Lecture Notes on Polynomials

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1 Introduction

These lecture notes give a very short introduction to polynomials with real and complex coefficients. They are a supplement to the book extract [1].

2 Definitions and Some Properties

Polynomials with complex coefficients are functions of a complex variable z of a particularly simple form. Examples are

$$z^2 + (8+i)z + 4, \quad z^{16} - 64, \quad (7-8i)z^3 - (4+4i)z^2 - \sqrt{17}, \quad 232, \quad \text{and} \quad z - 1. \quad (2.1)$$

The formal definition is as follows.

Definition 2.1. A polynomial with complex coefficients is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad (2.2)$$

where $a_j \in \mathbf{C}$, $j = 0, 1, \ldots, n$, and z is a complex variable. If $a_n \neq 0$, then n is the *degree* of $p(z)$, which is written as $\deg(p(z)) = n$. In general, the degree of a polynomial $p(z)$ is the largest k such that $a_k \neq 0$. The polynomial with all coefficients equal to zero is called the *zero polynomial*. The degree of the zero polynomial is defined to be zero[1].

Looking at the examples in (2.1), we see that the degree of the first polynomial is 2, the second one has degree 16, etc.

A number of operations can be performed with polynomials. Given a polynomial $p(z)$ and a complex number c, the polynomial $c\, p(z)$ is obtained by multiplying each coefficient in $p(z)$ by c. Given two polynomials $p(z)$ and $q(z)$, their sum is defined by adding the coefficients of corresponding power. Some examples will illustrate these definitions.

Given $p(z) = z^3 - iz + 1 + 7i$ and $c = 1 - i$, we have

$$c\, p(z) = (1-i)(z^3 - iz + 1 + 7i) = (1-i)z^3 + (1-i)(-i)z + (1-i)(1+7i)$$
$$= (1-i)z^3 + (-1-i)z + 8 + 6i.$$

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[1]This choice requires some care in certain computations, which however will not be needed there. In most cases one prefers to assign the degree $-\infty$ to the zero polynomial, but that also requires some care in computations

Let $q(z) = -z^3 + 4z^2 - 8z - 8$. Then the polynomial $p(z) + q(z)$ is given by

$$\begin{aligned} p(z) + q(z) &= (z^3 - iz + 1 + 7i) + (-z^3 + 4z^2 - 8z - 8) \\ &= (1 + (-1))z^3 + (0 + 4)z^2 + (-i + (-1 - i))z + (1 + 7i + (-8)) \\ &= 4z^2 + (-1 - 2i)z - 7 + 7i. \end{aligned}$$

Note that since the coefficient to the term z^2 in $p(z)$ is zero, it is not written explicitly in the usual expression for $p(z)$, but we have included it in the computation above to clarify the principle of addition.

Polynomials can be multiplied. Given $p_1(z) = z^2 - i$ and $q_1(z) = z^3 - z$, the product is obtained by multiplying out and collecting coefficients to the same power of z. We have

$$\begin{aligned} p_1(z)q_1(z) &= (z^2 - i)(z^3 - z) = z^2(z^3 - z) - i(z^3 - z) \\ &= z^5 - z^3 - iz^3 + iz = z^5 + (-1 - i)z^3 + iz. \end{aligned}$$

In general, we cannot divide polynomials and obtain a quotient, which is again a polynomial. But division with remainder can be carried out. The method is the same as used for integers. Given the integers $m = 9$ and $n = 4$, division of m by n with remainder means that we can write $m = kn + r$, where k is an integer, and the remainder r is an integer that satisfies $0 \le r < n$. Thus the result in the example is $9 = 2 \cdot 4 + 1$. The assumption needed to carry out this division with remainder is that $m \ge n$.

Given two polynomials $p_1(z)$ and $p_2(z)$, such that $\deg(p_1) \ge \deg(p_2) > 0$, division with remainder means to write

$$p_1(z) = q(z)p_2(z) + r(z).$$

Here $q(z)$ is a polynomial and $r(z)$ is a polynomial satisfying $0 \le \deg(r) < \deg(p_2)$.

Here are some examples to illustrate this procedure. First let us take $p_1(z) = z^4 - z^3 + z^2 - z$ and $p_2(z) = z^2 - 1$. Then the result is

$$z^4 - z^3 + z^2 - z = (z^2 - z + 2)(z^2 - 1) + (-2z + 2),$$

such that $q(z) = z^2 - z + 2$ and $r(z) = -2z + 2$.

For the next example we take $p_1(z) = 4z^4 - 64$ and $p_2(z) = z^2 - 4$. In this case

$$4z^4 - 64 = (4z^2 + 16)(z^2 - 4) + 0.$$

Thus in this case the remainder is zero.

There are various ways of doing these divisions with remainder by hand. At least one of the methods will be illustrated during the lectures. It is also possible to use Maple to carry out the computation of the quotient and the remainder. The functions are called `quo` and `rem`, respectively. See the Maple documentation for their use.

3   Roots of Polynomials

We introduce the following definition.

Definition 3.1. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \ge 1$. A complex number $z_0 \in \mathbf{C}$ is called a *root* of $p(z)$, if $p(z_0) = 0$.

Thus a root of the polynomial $p(z)$ is just a different name for a *zero* of $p(z)$ as a function. The reason for using a special name is that roots of a polynomial have many nice properties not shared by zeroes of general functions.

We have the following important result.

Proposition 3.2. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then $z_0 \in \mathbf{C}$ is a root of $p(z)$, if and only if there exists a polynomial $q(z)$ (of degree $n-1$), such that*

$$p(z) = q(z)(z - z_0). \tag{3.1}$$

Proof. If (3.1) holds, then it is obvious that $p(z_0) = 0$. Conversely, assume that z_0 is a root of $p(z)$. Then we can use the division with remainder described in the previous section to write $p(z) = q(z)(z - z_0) + c$, where c is the remainder, a polynomial of degree 0. Now if we use that $p(z_0) = 0$, it follows that $c = 0$ and (3.1) holds. $\qquad\square$

Definition 3.3. Let $p(z)$ be a polynomial of degree $n \geq 1$. Assume that z_0 is a root of $p(z)$. We define the *multiplicity* of the root z_0 to be the integer m that satisfies

$$p(z) = q(z)(z - z_0)^m \quad \text{and} \quad q(z_0) \neq 0. \tag{3.2}$$

The most important result about polynomials is the following result, which is called the Fundamental Theorem of Algebra. This theorem is not easy to prove, so we will state it without proof.

Theorem 3.4 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then $p(z)$ always has a root $z_0 \in \mathbf{C}$.*

It is important to note that this theorem states that there always exists a root in any polynomial of degree greater than or equal to one. But the theorem does not give a method or an algorithm to find a root. Actually there is no general algorithm to find the exact roots of a polynomial of degree five or higher.

One can apply the Fundamental Theorem of Algebra repeatedly to obtain the following result.

Corollary 3.5. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex numbers $z_1, z_2, \ldots, z_n$, such that*

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n). \tag{3.3}$$

Proof. We use the Fundamental Theorem of Algebra to write $p(z) = q_1(z)(z - z_1)$ for some complex number z_1. Now $q_1(z)$ is a polynomial of degree $n - 1$. If $n - 1 > 0$, we can apply the Fundamental Theorem of Algebra once more to write $q_1(z) = q_2(z)(z - z_2)$ for some complex number z_2. Repeating this argument the result follows. $\qquad\square$

We can use Definition 3.3 and Corollary 3.5 to obtain the following result, by grouping together repeated roots in (3.3).

Corollary 3.6. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex numbers $\zeta_1, \zeta_2, \cdots, \zeta_k,\ \zeta_j \neq \zeta_{j'},\ j \neq j',\ and\ integers\ m_1, m_2, \cdots, m_k,\ satisfying\ 1 \leq m_j \leq n,\ j = 1, 2, \ldots, k\ and\ m_1 + m_2 + \cdots + m_k = n,\ such\ that*

$$p(z) = a_n(z - \zeta_1)^{m_1}(z - \zeta_2)^{m_2} \cdots (z - \zeta_k)^{m_k}. \tag{3.4}$$

We note that m_j is the multiplicity of the root ζ_j.

We conclude this section with a few examples of factorizations. We consider first $p(z) = z^4 + 2z^2 + 1$. We have $p(z) = (z - i)^2(z + i)^2$. Thus this polynomial has two different complex roots, $+i$ and $-i$, and each of these roots has multiplicity 2.

Next we take $p(z) = 6z^3 - 6iz^2 + 12z - 6z^2 + 6iz - 12$. In this case one can show that $p(z) = 6(z - 1)(z + i)(z - 2i)$. Thus the roots are 1, $-i$, and $2i$, and all three roots have multiplicity one.

4   Roots in Polynomials of Degree One and Two

Let us start with the easy case of a polynomial of degree one, $p(z) = a_1 z + a_0$, $a_1 \neq 0$. The root is given by $z_1 = -a_0/a_1$ and has multiplicity one.

Next we look at a special type of polynomial of degree two, $p(z) = z^2 - a$. We have the following result.

Proposition 4.1. *Let* $p(z) = z^2 - a$, *where* $a = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$. *Let*

$$\operatorname{sgn}(\beta) = \begin{cases} 1, & \text{if } \beta \geq 0, \\ -1, & \text{if } \beta < 0. \end{cases} \tag{4.1}$$

Let $r = |a| = \sqrt{\alpha^2 + \beta^2}$. *Then the two roots of* $p(z) = z^2 - a$ *are given by*

$$z_1 = \sqrt{\frac{r + \alpha}{2}} + i \operatorname{sgn}(\beta) \sqrt{\frac{r - \alpha}{2}}, \tag{4.2}$$

$$z_2 = -\sqrt{\frac{r + \alpha}{2}} - i \operatorname{sgn}(\beta) \sqrt{\frac{r - \alpha}{2}}. \tag{4.3}$$

Proof. The proof is very simple. One needs to verify that $(z_1)^2 = a$ and $(z_2)^2 = a$. Let us verify the first equality. We have

$$(z_1)^2 = \left(\sqrt{\frac{r + \alpha}{2}} + i \operatorname{sgn}(\beta) \sqrt{\frac{r - \alpha}{2}} \right)^2$$

$$= \tfrac{1}{2}(r + \alpha) - \tfrac{1}{2}(r - \alpha) + 2i \operatorname{sgn}(\beta) \sqrt{\tfrac{1}{4}(r + \alpha)(r - \alpha)}$$

$$= \alpha + i \operatorname{sgn}(\beta) \sqrt{\beta^2} = \alpha + i \operatorname{sgn}(\beta) |\beta|$$

$$= \alpha + i\beta = a.$$

In this computation we have used that $(r + \alpha)(r - \alpha) = r^2 - \alpha^2 = (\alpha^2 + \beta^2) - \alpha^2 = \beta^2$ and $\beta = \operatorname{sgn}(\beta)|\beta|$. $\qquad \square$

Based on this result we can now find the roots in a general polynomial of degree two. We have the following result.

Proposition 4.2. *Let* $p(z) = az^2 + bz + c$, *where* $a, b, c \in \mathbf{C}$ *with* $a \neq 0$. *Let* $D = b^2 - 4ac$, *and let* w *be one of the solutions to* $z^2 - D = 0$. *Then the roots of* $p(z)$ *are given by*

$$\frac{-b \pm w}{2a}. \tag{4.4}$$

D is called the discriminant *of the polynomial.*

Proof. Since $a \neq 0$, we can rewrite the polynomial $p(z)$ as follows.

$$p(z) = az^2 + bz + c = a\left(z^2 + \frac{b}{a}z + \frac{c}{a} \right)$$

$$= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \right)$$

$$= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} \right)$$

$$= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{w^2}{4a^2} \right)$$

$$= a\left(z + \frac{b}{2a} - \frac{w}{2a} \right)\left(z + \frac{b}{2a} + \frac{w}{2a} \right),$$

which shows that the two roots are given by (4.4). $\qquad \square$

We now give a few examples of the use of these two results. First we solve the equation $z^2 = 2 - 2i$. Thus we have $\alpha = 2$, $\beta = -2$, $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$ and $\operatorname{sgn}(\beta) = -1$. Thus

$$z_1 = \sqrt{\tfrac{1}{2}(2\sqrt{2}+2)} + i(-1)\sqrt{\tfrac{1}{2}(2\sqrt{2}+2)} = \sqrt{\sqrt{2}+1} - i\sqrt{\sqrt{2}-1}.$$

The other root is of course $z_2 = -z_1$.

Next we find the roots of the polynomial $2\,z^2 - 10i\,z - 12$. First we compute the discriminant:

$$D = (-10i)^2 - 4 \cdot 2 \cdot (-12) = -4.$$

One of the solutions to $w^2 = -4$ is $w = 2i$. Thus the roots are

$$\frac{-(-10i) \pm 2i}{2 \cdot 2} = \begin{cases} 3i, \\ 2i. \end{cases}$$

Remark 4.3. There is an easy way to test whether one has found the correct roots of a polynomial of degree two. Assume for simplicity that $a = 1$, such that we have the roots z_1 and z_2 of the polynomial $z^2 + bz + c$. These two roots must then satisfy

$$z_1 + z_2 = -b \quad \text{and} \quad z_1 z_2 = c. \tag{4.5}$$

The verification of these results is left to the reader.

5 Roots of $z^m - a$

In this section we review the results from [1] concerning roots of the polynomial $z^m - a$, or equivalently, solutions to the equation $z^m = a$, where $m \geq 1$ is an integer. The method is to write a in polar form

$$a = re^{i\theta}, \quad r = |a|, \quad \theta = \operatorname{Arg} a.$$

The m different solutions are then given by

$$z_k = r^{1/m}\left(\cos(\frac{\theta + 2\pi k}{m}) + i\sin(\frac{\theta + 2\pi k}{m})\right), \quad k = 0, 1, \ldots, m-1. \tag{5.1}$$

Examples and further comments can be found in [1].

6 Factorization of Polynomials

The results in Corollaries 3.5 and 3.6 show that once we know the roots in a polynomial, then we can factor it. Even for polynomials with real coefficients we may get non-real numbers in the factorization, as in

$$4\,z^2 + 16 = 4(z - 2i)(z + 2i).$$

However, if we are satisfied with a factorization in factors that are of degree one or two, then it can be achieved with real coefficients only. Before we state this result, we need the following important result.

Proposition 6.1. *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with real coefficients $a_j \in \mathbf{R}$, $j = 0, 1, 2, \ldots, n$. If z_0 is a root of $p(z)$, then the conjugate $\overline{z_0}$ is also a root of $p(z)$.*

Proof. We have $p(z_0) = a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0 = 0$. Taking the complex conjugate we get, using the facts that the conjugate of a sum is the sum of the sum of the conjugates, and the conjugate of a product is the product of the conjugate of each factor,

$$\overline{p(z_0)} = \overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0}$$
$$= \overline{a_n z_0^n} + \overline{a_{n-1} z_0^{n-1}} + \cdots + \overline{a_1 z_0} + \overline{a_0}$$
$$= a_n \overline{z_0}^n + a_{n-1} \overline{z_0}^{n-1} + \cdots + a_1 \overline{z_0} + a_0$$
$$= p(\overline{z_0}).$$

In the computation above we have used that the coefficients a_j are real, such that $\overline{a_j} = a_j$, $j = 0, 1, \ldots, n$. Thus $p(z_0) = 0$ implies $p(\overline{z_0}) = 0$. $\qquad\square$

Proposition 6.2. *Let* $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ *with real coefficients* $a_j \in \mathbf{R}$, $j = 0, 1, 2, \ldots, n$. *Let* ξ_j, $j = 1, 2, \ldots, J$ *denote the distinct real roots of* $p(z)$, *and let* ζ_k, $k = 1, 2, \ldots, K$ *be complex numbers with* $\mathrm{Im}\, \zeta_k \neq 0$, *such that* $\zeta_k, \overline{\zeta_k}$ *are the remaining distinct roots of* $p(z)$, $k = 1, \ldots, K$. *Write* $\zeta_k = \alpha_k + i\beta_k$ *with* α_k *and* β_k *real. Then we have*

$$p(z) = a_n (z - \xi_1)^{n_1} \cdots (z - \xi_J)^{n_J} ((z - \alpha_1)^2 + \beta_1^2)^{m_1} \cdots ((z - \alpha_K)^2 + \beta_K^2)^{m_K}. \qquad (6.1)$$

Here n_j *is the multiplicity of the root* ξ_j, $j = 1, \ldots, J$ *and* m_k *is the multiplicity of the root* ζ_k, $k = 1, \ldots, K$. *We have* $J + 2K = n$.

Proof. The result is a consequence of Corollary 3.6 and Proposition 6.1. For the real roots this is immediate. For the pairs of complex conjugate roots we use that

$$(z - \zeta_k)(z - \overline{\zeta_k}) = (z - \alpha_k - i\beta_k)(z - \alpha_k + i\beta_k) = (z - \alpha_k)^2 + \beta_k^2.$$

$\qquad\square$

Let us give some examples. First we consider $p(z) = z^4 - 1$. Here we can use the result from Section 5 to find the roots. The roots are $z_1 = 1$, $z_2 = -1$, $z_3 = i$, and $z_4 = -i$. Thus we have two real roots and one pair of complex conjugate roots. Therefore we have

$$p(z) = (z - 1)(z + 1)(z - i)(z + i) = (z - 1)(z + 1)(z^2 + 1).$$

Next let us look at $p_1(z) = z^8 - 2z^4 + 1$. If we note that $p_1(z) = (z^4 - 1)^2$, we can use the previous factorization to get

$$p_1(z) = (z - 1)^2 (z + 1)^2 (z - i)^2 (z + i)^2 = (z - 1)^2 (z + 1)^2 (z^2 + 1)^2.$$

Thus the roots are the same, but their multiplicities are different.

We now give a somewhat more complicated example. We let

$$p_2(z) = z^5 - 3z^4 + 8z^3 - 14z^2 + 16z - 8.$$

The roots of this polynomial are

$$z_1 = 1, \; z_2 = 1 + i, \; z_3 = 1 - i, \; z_4 = 2i, \; z_5 = -2i.$$

Thus there is one real root and two pairs of complex conjugate roots. The factorization in Proposition 6.2 becomes

$$p_2(z) = (z - 1)((z - 1)^2 + 1)(z^2 + 4) = (z - 1)(z^2 - 2z + 2)(z^2 + 4).$$

Remark 6.3. The result in Remark 4.3 can be generalized to an arbitrary polynomial. Again to simplify the statement we assume that the coefficient to the highest power is equal to one. Thus we consider a polynomial

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with roots (repeated with multiplicity) $z_1, z_2, \ldots, z_n$. These roots then satisfy

$$z_1 + z_2 + \cdots + z_n = -a_{n-1} \quad \text{and} \quad z_1 z_2 \cdots z_n = (-1)^n a_0. \qquad (6.2)$$

7 Techniques for Finding Roots in a Polynomial

In the examples above we have not given many details on how to find the roots in a given polynomial. For polynomials of degree one and two, and for polynomials of the form $z^m - a$, we have given explicit formulas for finding roots. For polynomials of degree three or four there exist general formulas, but they are very complicated to state, and to use.

To illustrate this, we consider the following polynomial of degree three,

$$p(z) = z^3 - 2\,z^2 - 5\,z - 11.$$

From Proposition 6.2 we can conclude that $p(z)$ must have one real root. The real root is

$$z_1 = \frac{1}{6}\sqrt[3]{1612 + 12\sqrt{14997}} + \frac{38}{3}\,\frac{1}{\sqrt[3]{1612 + 12\sqrt{14997}}} + \frac{2}{3}.$$

It turns out that the remaining two roots are complex. One of the complex roots is

$$z_2 = -i\frac{1}{12}\sqrt[3]{1612 + 12\sqrt{14997}} - \frac{19}{3}\,\frac{1}{\sqrt[3]{1612 + 12\sqrt{14997}}} + \frac{2}{3}$$
$$+ \frac{1}{2}i\sqrt{3}\left(\frac{1}{6}\sqrt[3]{1612 + 12\sqrt{14997}} - \frac{38}{3}\,\frac{1}{\sqrt[3]{1612 + 12\sqrt{14997}}}\right).$$

We conclude that rather innocent looking polynomials of degree three with integer coefficients can have very complicated roots.

There are some general results and guidelines that can be given for finding roots. We state the consequence of Proposition 6.2 used above.

Proposition 7.1. *Let $p(z)$ be a polynomial of odd degree with real coefficients. Then $p(z)$ always has at least one real root.*

For the special case of polynomials with integer coefficients we have the following result. We recall that a rational number $\frac{k}{\ell}$ is said to be irreducible, if k and ℓ have no common factors. Thus $\frac{2}{4}$ is not irreducible, but $\frac{1}{2}$ is irreducible.

Proposition 7.2. *Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial, such that the coefficients a_j, $j = 0, 1, \ldots, n$ all are integers. Assume that $p(z)$ has a rational root $z_0 = \frac{k}{\ell}$, which is irreducible. Then a_n is divisible by ℓ and a_0 is divisible by k.*

Proof. Assume that the irreducible rational number $z_0 = \frac{k}{\ell}$ is a root of $p(z)$. We first write out $p(z_0) = 0$.

$$a_n\Big(\frac{k}{\ell}\Big)^n + a_{n-1}\Big(\frac{k}{\ell}\Big)^{n-1} + \cdots + a_1\Big(\frac{k}{\ell}\Big) + a_0 = 0.$$

Then we multiply on both sides of the equation by ℓ^n.

$$a_n k^n + a_{n-1}k^{n-1}\ell + \cdots + a_1 k\ell^{n-1} + a_0\ell^n = 0.$$

We now move the term $a_n k^n$ to the right hand side to get

$$a_{n-1}k^{n-1}\ell + \cdots + a_1 k\ell^{n-1} + a_0\ell^n = -a_n k^n.$$

Since all terms on the left hand side contain a factor ℓ, the left hand side is divisible by ℓ. Since k by assumption is not divisible by ℓ, it follows that a_n must be divisible by ℓ. The other statement is proved in the same manner. $\square$

Let us now show how to use this result. We take

$$p(z) = z^3 - 6\,z^2 + 13\,z - 10.$$

Thus $a_3 = 1$ and $a_0 = -10$. Now $a_3 = 1$ is divisible by ± 1, whereas $a_0 = -10$ is divisible by $\pm 1, \pm 2, \pm 5$, and ± 10. Thus the only possible rational (in this case actually integer) roots are $\pm 1, \pm 2, \pm 5$, and ± 10. Now one has to insert them in the polynomial and check whether they are roots. One finds one root, $z_1 = 2$. The other seven integers are not roots.

The next step is to use division by polynomials, as described in Section 2. The result is that we have

$$z^3 - 6\,z^2 + 13\,z - 10 = (z - 2)(z^2 - 4\,z + 5).$$

Since $z^2 - 4\,z + 5$ is a polynomial of degree two, we can use the result from Section 4 to find the roots. We will not give the details. The result is that $z_2 = 2 + i$ and $z_3 = 2 - i$.

Sometimes one can find the roots by a two step procedure. We take as an example the polynomial

$$p_1(z) = z^8 - 2\,z^4 + 1,$$

which was also considered above. The first step is a change of variable. We let $w = z^4$. Substituting into $p_1(z)$ we get $w^2 - 2\,w + 1$. This is a simple polynomial of degree two, actually we have $w^2 - 2\,w + 1 = (w - 1)^2$. Thus the only solution is $w = 1$. Going back to the variable z we have to find the roots of $z^4 - 1$, which can be done using the result from Section 5. See above for the result.

The lesson to be learned from the examples in this section is that in general for polynomials of degree three or greater, finding the roots exactly is a matter of skill, if it is possible at all, at least if one does the computations by hand.

8 Maple and Roots in Polynomials

As mentioned several times, there is no general formula or algorithm to find roots in polynomials of degree five or greater. Therefor Maple cannot in general find the roots in a polynomial of degree five or greater. By default, for polynomials of degree four, Maple does not try to find the roots. The algorithm is very demanding computationally, and often leads to many pages of output, which hardly is useful. One can force Maple to use the algorithm, but one should only do so, if there is a good reason for this.

For polynomials of degree three the general formula is used. The roots in the polynomial $p(z) = z^3 - 2\,z^2 - 5\,z - 11$ considered above were found using Maple.

Even for the simple polynomials $z^m - a$ it is often more efficient and simpler to use the formula given above, than to use Maple. As an example, consider finding the roots of unity of order 6, i.e. the roots of the polynomial $z^6 - 1$. Using the Maple command `solve(z^6-1=0,z)` leads to the result

$$-1,\; 1,\; -\tfrac{1}{2}\sqrt{-2 + 2\,i\sqrt{3}},\; \tfrac{1}{2}\sqrt{-2 + 2\,i\sqrt{3}},\; -\tfrac{1}{2}\sqrt{-2 - 2\,i\sqrt{3}},\; \tfrac{1}{2}\sqrt{-2 - 2\,i\sqrt{3}}.$$

However, this is not in the form that you are usually asked to find the roots of a polynomial. The roots must be given in the form $a + ib$ with a and b real numbers. So further Maple commands need to be applied, before the final answer is found. In contrast, the formula (5.1) immediately gives the result in the form $a + ib$.

If Maple cannot find the roots explicitly, it gives a reply involving the Maple notation for the roots of a polynomial. As an example, we consider the polynomial $z^5 + z^2 + z + 1$. The

command `solve(z^5+z^2+z+1=0,z)` leads to the result

$$-1, RootOf\left(\_Z^4 - \_Z^3 + \_Z^2 + 1, index = 1\right),$$
$$RootOf\left(\_Z^4 - \_Z^3 + \_Z^2 + 1, index = 2\right),$$
$$RootOf\left(\_Z^4 - \_Z^3 + \_Z^2 + 1, index = 3\right),$$
$$RootOf\left(\_Z^4 - \_Z^3 + \_Z^2 + 1, index = 4\right).$$

Thus in this case Maple as its default has not found the roots in the polynomial $z^4 - z^3 + z^2 + 1$. If one forces Maple to find these roots, one will see that they are very complicated.

9 Finding Roots of a Polynomial Numerically

One can use numerical methods to find the roots of a polynomial with a given number of significant digits. However, many numerical routines will only find some of the roots, and furthermore, working with a finite number of digits can lead to large deviations between the approximate roots and the exact roots. Thus one should only use numerical methods, if one is familiar with them, and knows how to work with floating point numbers. Both Maple and Matlab have routines for finding roots numerically.

We will give one example of what can happen. Consider the polynomial of degree 20 given by

$$p(z) = (z + 1)(z + 2) \cdots (z + 19)(z + 20).$$

Obviously the roots are $-1, -2, \ldots, -19, -20$. Suppose that we add a term which looks small. We take

$$p_1(z) = p(z) + 2^{-23}z^{19}.$$

One would probably guess that this polynomial also has 20 real roots, and that they are close to the roots of the polynomial $p(z)$. But this completely wrong. This polynomial has 10 real roots and 5 pairs of complex conjugate roots. One real root is -20.84690810 and a pair of complex conjugate roots is $-19.50243940 \pm i\,1.940330347$. One should note that the imaginary part is quite large.

The example shows very clearly that numerical computation of roots can be difficult, since small changes in a coefficient can change the roots a lot. In the example the coefficient to z^{19} in $p(z)$ is 210, which we have then changed to $210 + 2^{-23} \approx 210.000000119$.

References

[1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, Chapter 1 reprinted in Pearson Custom Publication 2008.