Exercises for Mathematics for Computer Graphics

On the following pages you will find a number of exercises to be used in the course “Mathematics for Computer Graphics” at MED5 (Medialogy) and SP1 (Computer Science), Aalborg University 2008. The course uses the very nice book


Although the CD-ROM of the book contains some exercises below you will find supplementary ones.

The present file will be updated throughout the course Mathematics for Computer Graphics. So stay updated.

1 Exercises for Chapter 2 of [VB]

**Exercise 1** Consider the vectors \( \vec{v} = (3, 1) \) and \( \vec{w} = (0, 2) \). Calculate \( \vec{v} - \vec{w} \) and \( \vec{v} + \vec{w} \). For these vectors make a drawing like Figure 2.3 at page 38 of [VB]. Check that the vectors \( \vec{v} - \vec{w} \) and \( \vec{v} + \vec{w} \) in the drawing corresponds to the vectors you calculated above.

**Exercise 2** Consider the vectors \( \vec{u} = (1, 1, 0) \), \( \vec{v} = (0, 2, 3) \) and \( \vec{w} = (2, 4, 3) \).

1. Express \( \vec{u} \) as a linear combination of \( \vec{v} \) and \( \vec{w} \).
2. Express \( \vec{v} \) as a linear combination of \( \vec{u} \) and \( \vec{w} \).
3. Express \( \vec{w} \) as a linear combination of \( \vec{u} \) and \( \vec{v} \).
4. Express \( \vec{0} \) as a non-trivial linear combination of \( \vec{u} \), \( \vec{v} \) and \( \vec{w} \).
5. Are \( \vec{u}, \vec{v}, \vec{w} \) linearly dependent or are they linearly independent?
6. Which geometric object does the set of vectors \( \{ \vec{u}, \vec{v}, \vec{w} \} \) span?
7. Find a basis for above mentioned span.
8. Are some two of the vectors \( \vec{u}, \vec{v}, \vec{w} \) parallel?

**Exercise 3**

1. Are the vectors \( \vec{u} = (1, 0, 1, 1) \), \( \vec{v} = (0, 0, 1, 0) \) and \( \vec{w} = (1, 0, 0, 1) \) linearly dependent?
2. Are the vectors $\vec{u} = (1, 0, 1, 1)$, $\vec{v} = (2, 4, 7, 12)$ linearly dependent?

Exercise 4

1. Find the $l_1$-norm of $(-1, 2, 4)$ (the $l_1$-norm is also known as the Manhattan norm).

2. Find the Euclidean norm of $(-1, 2, 4)$.

Exercise 5 In this exercise we consider the $l_1$-norm (also known as the Manhattan norm). Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. Show that

$$||\vec{v} + \vec{w}||_{l_1} \leq ||\vec{v}||_{l_1} + ||\vec{w}||_{l_1}$$

holds. You will need the well-known result that

$$|a + b| \leq |a| + |b|$$

holds for any real numbers $a, b$.

Exercise 6 Normalize the vector $\vec{v} = (1, 2, 1, 3, 1, 3)$.

Exercise 7 This exercise requires a calculator. Find the angle between $(1, 2, 3)$ and $(0, 1, 4)$.

Exercise 8

1. Are the vectors $\vec{u} = (1, 2, 3), \vec{v} = (3, 0, -1)$ orthogonal (also called perpendicular)?

2. Are the vectors $\vec{u} = (2, 4, 0), \vec{v} = (1, -3, 7)$ orthogonal (also called perpendicular)?

3. Is the angle between $\vec{u} = (2, 4, 0)$ and $\vec{v} = (3, 0, -1)$ greater or smaller than $90^\circ$? (you are not supposed to actually calculate the angle)

Exercise 9 In this exercise we consider a game where an agent wants to detect an enemy. The agent is in position $A = (1, 0, 0)$ and the enemy is in position $E = (2, -2, 0)$. Hence, the vector pointing from the agent to the enemy is $\vec{r} = E - A = (1, -2, 0)$. The agent is looking in the direction $\vec{v} = (1, 1, 1)$. Can the agent see the enemy? (The calculation you are going to do is the same as a computer would have to do when running a game).

Exercise 10 Let $\vec{u} = (1, 2)$. Find a vector $\vec{v}$ of the same length such that $\vec{u}$ and $\vec{v}$ are orthogonal.

Exercise 11 Let $\vec{u} = (a, b)$. Find a vector $\vec{v}$ of the same length such that $\vec{u}$ and $\vec{v}$ are orthogonal. How many possible choices are there for $\vec{v}$?
Exercise 12 Let \( \vec{u} = (1, 2, -1) \). Find a vector \( \vec{v} \) of the same length such that \( \vec{u} \) and \( \vec{v} \) are orthogonal. How many possible choices are there for \( \vec{v} \)?

Exercise 13 Let \( \vec{v} = (0, 1, 2) \) and \( \vec{w} = (3, 0, 4) \).

1. Find the projection of \( \vec{v} \) onto \( \vec{w} \)
2. Find the part of \( \vec{v} \) that is perpendicular to \( \vec{w} \)
3. Calculate the sum \( \text{proj}_\vec{w} \vec{v} + \text{perp}_\vec{w} \vec{v} \). Explain the result you get.

Exercise 14 Orthogonalize
\[ \{ \vec{v}_0 = (1, 0, 0), \vec{v}_1 = (1, 1, 0), \vec{v}_2 = (1, 1, 1) \} \]
by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 15 Orthogonalize
\[ \{ \vec{v}_0 = (1, 0, 1), \vec{v}_1 = (1, 1, 1), \vec{v}_2 = (0, 1, 1) \} \]
by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 16 In this exercise we change the enumeration of the vectors in Exercise 15 and investigate what is the implication. Orthogonalize
\[ \{ \vec{v}_0 = (1, 1, 1), \vec{v}_1 = (0, 1, 1), \vec{v}_2 = (1, 0, 1) \} \]
by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 17 Consider the dot product on \( \mathbb{R}^3 \). Show that the conditions 1.–5. on page 48 are satisfied.

Exercise 18 Define the dot product on \( \mathbb{R}^4 \) in an obvious way. Show that the conditions 1.–5. on page 48 are satisfied.

Exercise 19 In this exercise we want to extend
\[ \{ \vec{u} = (1, 2, 3), \vec{v} = (-3, 1, 1) \} \]
to a basis for \( \mathbb{R}^3 \) by adding a third vector \( \vec{w} \).

1. Perform this task such that the resulting basis is right handed
2. Perform this task such that the resulting basis is left handed

Exercise 20 Let \( \vec{u} = (-2, 3, 1), \vec{v} = (0, 4, 1) \). Find the area of the parallelogram bordered by \( \vec{u} \) and \( \vec{v} \).

Exercise 21 Using the cross product check if \( \vec{u} = (2, 6, 12) \) and \( \vec{v} = (1, 3, 6) \) are parallel.
Exercise 22  Let the corners of a triangle be $P = (0, 0, 0)$, $Q = (1, 1, 1)$ and $R = (0, 2, 0)$. Find a normal $\vec{n}$ for the triangle.

Exercise 23  Use the triple product method from Section 2.2.9 to derive from

$$\{\vec{u} = (1, 0, 1), \vec{v} = (0, 1, 1), \vec{w} = (1, 1, 1)\}$$

an orthogonal basis. If $\vec{u}$ corresponds to the view direction and $\vec{v}$ corresponds to upwards, which of the new vectors you find should be used as approximation for “upwards”?

Exercise 24  Sketch the parallelepiped (box) having as sides the vectors $\vec{u} = (1, 0, 2)$, $\vec{v} = (0, 1, 2)$ and $\vec{w} = (1, 1, 1)$. Find its volume.

Exercise 25  Use the scalar triple product to decide if the basis

$$\{(1, 0, 0), (0, 1, -1), (0, 1, 1)\}$$

is right handed or left handed. Interchange the order of the vectors to get a basis behaving opposite to this.

Exercise 26  In our reference frame upwards is defined by $\vec{u} = (0, 0, 1)$. A tank moves with velocity $\vec{v} = (1, 1, 0)$. We want it to move instead in the direction $\vec{d} = (3, 2, 0)$. Should we turn left or should we turn right?

Exercise 27  Determine the distance between $P_1 = (-2, 3, 0)$ and $P_0 = (4, 2, 3)$.

Exercise 28  As exercise 27 but now use instead of the Euclidean distance the Manhattan distance.

Exercise 29  Consider the triangle with corners $P = (1, 0, 0)$, $Q = (1, 1, 1)$, $R = (0, 2, 0)$. Find the centroid.

Exercise 30

1. Is a ball a convex set?
2. Is a banana a convex set?

Exercise 31  Let $P_0 = (1, 0, 0)$, $P_1 = (1, 1, 1)$, $P_2 = (0, 2, 0)$.

1. Show that $P_0, P_1, P_2$ is a simplex.
2. Find the barycentric coordinates of $P_0$
3. Find the barycentric coordinates of $P_1$
4. Which point has barycentric coordinates $(-1, 1, 1)$?
5. Consider the convex hull of $P_0, P_1, P_2$. What kind of geometric object do we have?
6. Find the centroid of the convex hull.

**Exercise 32** Consider the triangle having corners \( P_0 = (1,0,0), P_1 = (0,2,0), P_2 = (-1,-1,0) \). The triangle contains the point \( P = (0,0,0) \). Using the method from [VB] page 85 find the barycentric coordinates for \( P \).

**Exercise 33** Find the polar coordinates for the point \((3,-4)\)

**Exercise 34** Given the polar coordinates \( r = 2 \) and \( \theta = \frac{3\pi}{4} \) find the corresponding Cartesian coordinates.

**Exercise 35** Given the point \((1,1,1)\) find the corresponding spherical coordinates.

**Exercise 36** Given the spherical coordinates \( \rho = 2, \phi = \frac{\pi}{4} \) and \( \theta = \frac{\pi}{4} \) find the corresponding Cartesian coordinates.

**Exercise 37** Find at least three different parametric equations for the line through \((1,1,1)\) and \((-2,4,16)\).

**Exercise 38** Given the line \( L(t) = (1,1) + t(1,-1), t \in \mathbb{R} \) and the point \( Q = (4,4) \) find the distance between them.

**Exercise 39** Show that the points \((1,0,1), (2,1,-7), (2,3,4)\) are not collinear.

**Exercise 40** A plane contains the point \( P_0 = (-2,1,-1) \) and has normal vector \( \vec{n} = (1,2,3) \). Find a generalized plane equation for the plane.

**Exercise 41** The points \( P_1 = (0,0,0), P_2 = (1,1,1), P_3 = (0,1,0) \) are not collinear. Hence, they define a plane.

1. Find a generalized plane equation for the plane
2. Find a parametric equation for the plane

**Exercise 42** Are the points \( P_1 = (0,0,0), P_2 = (1,1,1), P_3 = (0,1,0), P_4 = (1,0,0) \) coplanar? Make sure to explain why or why not.

**Exercise 43** Consider the triangle having corners \( P_0 = (1,0,0), P_1 = (0,2,0), P_2 = (-1,-1,0) \). The point \( P = (0,0,0) \) is contained in the plane defined by \( P_0, P_1, P_2 \). Using the method from [VB] pages 83-84 show that the point \( P = (0,0,0) \) is contained in the triangle.
2  Partial solutions to exercises for Chapter 2 of [VB]

Solution to Exercise 2:
1. $\vec{u} = -\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$
2. $\vec{v} = -2\vec{u} + \vec{w}$
3. $\vec{w} = 2\vec{u} + \vec{v}$
4. $\vec{0} = 2\vec{u} + \vec{v} - \vec{w}$
5. Linearly dependent
6. A plane
7. Any two of the vectors $\vec{u}, \vec{v}, \vec{w}$ will do
8. No

Solution to Exercise 3:
1. Yes
2. No

Solution to Exercise 4:
1. 7
2. $\sqrt{21}$

Solution to Exercise 6: $(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5})$.

Solution to Exercise 7: 0.434 rad or 24, 84°.

Solution to Exercise 8:
1. Yes
2. No
3. Smaller

Solution to Exercise 9: No.

Solution to Exercise 10: (2, −1) or (−2, 1)

Solution to Exercise 11: Assuming not both $a$ and $b$ are zero we have two possibilities $(b, -a)$ and $(-b, a)$.

Solution to Exercise 12: Infinitely many choices.

Solution to Exercise 13:
1. \((\frac{24}{25}, 0, \frac{32}{25})\)
2. \((-\frac{24}{25}, 1, \frac{18}{25})\)

Solution to Exercise 14:

\[
\begin{align*}
\hat{w}_0 &= \vec{w}_0 = (1, 0, 0) \\
\hat{w}_1 &= \vec{w}_1 = (0, 1, 0) \\
\hat{w}_2 &= \vec{w}_2 = (0, 0, 1)
\end{align*}
\]

Solution to Exercise 15:

\[
\begin{align*}
\vec{w}_0 &= (1, 0, 1) \\
\vec{w}_1 &= (0, 1, 0) \\
\vec{w}_2 &= (-\frac{1}{2}, 0, \frac{1}{2}) \\
\hat{w}_0 &= (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \\
\hat{w}_1 &= (0, 1, 0) \\
\hat{w}_2 &= (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})
\end{align*}
\]

Solution to Exercise 16:

\[
\begin{align*}
\vec{w}_0 &= (1, 1, 1) \\
\vec{w}_1 &= (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \\
\vec{w}_2 &= (0, -\frac{1}{2}, \frac{1}{2}) \\
\hat{w}_0 &= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\
\hat{w}_1 &= (-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \\
\hat{w}_2 &= (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})
\end{align*}
\]

Solution to Exercise 19:

1. \(\vec{u} \times \vec{w} = (-1, -10, 7)\)
2. \((1, 10, -7)\)
Solution to Exercise 20: \( \sqrt{69} \)

Solution to Exercise 21: Start by checking if the cross product is close to \( \vec{0} \). If this is the case then the area of the parallelogram bordered by \( \vec{u} \) and \( \vec{v} \) is small, meaning that \( \vec{u} \) and \( \vec{v} \) are either close to being parallel or are close to being antiparallel. Finally, we check if the angle between \( \vec{u} \) and \( \vec{v} \) is greater than 90°.

Solution to Exercise 22: More possibilities. One is \( \vec{n} = (-2, 0, 2) \).

Solution to Exercise 23: \( \vec{w}_0 = \vec{u}, \vec{w}_1 = \vec{u} \times \vec{w}, \vec{w}_2 = \vec{u} \times (\vec{u} \times \vec{w}) \).
\( \vec{w}_1 \) “corresponds to upwards”.

Solution to Exercise 24: 3

Solution to Exercise 25:
\[
\{(1, 0, 0), (0, 1, -1), (0, 1, 1)\}
\]
is right handed.
To get left handed, interchange the last two vectors.

Solution to Exercise 26: \( \vec{u} \cdot (\vec{v} \times \vec{d}) < 0 \) and therefore we must turn right.

Solution to Exercise 29: \( (\frac{2}{3}, 1, \frac{1}{3}) \)

Solution to Exercise 30:
1. Yes
2. No

Solution to Exercise 31:
1. The vectors \( P_1 - P_0 \) and \( P_2 - P_0 \) are linearly independent.
2. \( (1, 0, 0) \)
3. \( (0, 1, 0) \)
4. \( (0, 3, 1) \)
5. The triangle with corners \( P_0, P_1, P_2 \)
6. See solution to exercise 29

Solution to Exercise 32:
The barycentric coordinates are \( (a_0 = \frac{2}{5}, a_1 = \frac{1}{5}, a_2 = \frac{2}{5}) \) corresponding to \( (s = \frac{1}{5}, t = \frac{2}{5}) \).
Solution to Exercise 33: \( r = 5, \theta = -0.9272... \)

Solution to Exercise 34: \((-\sqrt{2}, \sqrt{2})\)

Solution to Exercise 35: \(\rho = \sqrt{3}, \phi = 0.9553..., \theta = \frac{\pi}{4}\)

Solution to Exercise 36: \((1,1,\sqrt{2})\)

Solution to Exercise 41:
First find the normal
\[ \vec{n} = (P_2 - P_1) \times (P_3 - P_1) = (-1, 0, 1). \]

Then find the plane equation as
\[ (x - 0, y - 0, z - 0) \cdot (-1, 0, 1) = 0 \]
\[ \downarrow \]
\[ -x + z = 0 \]

One parametric equation is
\[ P(s, t) = P_1 + s(P_2 - P_1) + t(P_3 - P_1) \]
\[ \downarrow \]
\[ P(s, t) = s(1, 1, 1) + t(0, 1, 0) \]
\[ \downarrow \]
\[ P(s, t) = (s, s + t, s) \]

3 Exercises for Chapter 3 of [VB]

Exercise 44 Consider the map \( \tau: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) given by
\[ \tau((a, b, c)^T) = (a + c, b + c, 0)^T \]
for all choices of \( a, b, c \).
1. Is it a linear transformation?
2. Determine the null space?
3. Determine nullity(\( \tau \)) and rank(\( \tau \)).

Exercise 45 This is a continuation of Exercise 44. Find the matrix \( A \) such that
\[ A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tau([x, y, z]^T) \]
Exercise 46  A linear transformation $\tau : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$\begin{align*}
\tau((1,0,0)^T) &= (1,1)^T \\
\tau((0,1,0)^T) &= (2,3)^T \\
\tau((0,0,1)^T) &= (0,1)^T
\end{align*}$$

1. Find $\tau((2,-7,4)^T)$.

2. Determine $\text{nullity}(\tau)$ and $\text{rank}(\tau)$.

3. Find the matrix $A$ such that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tau([x,y,z]^T)$$

Exercise 47  Let

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Find $A + B$, $A - B$ and $B - A$.

Exercise 48  Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ -2 & 4 \end{bmatrix}$$

Find $A^T + B$ and $A + B^T$.

Exercise 49  Determine the product

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 & 7 \\ 0 & -2 & 4 & 0 \\ 1 & 7 & 2 & 2 \\ -1 & 2 & -1 & 1 \end{bmatrix}$$

using a method similar to the one described on the top of page 95 in [VB]. You should set $A = B = C = I_2$. Then do the same for the product

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & -2 & 4 \\ 1 & 7 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

Exercise 50

1. Find $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

2. Find $\begin{bmatrix} 7 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3. How are the results in 1. and 2. related?

Exercise 51 Let
\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 7 \end{bmatrix} \]

Find \( BA^T \) and \( AB^T \).

Exercise 52 Let
\[ A = \begin{bmatrix} 0 & 2 & 1 & 5 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} \]

Find \( AB \).

Exercise 53 Let
\[ A = \begin{bmatrix} 2 & 3 \\ 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \]

Find \( AB - BA \).

Exercise 54 Using the method from page the top of page 95 determine \( AB \), where
\[ A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \]

Exercise 55 In this exercise we use the tensor product to project onto the vector \( \hat{v} = (\sqrt{2}, 0, -\sqrt{2})^T \). Project the vectors \( \vec{u}_1 = (1, 2, 0)^T \), \( \vec{u}_2 = (\sqrt{2}, \sqrt{2}, \sqrt{2})^T \).

Exercise 56 Let \( \vec{v} = (1, 2, -2)^T \). Find the matrix \( \vec{v} \). Using \( \vec{v} \) calculate \( \vec{v} \times \vec{w} \), where \( \vec{w} = (0, 1, 2)^T \).

Exercise 57 Let
\[ A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

and let
\[ A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \]

Consider the linear transformation \( T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) given by
\[ T_1([x, y, z]^T) = A_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]
and consider the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_2([x, y]^T) = A_2 \begin{bmatrix} x \\ y \end{bmatrix}.$$ 

Find the matrix $A_3$, such that

$$T_2(T_1([x, y, z]^T)) = A_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$ 

Find $T_2(T_1([1, 2, 1]^T))$.

**Exercise 58** Give the augmented matrix corresponding to the following system of linear equations.

$$\begin{cases} 2x + 3y + 2z = 18 \\ x - y - z = 0 \\ x + y + z = 8 \end{cases}$$

Solve the system.

**Exercise 59** Give the augmented matrix corresponding to the following system of linear equations.

$$\begin{cases} x + 2y + 4z - u = 2 \\ y - z = 4 \\ 2x + 12y + z + u = 2 \\ 4x + 19y + 6z - u = 19 \end{cases}$$

Solve the system.

**Exercise 60** Give the augmented matrix corresponding to the following system of linear equations.

$$\begin{cases} x + 2y + 3z = 4 \\ 4y - z = 1 \\ 2x + 12y + 4z = 10 \end{cases}$$

Solve the system.

**Exercise 61** Give the augmented matrix corresponding to the following system of linear equations.

$$\begin{cases} x + 2y + 3z + u = 4 \\ 4y - z + u = 1 \\ 2x + 12y + 4z + 4u = 10 \end{cases}$$

Solve the system.
Exercise 62 Let

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix} \]

Determine \( A^{-1} \). Solve the equations \( A\vec{x} = [2, 0, 3]^T \), \( A\vec{x} = [1, 0, 7]^T \), \( A\vec{x} = [a, b, c]^T \), \( A\vec{x} = [0, 0, 0]^T \). Let \( T \) be the linear transformation corresponding to \( A \). Determine \( \text{Nullity}(T) \) and \( \text{rank}(T) \)?

Exercise 63 Consider

\[ A = \begin{bmatrix} 4 & 0 & 1 \\ 7 & 2 & 1 \\ 6 & 4 & 0 \end{bmatrix} \]

Solve the equation \( A\vec{x} = \vec{0} \). Let \( T \) be the linear transformation corresponding to \( A \). What is \( \text{N}(T) \), \( \text{Nullity}(T) \) and \( \text{rank}(T) \)? Using elementary row operation on \( A \) derive a row echelon form (you might have done this already) and show that \( \text{rank}(T) \) equals the number of non-zero rows. Is this a coincidence?

Exercise 64 Find the inverse of the following matrices

\[ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix} \]

Exercise 65 Find (if possible) the inverse of the following matrix

\[ A = \begin{bmatrix} 4 & 7 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

Exercise 66 Let

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \]

Show by direct calculations that \( \det(AB) = \det(A)\det(B) \) holds.

Exercise 67 Using the definition on page 123 determine

\[ \det \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 7 & 1 & 1 \end{bmatrix} \]

Exercise 68 Using a clever method determine \( \det(A) \), where

\[ A = \begin{bmatrix} 2 & 0 & 1 & -2 \\ 7 & 0 & 1 & 0 \\ 12 & 0 & 1 & 0 \\ 4 & 1 & 1 & 4 \end{bmatrix} \]

Does \( A \) posses an inverse?
Exercise 69  *Using a clever method determine \( \det(A) \), where*

\[
A = \begin{bmatrix}
2 & 1 & 1 & 2 \\
0 & 0 & 7 & 4 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

*Does \( A \) posses an inverse?*

Exercise 70  *Find the volume of the parallelepiped bounded by basis vectors \((0, 1, 2), (2, 1, -1), (4, 7, 0)\).*

Exercise 71  *Using the method described at page 127 and page 128 determine \( \det(A) \), where*

\[
A = \begin{bmatrix}
1 & 0 & 1 & 1 \\
4 & 1 & 1 & 0 \\
5 & 1 & 2 & 7 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

4  Partial solutions to exercises for Chapter 3 of [VB]

Solution to Exercise 44:
1. Yes
2. \( \{(-s, -s, s) \mid s \in \mathbb{R}\} \)
3. 1,2

Solution to Exercise 58:  \( (x, y, z) = (4, 2, 2) \)
Solution to Exercise 59:  \( \emptyset \)
Solution to Exercise 60:  \( (x, y, z) = (3\frac{1}{2} - 3\frac{1}{2}s, \frac{1}{4} + \frac{1}{4}s, s), s \in \mathbb{R} \)
Solution to Exercise 65:  *Not possible*

Solution to Exercise 67:  27

Solution to Exercise 68:  10. Yes.

Solution to Exercise 69:  70. Yes.

Solution to Exercise 71:  -24.

5  Exercises for Chapter 4 of [VB]
Exercise 72  Let $V = \mathbb{R}^3$ be described as an affine space by $O = (0,0,0)^T$, 
$\{\vec{v}_0 = (1,0,0)^T, \vec{v}_1 = (0,1,0)^T, \vec{v}_2 = (0,0,1)^T\}$. Let $W = \mathbb{R}^3$ be described as an 
affine space by $O = (0,0,0)^T$, $\{\vec{w}_0 = (1,0,0)^T, \vec{w}_1 = (0,1,0)^T, \vec{w}_2 = (0,0,1)^T\}$. Consider the affine transformation $T : V \rightarrow W$ given by

$$T(O) = 2\vec{w}_0 + \vec{w}_1 - \vec{w}_2 + O$$
$$T(\vec{v}_0) = \vec{w}_0 - \vec{w}_2$$
$$T(\vec{v}_1) = \vec{w}_1 + \vec{w}_2$$
$$T(\vec{v}_2) = \vec{w}_0 + \vec{w}_1 + 2\vec{w}_2$$

Describe the corresponding matrix (use the description at bottom of page 138). What is the transformation of the point $(1,2,1)^T$?

Exercise 73  Let all values be as in exercise 72 with the only exception, that we replace the above values of $\vec{w}_0$, $\vec{w}_1$, $\vec{w}_2$ with the following: $\vec{w}_0 = (1,0,1)^T$, $\vec{w}_1 = (0,1,0)^T$, $\vec{w}_2 = (0,1,1)^T$. What is the the transformation of the point $(1,2,1)^T$?

Exercise 74  Consider the affine transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ corresponding to a 
translation of all points by $\vec{t} = (1,-2,7)^T$. Find the corresponding matrix.

Exercise 75  Find the matrix that describes the affine transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ 
given by a scaling along the $x$-axis with a factor 2, a scaling along the $y$-axis 
with a factor 4, and finally by a scaling along the $z$-axis with a factor $1/3$. What is the point $(0,0,0)^T$ transformed to? What is the point $(1,1,1)^T$ transformed to?

Exercise 76  This is a continuation of Exercise 74 and Exercise 75. Find the matrix that describes the affine transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by a scaling along 
the $x$-axis with a factor 2, a scaling along the $y$-axis with a factor 4, a scaling 
along the $z$-axis with a factor $1/3$, and finally a translation of all points by $\vec{t} = (1,-2,7)^T$.

Exercise 77  This is a continuation of Exercise 74 and Exercise 75. Find the matrix that describes the affine transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by a translation 
of all points by $\vec{t} = (1,-2,7)^T$, a scaling along the $x$-axis with a factor 2, a 
scaling along the $y$-axis with a factor 4, and finally a scaling along the $z$-axis 
with a factor $1/3$.

Exercise 78  Consider the inverse of the transformation treated in Exercise 74. Find the corresponding matrix.

Exercise 79  Consider the inverse of the transformation treated in Exercise 75. Find the corresponding matrix.

Exercise 80  How are the matrices in Exercise 76 and Exercise 77 related?
Exercise 81 Determine the matrix $A$ that describes a rotation around the $x$-axis with $-45^\circ$. Find in an easy way $A^{-1}$.

Exercise 82 Determine the matrix $A$ that describes the rotation around the $z$-axis with $45^\circ$ followed by a rotation around the $y$-axis with $180^\circ$ followed by a rotation around the $x$-axis with $90^\circ$.

Exercise 83 Determine the matrix $A$ that describes a rotation around the $z$-axis with $-45^\circ$ followed by a rotation around the $y$-axis with $-45^\circ$ followed by a rotation around the $x$-axis with $45^\circ$. Find $A^{-1}$.

Exercise 84 Determine the matrix that describes rotation around the axis $(\sqrt{2}/2, \sqrt{2}/2, 0)^T$ with $45^\circ$.

Exercise 85 Determine the matrix that describes rotation around the axis $(1, 1, 1)^T$ with $90^\circ$ (be careful!)

Exercise 86 Consider the affine transformation given by

$$
\begin{bmatrix}
2 & 0 & 0 & 2 \\
0 & 3 & 0 & 3 \\
0 & 0 & 2 & 7 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Find the matrix that corresponds to the inverse affine transformation.

Exercise 87 Consider the affine transformation given by

$$
\begin{bmatrix}
1 & 0 & 2 & -2 \\
0 & 1 & 3 & -2 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Find the matrix that corresponds to the inverse affine transformation.

Exercise 88 Consider the line in $\mathbb{R}^3$ given by

$$L(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $Q_1$ and $Q_2$ be two points on this line that are at distance $\sqrt{2}$ apart from each other. We now transform the affine by applying the transformation $T$ given by:

$$
\begin{bmatrix}
2 & 0 & 0 & 2 \\
0 & 2 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

What is the distance between $T(Q_1)$ and $T(Q_2)$?
Exercise 89  Consider the matrix
\[ A = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

We have \( AA^T = I \) (please check); but \( \det(A) = -1 \neq 1 \) (please check). Hence, \( A \) is not a rotation matrix. Give a physical explanation of what \( A \) is doing.

Exercise 90  Determine the matrix
\[ \begin{bmatrix} A & \vec{y} \\ \vec{0}^T & 1 \end{bmatrix} \]

that corresponds to a rotation with 90° around an axis through the point
\[ P_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

and pointing in the direction
\[ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

.

Exercise 91  Consider the plane through origo with normal \( \vec{n} = [3, 4, 0]^T \). We consider shear defined from this plane and from the shear vector \( \vec{s} = [-4, 3, 1]^T \). Determine the matrix, that describes the shear as an affine transformation (be careful). Find the transformation of the point \((1, 1, 1)^T\). Find the transformation of the point \((0, 0, 0)^T\).

Exercise 92  Calculate the distance from the point \((-23/5, 26/5, 12/5)^T\) to the plane through origo with normal \( \hat{n} = [3, 4, 5, 0]^T \). Calculate the distance from \((1, 1, 1)^T\) to the same plane. The result hopefully indicates that your calculations in Exercise ?? are correct. Explain

Exercise 93  Consider the plane through the point \((1, 1, 1)^T\) with normal \( \vec{n} = [0, 4, 3]^T \). We now perform a shear with respect to this plane and the shear vector \( \vec{s} = [2, 3, -4]^T \). Determine the matrix that describes this shear. Find the transformation of the point \((1, 1, 1)^T\). Find the transformation of the point \((0, 0, 0)^T\).

Exercise 94  Let an affine transformation be given by the matrix
\[ M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Find $\text{det}(M)$ and $M^{-1}$. Consider the plane through $P = (1, 1, 1)^T$ with normal $\vec{n} = [1, 0, 0]^T$. Find the normal of the plane after the affine transformation. Find the transformation of the point $P$. Give the equation for the new plane.

Exercise 95 Consider the reflection across the plane through origo with normal $\vec{n} = [0, 3, 4]^T$. Find the matrix that describes this transformation. Find the reflection of the point $(1, 1, 1)^T$.

Exercise 96 Consider the reflection through origo. Find the reflection of the point $(2, 3, 2)^T$.

6 Partial solutions to exercises for Chapter 4 of [VB]

Solution to Exercise 74:
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 7 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Solution to Exercise 75:
\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

$(0, 0, 0)^T, (2, 4, \frac{1}{3})^T$

Solution to Exercise 76:
\[
\begin{bmatrix}
2 & 0 & 0 & 1 \\
0 & 4 & 0 & -2 \\
0 & 0 & 1 & 7 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Solution to Exercise 77:
\[
\begin{bmatrix}
2 & 0 & 0 & 2 \\
0 & 4 & 0 & -8 \\
0 & 0 & 1 & 7 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Solution to Exercise 82:
\[
\begin{bmatrix}
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 1 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0
\end{bmatrix}
\]
Solution to Exercise 83:

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}-2}{4} & \frac{2-\sqrt{2}}{4} & -\frac{1}{2} \\
\frac{\sqrt{2}-2}{4} & \frac{2+\sqrt{2}}{4} & \frac{1}{2}
\end{bmatrix}
\]

A\(^{-1}\) = A\(^T\).

Solution to Exercise 84:

\[
\begin{bmatrix}
\frac{\sqrt{2}+2}{4} & \frac{2-\sqrt{2}}{4} & \frac{1}{2} \\
\frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix}
\]

Solution to Exercise 85:  Hint: start by normalizing \(\vec{r}\). If you forget to do so you will get a wrong result!

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} \\
\frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{-1-\sqrt{3}}{3} \\
\frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3}
\end{bmatrix}
\]

Solution to Exercise 86:

\[
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & -1 \\
0 & \frac{1}{3} & 0 & -1 \\
0 & 0 & \frac{1}{2} & -\frac{7}{2} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Solution to Exercise 90:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
= \cdots
\]

Solution to Exercise 91:

\[
\begin{bmatrix}
-\frac{7}{5} & -\frac{16}{5} & 0 & 0 \\
\frac{9}{5} & \frac{17}{5} & 0 & 0 \\
\frac{3}{5} & \frac{4}{5} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\((-\frac{23}{5}, \frac{26}{5}, \frac{12}{5}), \ (0,0,0)\)

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Solution to Exercise 92: \( \frac{7}{5}, \frac{7}{5} \). Use the fact that \( \vec{s} \) is contained in the plane.

Solution to Exercise 93:

\[
\begin{bmatrix}
1 & \frac{8}{5} & \frac{6}{5} & -\frac{14}{5} \\
0 & \frac{17}{5} & \frac{9}{5} & -\frac{21}{5} \\
0 & -\frac{16}{5} & -\frac{7}{5} & \frac{28}{5} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\((1, 1, 1), (-\frac{14}{5}, -\frac{21}{5}, \frac{28}{5})\)

Solution to Exercise 94:

\[
M^{-1} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\(\vec{n}' = [1, 0, 0]^T, (4, 3, 2), x - 4 = 0.\)

Solution to Exercise 95:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{7}{25} & -\frac{24}{25} & 0 \\
0 & -\frac{24}{25} & -\frac{7}{25} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\((1, -\frac{17}{25}, -\frac{31}{25})\)

Solution to Exercise 96: You answered the exercise correctly...

7 Exercises for Chapter 5 of [VB]

Exercise 97 Consider the matrix

\[
R = \begin{bmatrix}
\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & 1 \\
\frac{\sqrt{6}+2\sqrt{2}}{8} & -\frac{\sqrt{6}+2\sqrt{2}}{8} & -\frac{3}{4} \\
-\frac{\sqrt{2}+2\sqrt{6}}{8} & \frac{\sqrt{2}}{2}+2\sqrt{6} & -\frac{\sqrt{3}}{4}
\end{bmatrix}
\]

Find angles \( \theta_x, \theta_y, \theta_z \) such that \( R = R_xR_yR_z \). How many choices are there of \( \theta_x, \theta_y, \theta_z \)? List them.

Exercise 98 Consider the matrix

\[
R = \begin{bmatrix}
0 & 0 & 1 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0
\end{bmatrix}
\]
Find angles $\theta_x, \theta_y, \theta_z$ such that $R = R_x R_y R_z$. Are there more choices of $\theta_x, \theta_y, \theta_z$?

**Exercise 99** Consider the matrix

$$
A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0
\end{bmatrix}
$$

1. Show that $A$ is a rotation matrix.
2. What happens to the point $[-1, -1, 0]^T$ under the rotation?
3. Determine rotation angle and rotation axis.

**Exercise 100** Consider the rotation matrix

$$
A = \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

Determine rotation angle and rotation axis.

**Exercise 101** Find the quaternion that corresponds to rotation around the axis $[1, 2, 2]^T$ with $\theta = 90^\circ$.

**Exercise 102** Determine the quaternion that corresponds to rotation around the axis $[-1, -2, -2]^T$ with $\theta = 270^\circ$.

**Exercise 103** Determine the quaternion that corresponds to rotation around the axis $[1, 1, 1]^T$ with $\theta = 60^\circ$.

**Exercise 104** Determine the quaternion that corresponds to the rotation matrix $A$ from Exercise 99 in the following two ways

1. Use the result from Exercise 99
2. Use the theory from page 191

**Exercise 105** Determine the quaternion that corresponds to the matrix $A$ from Exercise 100 in the following two ways

1. Use the result from Exercise 100
2. Use the theory from page 191

**Exercise 106** Given the quaternion $q = \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}, 0, 0\right)$ find all possible corresponding fixed angles.

**Exercise 107** Given the quaternion $q = \frac{\sqrt{3}}{2} + \left(\frac{1}{2}, \frac{1}{2}, 0\right)$
1. Show that $||q|| = 1$.

2. Determine the corresponding rotation matrix

**Exercise 108** Determine the quaternion $q_z$ that corresponds to rotation around the $z$-axis with $\theta_z$. Determine the quaternion $q_y$ that corresponds to rotation around the $y$-axis with $\theta_y$. Determine the quaternion $q_x$ that corresponds to rotation around the $x$-axis with $\theta_x$. Calculate $q_xq_yq_z$ and compare to the theory at the top of page 193.

**Exercise 109** Consider the quaternions $p = 2 + (1, 0, 1)$ and $q = 1 + (1, 3, 0)$.

1. Find $pq$

2. Find $p^{-1}$

**Exercise 110** Find the quaternion $q$ that corresponds to rotation around the axis $(1, 1, 0)$ with $90^\circ$. Consider the point $(1, 1, 1)$. Let $p_1 = 0 + (1, 1, 1)$ be the corresponding quaternion. Calculate $qp_1q^{-1}$. Let $p_2 = 0 + (1, 1, 0)$. Argue that $qp_2q^{-1}$ equals $p_2$. Show this by performing the exact calculations.

**Exercise 111** Consider the rotation with rotation axis $(1, 0, 1)$ and rotation angle $90^\circ$. Given the point $(x, y, z)$ consider the quaternion $p = 0 + (x, y, z)$. The above rotation sends $(x, y, z)$ to $(x', y', z')$ corresponding to the quaternion $p' = 0 + (x', y', z')$.

1. Determine the quaternion $q$ such that $p' = qpq^{-1}$

2. Find the matrix $A$ such that

$$
\begin{bmatrix}
  x' \\
  y' \\
  z'
\end{bmatrix} = A
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
$$

**Exercise 112** Consider the rotation with rotation axis $(4, 0, 3)$ and rotation angle $90^\circ$. Given the point $(x, y, z)$ consider the quaternion $p = 0 + (x, y, z)$. The above rotation sends $(x, y, z)$ to $(x', y', z')$ corresponding to the quaternion $p' = 0 + (x', y', z')$.

1. Determine the quaternion $q$ such that $p' = qpq^{-1}$.

2. Demonstrate that $||q|| = 1$

3. What is $(1, 0, 0)$ being rotated to?
8 Partial solutions to exercises for Chapter 5 of [VB]

Solution to Exercise 97: \((120^\circ, 30^\circ, 45^\circ), (300^\circ, 150^\circ, 225^\circ)\).

Solution to Exercise 98: One possibility is \((45^\circ, 90^\circ, 0^\circ)\). This is not the only possibility.

Solution to Exercise 99: \([-1, -1, 0]^T, 90^\circ, [1, 1, 0]^T\) (your answer may differ by a positive scaling factor).

Solution to Exercise 100: \(180^\circ, [1, 0, 1]^T\) (your answer may differ by a positive scaling factor).

Solution to Exercise 101: \(\sqrt{2}^2 + (\sqrt{2}/2, \sqrt{2}/2, 0)

Solution to Exercise 102: \(-\sqrt{2}^2 + (\sqrt{2}/2, -\sqrt{2}/2, 0)

Solution to Exercise 103: \(\sqrt{3}^2 + (1/2\sqrt{3}, 1/2\sqrt{3}, 1/2\sqrt{3})

Solution to Exercise 104: \(\sqrt{2}^2 + (1/2, 1/2, 0)

Solution to Exercise 105: \(0 + (\sqrt{2}/2, 0, 0)

Solution to Exercise 106: \((\theta_x, \theta_y, \theta_z) = (90^\circ, 0^\circ, 0^\circ)\) og \((\theta_x, \theta_y, \theta_z) = (90^\circ, 180^\circ, 180^\circ)\)

Solution to Exercise 107:
\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0
\end{bmatrix}
\]

Solution to Exercise 108: \(q_z = \cos(\theta_z/2) + \sin(\theta_z/2)(0, 0, 1), q_y = \cos(\theta_y/2) + \sin(\theta_y/2)(0, 1, 0), q_x = \cos(\theta_x/2) + \sin(\theta_x/2)(1, 0, 0)\).

Solution to Exercise 109: \(1 + (0, 7, 4), \frac{1}{3} - (1/6, 0, 1/6)\).

Solution to Exercise 110: \(\sqrt{2}^2 + (\frac{1}{2}, \frac{1}{2}, 0), (\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}, 0)\).

Solution to Exercise 111: \(\sqrt{2}^2 + (\frac{1}{2}, 0, \frac{1}{2})

\[
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2}
\end{bmatrix}
\]
9 Exercises for Chapter 10 of [VB]

Exercise 113 In this exercise we use linear interpolation to fit a curve \( P(t) \) through the points \( P_0 = (1, 0, 1) \) og \( P_1 = (-1, 2, -7) \). We want to pass \( P_0 \) at time \( t_0 = 2 \) and to pass \( P_1 \) at time \( t_1 = 10 \). Which point do the curve pass at time \( t = 4 \)? Determine \( P(t) \).

Exercise 114 Find the Hermite curve \( Q(u) \), \( u = 0, \ldots , 1 \) with \( Q(0) = (0, 1, 2) \), \( Q'(0) = (2, 1, 0) \), \( Q(1) = (2, 2, 2) \) and \( Q'(1) = (2, 1, 0) \). Explain the nice answer.

Exercise 115 Find the Hermite curve \( Q(u) \), \( u = 0, \ldots , 1 \) with \( Q(0) = (0, 1, 2) \), \( Q'(0) = (1, 0, 0) \), \( Q(1) = (2, 2, 2) \) and \( Q'(1) = (1, 0, 1) \).

Exercise 116 Find the piecewise Hermite curve \( Q_0(u), Q_1(u) \) such that \( Q_0(0) = (0, 0, 0) \), \( Q'_0(0) = (1, 0, 0) \), \( Q_0(1) = Q_1(0) = (1, 1, 0) \), \( Q'_0(1) = Q'_1(0) = (0, 0, 1) \), \( Q_1(1) = (1, 0, 0) \), \( Q'_1(1) = (0, 1, 0) \) hold.

Exercise 117 Find the piecewise Hermite curve \( Q_0(u), Q_1(u) \) such that \( Q_0(0) = (0, 0, 0) \), \( Q'_0(0) = (1, 0, 0) \), \( Q_0(1) = Q_1(0) = (1, 1, 0) \), \( Q'_0(1) = Q'_1(0) = (0, 0, 1) \), \( Q_1(1) = (1, 0, 0) \), \( Q'_1(1) = (0, 1, 0) \) hold.

Exercise 118 Find the piecewise Hermite curve \( Q_0(u), Q_1(u) \) such that \( Q_0(0) = (0, 0, 0) \), \( Q_0(1) = Q_1(0) = (1, 1, 0) \), \( Q'_0(1) = Q'_1(0) \), \( Q_0(1) = (1, 0, 0) \), \( Q'_0(1) = Q'_1(0) \) and \( Q'_1(0) \) hold.

Exercise 119 Consider the Bezier curve with control points \( P_0 = (1, 0, 1) \), \( P_1 = (2, 0, 1) \), \( P_2 = (2, 2, 3) \) and \( P_3 = (2, 2, 2) \). Find \( Q(u) \). What are the tangent to the curve at the end points \( P_0 \) and \( P_3 \)?

Exercise 120 Consider the quaternions \( p = \frac{\sqrt{2}}{2} + (0, \frac{\sqrt{2}}{2}, 0) \) and \( q = \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2}, 0, 0) \). Determine the function \( \text{slerp}(p, q, t) \). Calculate (in floating point) \( \text{slerp}(p, q, \frac{1}{4}) \).

Exercise 121 Let the quaternions \( p \) and \( q \) be as in Exercise 120. Calculate (in floating point) \( \text{lerp}(p, q, \frac{1}{4}) \).

10 Solutions to exercises for Chapter 10

Solution to Exercise 113: \( P(t) = (3/2, -1/2, 3) + t(-1/4, 1/4, -1) \), \( P(4) = (1/2, 1/2, -1) \).

Solution to Exercise 114: \( Q(u) = u(2, 1, 0) + (0, 1, 2) \).

Solution to Exercise 115: \( Q(u) = u^3(-2, -2, 1) + u^2(3, 3, -1) + u(1, 0, 0) + (0, 1, 2) \).
Solution to Exercise 116: $Q_0(u) = u^3(-1, -2, 1) + u^2(1, 3, -1) + u(1, 0, 0) + (0, 0, 0), Q_1(u) = u^3(0, 3, 1) + u^2(0, -4, -2) + u(0, 0, 1) + (1, 1, 0)$

Solution to Exercise 117: (mellemresultat: $\vec{P}_0' = (1/2, -1/4, 0)$). $Q_0(u) = u^3(-1/2, -9/4, 0) + u^2(1/2, 13/4, 0) + u(1, 0, 0) + (0, 0, 0), Q_1(u) = u^3(1/2, 11/4, 0) + u^2(-1, -7/2, 0) + u(1/2, -1/4, 0) + (1, 1, 0)$

Solution to Exercise 118: (mellemresultat: $\vec{P}_0' = (5/4, 3/2, 0), \vec{P}_1' = (1/2, 0, 0)$, $\vec{P}_2' = (-1/4, -3/2, 0)$) $Q_0(u) = u^3(-1/4, -1/2, 0) + u(5/4, 3/2, 0) + (0, 0, 0), Q_1(u) = u^3(1/4, 1/2, 0) + u^2(-3/4, -3/2, 0) + u(1/2, 0, 0) + (1, 1, 0)$

Solution to Exercise 119: $(u^3 - 3u^2 + 3u + 1, 4u^3 + 6u^2, -5u^3 + 6u^2 + 1)$. $3(P_1 - P_0) = (3, 0, 0), 3(P_3 - P_2) = (0, 0, -3)$.

Solution to Exercise 120:

$$\sin((1-t)60^\circ)p + \sin(t60^\circ)q \over \sqrt{3}$$

$(0.7887, 0.2113, 0.5774, 0)$

Solution to Exercise 121:

$(0.7845, 0.1961, 0.5883, 0)$