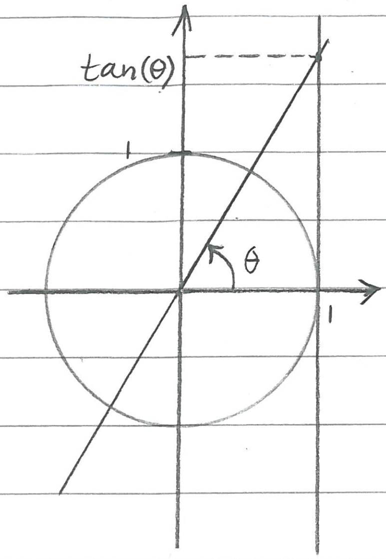
1. kursusgang: Trigonometriske funktioner og deres inverse II

Invers tangens



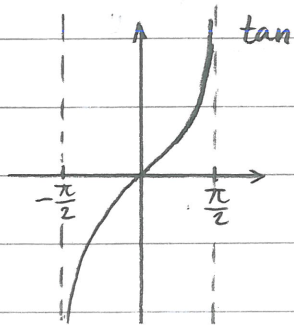
Funktionen $\tan(x)$ har ingen invers funktion, da f.eks $\tan\left(\frac{\pi}{4}\right) = 1 = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right)$. Men ved restriktion af definitionsmængden til intervallet $]-\frac{\pi}{2}, \frac{\pi}{2}[$, fås en voksende og surjektiv funktion

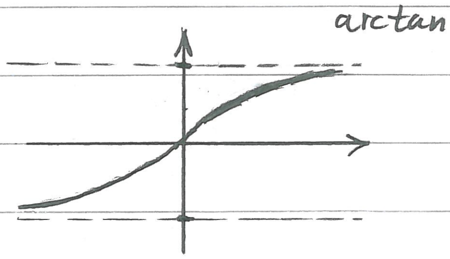
$$\tan : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow]-\infty, \infty[\,.$$

Arcustangens er den inverse af denne restriktion

$$\arctan :]-\infty, \infty[\longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Def. $y = \arctan(x) \iff \tan(y) = x \;\wedge\; -\frac{\pi}{2} < y < \frac{\pi}{2}$.





(grafen for den inverse fremkommer ved spejling i linjen $y=x$)

Bemærk: $\tan(\arctan(x)) = x \quad , \quad x \in]-\infty, \infty[$ og

$\arctan(\tan(x)) = x \quad , \quad x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$.

Husk: $\frac{d}{dx} \tan(x) = 1 + \tan^2(x)$

Sætning: $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

Bevis: Da $\frac{d}{dx}\tan(x) = 1 + \tan^2(x) > 0 \quad , \quad x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ er arctan differentiabel (overvej evt. dette via graferne ovenfor). Ved kædereglen fås

$$x = \tan(\arctan(x)) \Rightarrow$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(\arctan(x))) \Rightarrow$$

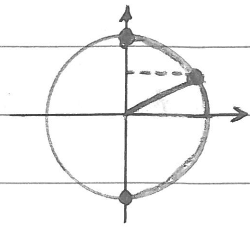
$$1 = (1+\tan^2(\arctan(x)))\frac{d}{dx}\arctan(x) = (1+x^2)\frac{d}{dx}\arctan(x)$$

$$\Rightarrow \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \qquad \text{q.e.d.}$$

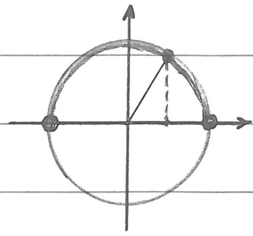
Andre inverse trigonometriske funktioner

$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$,

$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$



$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad , \ -1 < x < 1 .$$



$\cos : [0, \pi] \to [-1, 1]$,

$\arccos : [-1, 1] \to [0, \pi]$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \quad , \ -1 < x < 1 .$$

Bemærk:

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C .$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \quad , \ -1 < x < 1$$