mod 5$

More variables:
$(2, 3) \in \mathbb{Z}_5 \times \mathbb{Z}_5$ is a zero of $F(X, Y) = XY + X + 3Y + 3$ as $2 \cdot 3 + 2 + 3 \cdot 3 + 3 \equiv 0 \mod 5$.

Main question:
How do we estimate how many zeros a given polynomial in more variables can have?

Zeros of a polynomial - continued

Example:
$X^3 - Y^2 - Y$ has infinitely many zeros in $\mathbb{R} \times \mathbb{R}$. For every choice of b let $\sqrt[3]{b^2 + b}$ then (a, b) is a zero.

Example:
$X^3 - Y^2 - Y$ and $Y^4 - Y$ has only finitely many zeros in common in $\mathbb{R} \times \mathbb{R}$.

The footprint bound

One variable case:
$F(X) = X^3 + X + 7$ has at most 3 zeros as $\deg(F) = 3$. Note that $\mathrm{lm}(F) = X^3$ and that also number of $*$'s is 3.

$$\begin{array}{ccccccc} * & * & * & \square & \cdot & \cdot & \cdot \\ 1 & X & X^2 & X^3 & X^4 & X^5 & X^6 \end{array}$$

To deal with more variables and more polynomials we will draw pictures as above.

The footprint bound - continued

Example:
We look for solutions (a, b) to $X^2Y - XY$ in $\mathbb{F}_3 \times \mathbb{F}_3$. If $(a, b) \in \mathbb{F}_3 \times \mathbb{F}_3$ then solution to $X^3 - X$ and $Y^3 - Y$ as well.
We have the zeros

$$(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0).$$

Consider the lexicographic ordering. We have the leading monomials $\mathrm{lm}(X^2Y - XY) = X^2Y$, $\mathrm{lm}(X^3 - X) = X^3$ and $\mathrm{lm}(Y^3 - Y) = Y^3$ giving us the picture

$$\begin{array}{cccccc}
Y^4 & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & \square & \cdot & \cdot & \cdot & \cdot \\
Y^2 & * & * & \cdot & \cdot & \cdot \\
Y & * & * & \square & \cdot & \cdot \\
1 & * & * & * & \square & \cdot \\
 & 1 & X & X^2 & X^3 & X^4
\end{array}$$

Certainly, $\#$ zeros equals $\#$ of $*$'s.

The footprint bound - continued

Example:
By inspection $X^3 - Y^2 - Y$ has 8 zeros in $\mathbb{F}_4 \times \mathbb{F}_4$. Consider the weighted graded lexicographic ordering $\prec_w$ with $w(X) = 2$ and $w(Y) = 3$ and $X_1 = X$ and $X_2 = Y$. We have the leading monomials $\mathrm{lm}(X^3 - Y^2 - Y) = X^3$, $\mathrm{lm}(X^4 - X) = X^4$ and $\mathrm{lm}(Y^4 - Y) = Y^4$ giving us the picture

$$\begin{array}{ccccccc}
Y^5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^4 & \square & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & * & * & * & \cdot & \cdot & \cdot \\
Y^2 & * & * & * & \cdot & \cdot & \cdot \\
Y & * & * & * & \cdot & \cdot & \cdot \\
1 & * & * & * & \square & \square & \cdot \\
 & 1 & X & X^2 & X^3 & X^4 & X^5
\end{array}$$

...Ups...now more $*$'s than zeros. What is wrong?

The footprint bound - continued

If (a, b) is a common zero of $X^3 - Y^2 - Y, X^4 - X, Y^4 - Y$ then certainly also a zero of

$$X(X^3 - Y^2 - Y) + (X^4 - X) = XY^2 + XY + X.$$

This polynomial has leading monomial XY^2.

$$\begin{array}{ccccccc}
Y^5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^4 & \Box & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^2 & * & \Box & \cdot & \cdot & \cdot & \cdot \\
Y & * & * & * & \cdot & \cdot & \cdot \\
1 & * & * & * & \Box & \Box & \cdot \\
 & 1 & X & X^2 & X^3 & X^4 & X^5
\end{array}$$

Now, $\#$ zeros equals $\# *$'s.

Conclusion: we need to consider not only polynomials but also their consequences.

The footprint bound - continued

Definition:
Let $F_1, F_2, \ldots, F_s$ be polynomials. The set

$$\{K_1F_1 + K_2F_2 + \cdots + K_sF_s \mid K_1, K_2, \ldots, K_s \text{ are polynomials } \}$$

is called the ideal generated by $F_1, F_2, \ldots, F_s$. It is denoted $\langle F_1, F_2, \ldots, F_s \rangle$.

Example:
$XY^2+XY+X = X(X^3-Y^2-Y)+(X^4-X) \in \langle X^3-Y^2-Y, X^4-X, Y^4-Y \rangle = \langle X^3-Y^2-Y, X^4-X, Y^4-Y, XY^2+XY+X \rangle$.

Conclusion: The concept of an ideal plays a central role for estimating zeros.

The footprint bound - continued

Definition:
Given an ideal $I = \langle F_1, F_2, \ldots, F_s \rangle$ and a monomial ordering the footprint of I is:

$$\begin{aligned}\Delta_{\prec}(I) \quad = \quad \{M \text{ is a monomial } &\mid M \text{ can not be found as} \\ &\text{the leading monomial of any polynomial in } I\}\end{aligned}$$

Buchberger's algorithm add more polynomials (consequences) to the list $\{F_1, \ldots, F_s\}$ so that the footprint can be easily read of. Such an enlarged set is called a Gröbner basis.

The footprint bound - continued

Example:

$I = \langle X^3 - Y^2 - Y, X^4 - X, Y^4 - Y \rangle$ has as leading monomials

$\mathrm{lm}(X^iY^j(X^3 - Y^2 - Y)) = X^{3+i}Y^j$,
$\mathrm{lm}(X^iY^j(X^4 - X)) = X^{4+i}$,
$\mathrm{lm}(X^iY^j(Y^4 - Y)) = X^iY^{4+j}$ and
$\mathrm{lm}(X^iY^j(XY^2 + XY + X)) = X^{i+1}Y^{2+j}$

where (i, j) runs through all possibilities.

$$
\begin{array}{ccccccc}
Y^5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^4 & \square & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^2 & * & \square & \cdot & \cdot & \cdot & \cdot \\
Y & * & * & * & \cdot & \cdot & \cdot \\
1 & * & * & * & \square & \square & \cdot \\
 & 1 & X & X^2 & X^3 & X^4 & X^5
\end{array}
$$

The footprint bound - continued

Example:
As seen $X^3 - Y^2 - Y$ has 8 zeros in $\mathbb{F}_4 \times \mathbb{F}_4$.
Consider weighted graded lexicographic ordering with $w(X) = 2$, $w(Y) = 3$ and $X_1 = Y$, $X_2 = X$. The leading monomials with respect to $\prec_w$ are:
$\text{lm}(X^4 - X) = X^4$, $\text{lm}(Y^4 - Y) = Y^4$ and $\text{lm}(X^3 - Y^2 - Y) = Y^2$.

$$
\begin{array}{ccccccc}
Y^5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^2 & \square & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y & * & * & * & * & \cdot & \cdot \\
1 & * & * & * & * & \square & \cdot \\
 & 1 & X & X^2 & X^3 & X^4 & X^5
\end{array}
$$

$\#$ zeros equals $\#*$'s.

Key observation 1 generalized

Theorem (The footprint bound):
The number of common zeros of $F_1, F_2, \ldots, F_s$ is at most equal to $\#\Delta_{\prec}(\langle F_1, F_2, \ldots, F_s \rangle)$. (If $X_1^q - X_1, X_2^q - X_2, \ldots, X_m^q - X_m$ are among $F_1, F_2, \ldots, F_s$ then equality holds.)

Hence, if $\{P_1, \ldots, P_n\}$ is the common zeros of $F_1, \ldots, F_s$ and G is a polynomial then $w_H\left((G(P_1), \ldots, G(P_n)\right)$ is at least equal to $n - \#\Delta_{\prec}(\langle G, F_1, \ldots, F_s \rangle)$.

Key observation 2 generalized

Theorem:
Assume $X_1^q - X_1, X_2^q - X_2, \ldots, X_m^q - X_m$ are among $F_1, F_2, \ldots, F_s$. Let $P_1, P_2, \ldots, P_n$ be the common zeros of $F_1, F_2, \ldots, F_s$. The set

$$\{(M(P_1), M(P_2), \ldots, M(P_n)) \mid M \in \Delta_{\prec}(\langle F_1, F_2, \ldots, F_s \rangle)\}$$

constitutes a basis for $\mathbb{F}_q^n$ as a vectorspace over $\mathbb{F}_q$.

Key observation 2 - continued

Example:

The common solutions of $X^2Y - XY$, $X^3 - X$ and $Y^3 - Y$ are

$$P_1 = (0,0), P_2 = (0,1), P_3 = (0,2), P_4 = (1,0),$$

$$P_5 = (1,1), P_6 = (1,2), P_7 = (2,0).$$

And the footprint (corresponding to the lexicographic ordering) is

$$\begin{array}{cccccc}
Y^4 & \cdot & \cdot & \cdot & \cdot & \cdot \\
Y^3 & \square & \cdot & \cdot & \cdot & \cdot \\
Y^2 & * & * & \cdot & \cdot & \cdot \\
Y & * & * & \square & \cdot & \cdot \\
1 & * & * & * & \square & \cdot \\
 & 1 & X & X^2 & X^3 & X^4
\end{array}$$

Hence,

$$\{(M(P_1), M(P_2), \ldots, M(P_7)) \mid M \in \{1, X, X^2, Y, XY, Y^2, XY^2\}\}$$

is a basis for $\mathbb{F}_3^7$.

Reed-Solomon codes revisited

$P_1, P_2, \ldots, P_{16}$ the zeros of $X^{16} - X$ (the elements of $\mathbb{F}_{16}$).
Footprint: $\Delta_<(\langle X^{16} - X\rangle) = \{1, X, X^2, \ldots, X^{15}\}$. Hence,

$$\{M(P_1), M(P_2), \ldots, M(P_{16})) \mid M \in \Delta_<(\langle X^{16} - X\rangle)\}$$

constitutes a basis for $\mathbb{F}_{16}^{16}$.

$\Delta_<(\langle X^{16} - X\rangle)$						$\#\Delta_<(\langle X^i, X^{16}\rangle)$					
1	X	X^2	$\cdots$	X^{14}	X^{15}	0	1	2	$\cdots$	14	15

For F with $\mathrm{lm}(F) = X^i$ we have

$$\Delta_<(\langle F, X^{16} - X\rangle) \subseteq \Delta_<(\langle X^i, X^{16}\rangle).$$

By Key observation 1,

$$w_H((F(P_1), F(P_2), \ldots, F(P_1))) = 16 - \#\Delta_<(\langle F, X^{16} - X\rangle) \geq 16 - i.$$

We have shown $\dim(\mathrm{RS}_{16}(k)) = k$ for $k < 16$ and
$d(\mathrm{RS}_{16}(k)) \geq 16 - (k-1) = 16 - k + 1$.

Generalized Reed-Muller codes and hyperbolic codes

Let $P_1, P_2, \ldots, P_{25}$ be the common zeros of $X^5 - X, Y^5 - Y$.
We consider words of the form $(F(P_1), F(P_2), \ldots, F(P_{25}))$.
Let $\prec$ be any monomial ordering.

$$\Delta_\prec(\langle X^5 - X, Y^5 - Y \rangle) \qquad \#\Delta_\prec(\langle X^5, Y^5, X^i Y^j \rangle)$$

$$\begin{array}{ccccc} Y^4 & XY^4 & X^2Y^4 & X^3Y^4 & X^4Y^4 \\ Y^3 & XY^3 & X^2Y^3 & X^3Y^3 & X^4Y^3 \\ Y^2 & XY^2 & X^2Y^2 & X^3Y^2 & X^4Y^2 \\ Y & XY & X^2Y & X^3Y & X^4Y \\ 1 & X & X^2 & X^3 & X^4 \end{array} \qquad \begin{array}{ccccc} 20 & 21 & 22 & 23 & 24 \\ 15 & 17 & 19 & 21 & 23 \\ 10 & 13 & 16 & 19 & 22 \\ 5 & 9 & 13 & 17 & 21 \\ 0 & 5 & 10 & 15 & 20 \end{array}$$

$$G(X, Y) = XY + aX^2 + bY + cX + d$$

$$\#\Delta_\prec(\langle X^5 - X, Y^5 - Y, G(X, Y) \rangle) \le \#\Delta_\prec(\langle X^5, Y^5, XY \rangle \le 9$$

$$w_H(G(P_1), G(P_2), \ldots, G(P_{25})) \ge 25 - 9 = 16$$

Generalized Reed-Muller codes

New notation: $\varphi(G) = (G(P_1), G(P_2), \ldots, G(P_{25}))$.
$\mathrm{RM}_5(4,2) = \mathrm{Span}_{\mathbb{F}_5}\{\varphi(X^iY^j) \mid i+j \leq 4\}$

$$\Delta_{\prec}(\langle X^5 - X, Y^5 - Y\rangle) \qquad \#\Delta_{\prec}(\langle X^5, Y^5, X^iY^j\rangle)$$

Y^4	$*$	$*$	$*$	$*$	20	$*$	$*$	$*$	$*$
Y^4	XY^3	$*$	$*$	$*$	15	17	$*$	$*$	$*$
Y^2	XY^2	X^2Y^2	$*$	$*$	10	13	16	$*$	$*$
Y	XY	X^2Y	X^3Y	$*$	5	9	13	17	$*$
1	X	X^2	X^3	X^4	0	5	10	15	20

Worst case code word: $\mathrm{lm} = Y^4$ or $\mathrm{lm} = X^4$
$w_H(Y^4 + \cdots) \geq 25 - 20 = 5$
$[n,k,d] = [25,15,5]$

Hyperbolic codes

Choose X^iY^j's with $\#\Delta(\langle X^5, Y^5, X^iY^j\rangle)$ small.

$[25, 17, 5]$

20	$*$	$*$	$*$	$*$
15	17	19	$*$	$*$
10	13	16	19	$*$
5	9	13	17	$*$
0	5	10	15	20

$[25, 15, 6]$

$*$	$*$	$*$	$*$	$*$
15	17	19	$*$	$*$
10	13	16	19	$*$
5	9	13	17	$*$
0	5	10	15	$*$

$P_1, P_2, \ldots, P_{64}$ the common zeros of $X^8 - X, Y^8 - Y$

56	57	58	59	60	61	62	63
48	50	52	54	56	58	60	62
40	43	46	49	52	55	58	61
32	36	40	44	48	52	56	60
24	29	34	39	44	49	54	59
16	22	28	34	40	46	52	58
8	15	22	29	36	43	50	57
0	8	16	24	32	40	48	56

$\mathrm{RM}_8(7, 2)$ is $[64, 36, 8]$

Hyperbolic codes with $[64, 48, 8 = 64 - 56]$ and $[64, 37, 14 = 64 - 50]$

Generalized Reed-Muller codes and Hyperbolic codes

$\{P_1, \ldots, P_{q^m}\}$ the common zeros of $X_1^q - X_1, \ldots, X_m^q - X_m$.

If $G(X_1, \ldots, X_m)$ has leading monomial $X_1^{i_1} \cdots X_m^{i_m}$ then

$$\begin{aligned} w_H\left(G(P_1), \ldots, G(P_{q^m})\right) &= q^m - \#\Delta_\prec(\langle G, X_1^q - X_1, \ldots, X_m^q - X_m\rangle) \\ &\geq q^m - \#\Big(\Delta_\prec(\langle X_1^{i_1} \cdots X^{i_m}\rangle) \cap \\ &\qquad \Delta_{\prec_\prec}(\langle X_1^q - X_1, \ldots, X_m^q - X_m\rangle)\Big) \end{aligned}$$

For $X_1^{i_1} \cdots X_m^{i_m} \in \Delta_\prec(\langle X_1^q - X_1, \ldots, X_m^q - X_m\rangle)$ define

$$\begin{aligned} D(X_1^{i_1} \cdots X_m^{i_m}) &= \#\Delta_\prec(\langle X_1^{i_1} \cdots X_m^{i_m}, X_1^q, \ldots, X_m^q\rangle) \\ &= \#\Big(\Delta_\prec(\langle X_1^{i_1} \cdots X_m^{i_m}\rangle) \cap \Delta_\prec(I)\Big) \\ &= q^m - \prod_{s=1}^{m}(q - i_s) \end{aligned}$$

Generalized Reed-Muller codes and hyperbolic codes

If $\mathrm{lm}(F(X_1, \ldots, X_m)) = X_1^{i_1} \cdots X_m^{i_m}$ then

$$w_H(\varphi(F)) \geq q^m - D(X_1^{i_1} \cdots X_m^{i_m}) = \prod_{s=1}^{m} (q - i_s)$$

The polynomial $\prod_{t=1}^{m} \prod_{s=1}^{i_t} (X_t - P_s)$ has leading monomial equal to $X_1^{i_1} \cdots X_m^{i_m}$ (for ANY ordering) and has $D(X_1^{i_1} \cdots X_m^{i_m})$ zeros.

General constructions of generalized Reed-Muller codes and hyperbolic codes along the lines in above examples.

Codes from Hermitian curve

$J = \langle X^{q+1} - Y^q - Y, X^{q^2} - X, Y^{q^2} - Y \rangle$. Common zeros are $\{P_1, \ldots, P_{q^3}\}$.

Let $w(X^i Y^j) = iq + j(q+1)$ and define $\prec_w$ by: $X^\alpha Y^\beta \prec_w X^\gamma Y^\delta$ if (1) or (2) holds

$$\begin{array}{ll} (1) & w(X^\alpha Y^\beta) < w(X^\gamma Y^\beta) \\ (2) & w(X^\alpha Y^\beta) = w(X^\gamma Y^\beta) \text{ and } \beta < \delta \end{array}$$

To estimate $w_H\left((F(P_1), \ldots, F(P_{q^3})\right)$ we consider

$$\begin{aligned} & \#(\Delta_{\prec_w}(\langle F(X,Y), X^{q+1} - Y^q - Y, X^{q^2} - X, Y^{q^2} - Y \rangle)) \\ \leq \; & \#\left(\Delta_{\prec_w}(\langle X^{q+1} - Y^q - Y, F(X,Y)\rangle) \cap \Delta_{\prec_w}(J)\right) \end{aligned}$$

Codes from hermitian curve

We can show:

$$\begin{aligned} & \#\Big(\Delta_{\prec_w}(\langle X^{q+1}-Y^q-Y, F(X,Y)\rangle)\cap\Delta_{\prec_w}(J)\Big) \\ \leq\ & \#\Big(\Delta_{\prec_w}(\langle X^{q+1}-Y^q, \mathrm{lm}(F(X,Y))\rangle)\cap\Delta_{\prec_w}(J)\Big) \end{aligned}$$

Proof relies on the fact that $X^{q+1}-Y^q-Y$ has precisely two monomials of highest weight and that $F\in\mathrm{Span}_{\mathbb{F}_{q^2}}(J)$ has no two monomials of same weight.

(Run Buchberger's algorithm simultaneously for $\{X^{q+1}-Y^q, \mathrm{lm}(F(X,Y))\}$ and $\{X^{q+1}-Y^q-Y, F(X,Y)\}$.)

$D(X^iY^j)$

For $X^iY^j \in \Delta_{\prec_w}(J)$ define

$$D(X_1^{i_1}\cdots X_m^{i_m}) = \#\left(\Delta_{\prec_w}(\langle X^{q+1}-Y^q, \mathrm{Im}(F(X,Y))\rangle) \cap \Delta_{\prec_w}(J)\right)$$

We have shown $w_H(\varphi(F)) \geq n - D(\mathrm{Im}(F))$, where $\varphi(F) = (F(P_1),\ldots,F(P_{q^3}))$.

$J = \langle X^3 - Y^2 - Y, X^4 - X, Y^4 - Y \rangle$

The zeros are $\{P_1, \ldots, P_8\}$.

$w(X^iY^j)$				$D(X^iY^j)$			
3	5	7	9	3	5	6	7
0	2	4	6	0	2	4	6

Let $F(X, Y) = Y + aX + b$ then $w_H(\varphi(F)) \geq 8 - 3 = 5$.

$\Delta_{\prec_w}(J)$				$w(X^iY^j)$				$n-D(X^iY^j)$			
Y	XY	X^2Y	X^3Y	3	5	7	9	5	3	2	1
1	X	X^2	X^3	0	2	4	6	8	6	4	2

$$\begin{aligned} E(s) &= \mathrm{Span}_{\mathbb{F}_4}\{\varphi(X^iY^j) \mid w(X^iY^j) \le s, X^iY^j \in \Delta_{\prec_w}(J)\} \\ &= \mathrm{Span}_{\mathbb{F}_4}\{\varphi(X^iY^j) \mid w(X^iY^j) \le s\} \end{aligned}$$

$$\tilde{E}(s) = \mathrm{Span}_{\mathbb{F}_4}\{\varphi(X^iY^j) \mid n-D(X^iY^j) \ge s, X^iY^j \in \Delta_{\prec_w}(J)\}$$

$E(0)$ is $[8,1,8]$, $E(2)$ is $[8,2,6]$,...,$E(6)$ is $[8,6,2]$, $E(7)$ is $[8,7,2]$ and $E(9]$ is $[8,8,1]$

..., $\tilde{E}(5)$ is $[8,3,5]$, $\tilde{E}(2)$ is $[8,7,2]$, ...

Some observations on $D(X^iY^j)$

Observation 1:

$w(X^iY^j)$				$D(X^iY^j)$			
3	5	7	9	3	5	6	7
0	2	4	6	0	2	4	6

$w(X^iY^j) \geq D(X^iY^j)$

Observation 2:

$w(X^iY^j)$				$8 - D(X^iY^j)$			
3	5	7	9	5	3	2	1
0	2	4	6	8	6	4	2

$n - D(X^iY^j)$ counts what $w(X^iY^j)$ can hit. Meaning that:

$8 - D(Y) = 5$ as $3 + 0 = 3$, $3 + 2 = 5$, $3 + 3 = 6$, $3 + 4 = 7$ and $3 + 6 = 9$

$8 - D(XY) = 3$ as $5 + 0 = 0$, $5 + 2 = 7$ and $5 + 4 = 9$

Some observations on $D(X^iY^j)$ - continued

Observation 1:
$w(X^iY^j) \geq D(X^iY^j)$

Observation 2:
$n - D(X^iY^j)$ counts what $w(X^iY^j)$ can hit.

These observations can be shown to hold for general $I = \langle X^{q+1} - Y^q - Y, X^{q^2} - X, Y^{q^2} - Y \rangle$ as a consequence of the following facts:

Fact 1:
The polynomial $\{X^{q+1} - Y^q - Y\}$ has precisely two monomials of highest weight.

Fact 2: In $\Delta_{\prec_w}(\langle X^{q+1} - Y^q - Y, X^{q^2} - X, Y^{q^2} - Y \rangle)$ there are no two monomials of the same weight.

$J = \langle X^9 - X, Y^9 - Y, X^4 - Y^3 - Y \rangle$ has 27 common points.
$w(X) = 3, w(Y) = 4$

8	11	14	17	20	23	26	29	32
4	7	10	13	16	19	22	25	28
0	3	6	9	12	15	18	21	24

19	16	13	10	7	4	3	2	1
23	20	17	14	11	8	6	4	2
27	24	21	18	15	12	9	6	3

$E(23)$ is $[27, 21, 4]$
but
$\tilde{E}(4)$ is $[27, 22, 4]$

Hermitian codes

Our method gives true minimum distance for all codes $E(s)$ and all codes $\tilde{E}(s)$ coming from the Hermitian curve.

The estimations are even tight in general case of norm-trace curves.

Generalized RM codes and hyperbolic codes revisited

$w(X^iY^j) = (i,j) \in \mathbb{N}_0^2$. Choose some monomial ordering $\prec_{\mathbb{N}_0^2}$ on $\mathbb{N}_0^2$. Choose some monomial ordering $\prec_{\mathcal{M}}$ on $\mathcal{M}(X,Y)$ and define $\prec_w$ by: $X^\alpha Y^\beta \prec_w X^\gamma Y^\delta$ if (1) or (2) holds

$$\begin{array}{ll} (1) & w(X^\alpha Y^\beta) \prec_{\mathbb{N}_0^2} w(X^\gamma Y^\beta) \\ (2) & w(X^\alpha Y^\beta) = w(X^\gamma Y^\beta) \text{ and } X^\alpha Y^\beta \prec_{\mathcal{M}} X^\gamma Y^\delta \end{array}$$

$w(X^i, Y^j)$ $\quad$ $\#\Delta_\prec(\langle X^5, Y^5, X^iY^j\rangle)$

$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	20	21	22	23	24
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	15	17	19	21	23
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	10	13	16	19	22
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	5	9	13	17	21
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	0	5	10	15	20

$25 - \#\Delta(\langle X^5, Y^5, X^3Y^3\rangle) =$
$\#\{(3,3)+(0,0), (3,3)+(1,0), (3,3)+(0,1), (3,3)+(1,1)\}$

Generalized RM codes and hyperbolic codes revisited

$w(X^iY^j) = (i,j) \in \mathbb{N}_0^2$. Choose some monomial ordering $\prec_{\mathbb{N}_0^2}$ on $\mathbb{N}_0^2$. Choose some monomial ordering $\prec_{\mathcal{M}}$ on $\mathcal{M}(X,Y)$ and define $\prec_w$ by: $X^\alpha Y^\beta \prec_w X^\gamma Y^\delta$ if (1) or (2) holds

$$\begin{array}{ll} (1) & w(X^\alpha Y^\beta) \prec_{\mathbb{N}_0^2} w(X^\gamma Y^\beta) \\ (2) & w(X^\alpha Y^\beta) = w(X^\gamma Y^\beta) \text{ and } X^\alpha Y^\beta \prec_{\mathcal{M}} X^\gamma Y^\delta \end{array}$$

$w(X^i, Y^j)$					$\#\Delta_\prec(\langle X^5, Y^5, X^iY^j\rangle)$				
$(0,4)$	$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	20	21	22	23	24
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	15	17	19	21	23
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	10	13	16	19	22
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	5	9	13	17	21
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	0	5	10	15	20

$25 - \#\Delta(\langle X^5, Y^5, X^3Y^3\rangle) =$
$\#\{(3,3)+(0,0), (3,3)+(1,0), (3,3)+(0,1), (3,3)+(1,1)\}$

Forgetting about the $X^q - X, Y^q - Y$-part.

$J = \langle X^q - X, Y^q - Y \rangle$ and $I = \langle \ \rangle$

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & . \\
Y^4 & XY^4 & X^2Y^4 & X^3Y^4 & X^4Y^4 & \cdots \\
Y^3 & XY^3 & X^2Y^3 & X^3Y^3 & X^4Y^3 & \cdots \\
Y^2 & XY^2 & X^2Y^2 & X^3Y^2 & X^4Y^2 & \cdots \\
Y & XY & X^2Y & X^3Y & X^4Y & \cdots \\
1 & X & X^2 & X^3 & X^4 & \cdots
\end{array}$$

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & . \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & \cdots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & \cdots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & \cdots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & \cdots \\
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & \cdots
\end{array}$$

Forgetting about the $X^q - X, Y^q - Y$-part.

$J = \langle X^3 - Y^2 - Y, X^q - X, Y^q - Y \rangle$ and

$I = \langle X^3 - Y^2 - Y \rangle$

$$\begin{array}{cccccc} Y & XY & X^2Y & X^3Y & X^4Y & \cdots \\ 1 & X & X^2 & X^3 & X^4 & \cdots \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 7 & 9 & 11 & \cdots \\ 0 & 2 & 4 & 6 & 8 & \cdots \end{array}$$

Forgetting about the $X_1^q - X_1, \ldots, X_m^q - X_m$

- $\emptyset$ is a Gröbner basis for $\langle 0 \rangle$ and $\{X^{q+1} - Y^q - Y\}$ is a Gröbner basis for $\langle X^{q+1} - Y^q - Y \rangle$. Both with respect to some weighted degree monomial ordering.
- In examples so far the set of defining polynomials are $\emptyset$ respectively $\{X^{q+1} - Y^q - Y\}$. "All" defining polynomials have exactly two monomials of the same highest weight.
- Monomials in the big footprint are of different weights implying that so are the monomials in the small footprint.
- $\mathbb{F}_q[X, Y]$ and $\mathbb{F}_{q^2}[X, Y]/\langle X^{q+1} - Y^q - Y \rangle$ are examples of order domains.

Definition:
$w(X_1), \ldots, w(X_m) \in \mathbb{N}_0^r \backslash \{\vec{0}\}$, $\prec_{\mathbb{N}_0^r}$ a monomial ordering on $\mathbb{N}_0^r$, $\prec_{\mathcal{M}}$ a monomial ordering on $\mathcal{M}(X_1, \ldots, X_m)$. The generalized weighted degree ordering $\prec_w$ is given by: $M_1 \prec_w M_2$ if and only if one of the following two conditions holds:

(1) $w(M_1) \prec_{\mathbb{N}_0^r} w(M_2)$ (2) $w(M_1) = w(M_2)$ and $M_1 \prec_{\mathcal{M}} M_2$.

$\mathrm{wdeg}(F) = \max_{\prec_{\mathbb{N}_0^r}} \{w(M) \mid M \in \mathrm{Sup}(F)\}$

Order domain assumptions:
Given $\prec_w$, $I \subset \mathbb{F}[X_1, X_2, \ldots, X_m]$ and corresponding Gröbner basis $\mathcal{G}$. Suppose that the elements of the footprint $\Delta_{\prec_w}(I)$ have mutually distinct weights and that every element of $\mathcal{G}$ has exactly two monomials of highest weight in its support.

Another example

Let $I = \langle X^5 + Y^4 + Y, Y^5 + Z^4 + Z \rangle \subseteq \mathbb{F}_{16}[X, Y, Z]$.

Definition of $\prec_w$: $w(X) = 16, w(Y) = 20, w(Z) = 25 \in \mathbb{N}_0$.
$\prec_{\mathbb{N}_0} = <$ (the usual (and unique) monomial ordering on $\mathbb{N}_0$).
$\prec_{\mathcal{M}}$ the lexicographic ordering with $X \prec_{\mathcal{M}} Y \prec_{\mathcal{M}} Z$.

$\{X^5 + Y^4 + Y, Y^5 + Z^4 + Z\}$ is a Gröbner basis w.r.t. $\prec_w$.
Every defining monomial has precisely two monomials of highest weight.
Monomials in footprint $\Delta_{\prec}(I) = \{X^i Y^j Z^l \mid j < 4, l < 4\}$ is of different weights.

The order domain assumption is satisfied.

$D(X_1^{i_1} \cdots X_m^{i_m})$

Assume

$$I = \langle F_1, \dots, F_s \rangle \subseteq \mathbb{F}_q[X_1, \dots, X_m]$$

satisfy the order domain assumption and define

$$J = \langle F_1, \dots, F_s, X_1^q - X_1, \dots, X_m^q - X_m \rangle$$

Let $B_1, \dots, B_s$ be binomials, B_i being the difference of the two monomials of highest weight in F_s

Given $F \in \mathrm{Span}\{M \mid M \in \Delta_{\prec_w}(J)\}$ with $\mathrm{lm}(F) = N$ we have

$$\Delta_{\prec_w}(\langle N, B_1, \dots, B_s \rangle) \supseteq \Delta_{\prec_w}(\langle F, F_1, \dots, F_s \rangle)$$

Define

$$D(N) = \#\left(\Delta_{\prec_w}(\langle N, B_1, \dots, B_s \rangle) \cap \Delta_{\prec_w}(J)\right).$$

We conclude

$$w_H\left((F(P_1), \dots, F(P_n))\right) \geq n - D(N).$$

Some nice results

Result 1:

$$n - D(X_1^{i_1} \cdots X_m^{i_m}) \quad = \quad \#\{s \in w(\Delta_{\prec_w}(J)) \mid \\ s - w(X_1^{i_1} \cdots X_m^{i_m}) \in w(\Delta_{\prec_w}(J))\}$$

(we count what can be “hit”)

Result 2:
If weights are **numerical**, then

$$D(X_1^{i_1} \cdots X_m^{i_m}) \leq w(X_1^{i_1} \cdots X_m^{i_m}).$$

Code constructions

$E(s)$ and $\tilde{E}(s)$ codes along the lines of above examples.

Time does not permit to go in deeper detail here.

Some features of the theory

- Works for any one-point geometric Goppa code
- Gives improved one-point geometric Goppa codes
- Generalizations of one-point geometric Goppa codes to surfaces
- Easily extended to deal with generalized Hamming weights
- Connects nicely to Shibuya and Sakaniwa's nice theory
- Theory can be reformulated directly in "code-domain". Doing this allows for even more codes to be treated.
- Strong connection to Feng-Rao theory

Feng-Rao theory

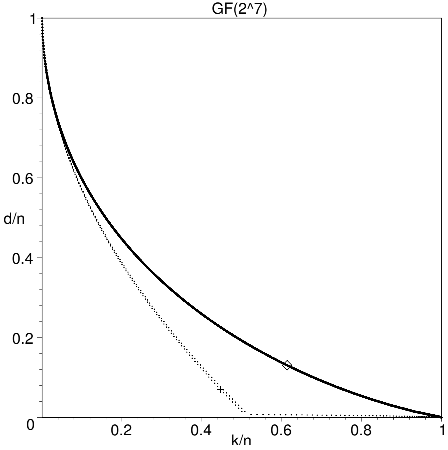
Are concerned with H instead of G (dual description).

Feng-Rao counts what can hit the weight under consideration.
We count what the weight under consideration can hit.
Feng-Rao investigate weights not used in code construction
We investigate weights used in code construction

When $\Delta_{\prec_w}(J)$ has the shape of a box (in some dimension) the two methods produce same estimates for the two classes of codes under consideration.

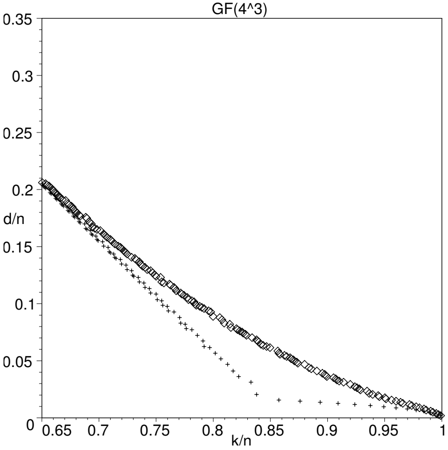
When not form of a box we get typically not similar estimates as Feng-Rao.

$$I = \langle X^{(q^r-1)/(q-1)} - Y^{q^{r-1}} - Y^{q^{r-2}} - \cdots - Y \rangle \subseteq \mathbb{F}_{q^r}[X, Y]$$



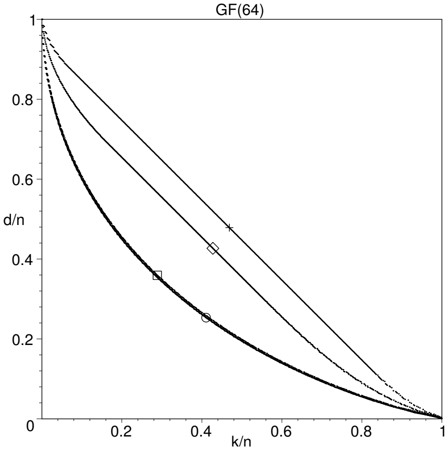
Alphabet=$\mathbb{F}_{q^r} = \mathbb{F}_{2^7}$, $n = 2^{13}$ Improved versus non-improved.

$$I = \langle X^{(q^r-1)/(q-1)} - Y^{q^{r-1}} - Y^{q^{r-2}} - \cdots - Y \rangle \subseteq \mathbb{F}_{q^r}[X, Y]$$



Alphabet=$\mathbb{F}_{q^r} = \mathbb{F}_{4^3}$, $n = 4^5$ Improved versus non-improved.

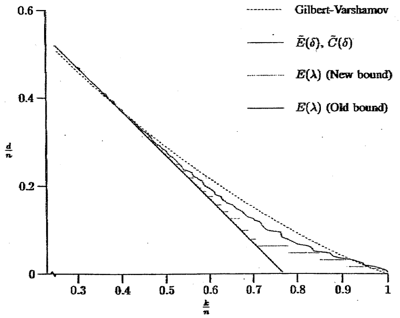
$$I = \langle X^{(q^r-1)/(q-1)} - Y^{q^{r-1}} - Y^{q^{r-2}} - \cdots - Y \rangle \subseteq \mathbb{F}_{q^r}[X,Y]$$



Alphabet=$\mathbb{F}_{64}$. From above: $64 = 8^2$ gives $n = 2^9$, $64 = 4^3$ gives $n = 2^{10}$, $64 = 2^6$ gives $n = 2^{11}$, $\mathrm{Hyp}_{64}(s,2)$ gives $n = 2^{12}$

$$I = \langle X^5 - Y^4 - Y,\ Y^5 - Z^4 - Z \rangle \subseteq \mathbb{F}_{16}[X,Y,Z]$$

$$w(X) = 16, \quad w(Y) = 20, \quad w(Z) = 25$$

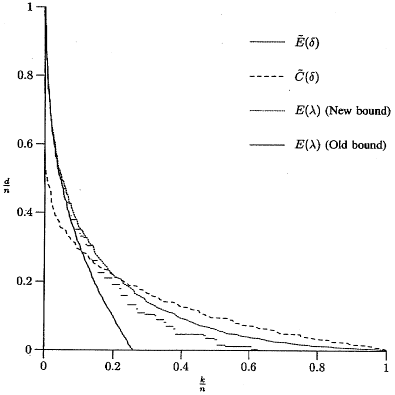


alphabet $= \mathbb{F}_{16}$, $n = 256$

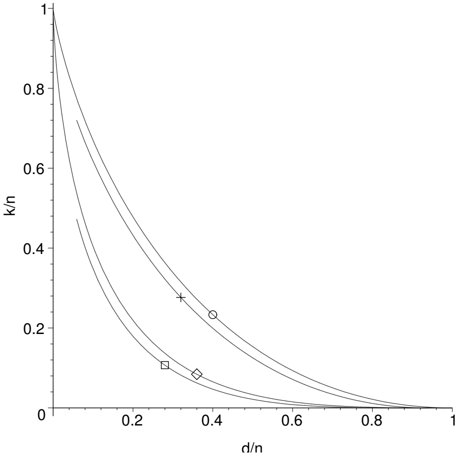
$I = \langle X^5 - Y^4 - Y,\ Y^5 - Z^4 - Z,\ Z^5 - U^4 - U^2 \rangle \subseteq \mathbb{F}_{16}[X, Y, Z, U]$

$w(X) = 64,\ w(Y) = 80,\ w(Z) = 100,\ w(U) = 125$

alphabet $= \mathbb{F}_{16}$, $n = 512$



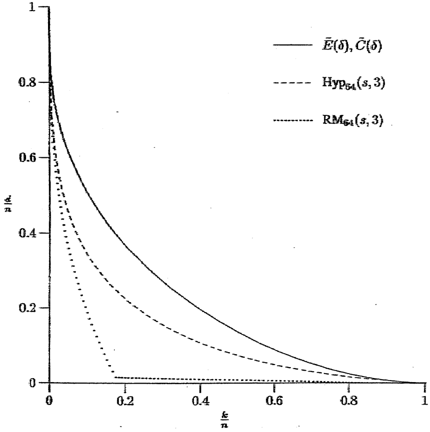
Tensor product of m Hermitian order domains involves weights in $\mathbb{N}_0^m$.



Alphabet=$\mathbb{F}_{256}$. From above: $\mathrm{Hyp}_{256}(s, 2)$ of length $n = 65536$, $\mathrm{Herm}_{256}(s, 2)$ of length $n = 16777216$, $\mathrm{Hyp}_{256}(s, 3)$ of length $n = 16777216$, $\mathrm{Herm}_{256}(s, 3)$ of length $n = 68719476736$.

$$I = \langle X^q + YZ^q - Y^q Z - X,\ U^q - Z^{q+1} + aX^q - aY^q Z + bY^{q+1} + U \rangle$$
$$\subseteq \mathbb{F}_{q^2}[X, Y, Z, U], \quad a, b \in \mathbb{F}_q$$

$$w(X) = (q, 1),\ w(Y) = (0, q),\ w(Z) = (q, 0),\ w(U) = (q+1, 0)$$



alphabet $= \mathbb{F}_{64}$, $n = 262144$

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