11. kursusgang : Repetition

Kædereglen

Sætning : Lad $w = f(x,y)$ hvor $x = g(t)$ og $y = h(t)$. Hvis f er kontinuert differentiabel og g samt h er differentiable, så er w en differentiabel funktion af t og

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}.$$

Dvs.

$$(f(g(t),h(t)))' = f_x(g(t),h(t)) \cdot g'(t) + f_y(g(t),h(t)) \cdot h'(t).$$

Eks. $w = e^{-x^2-y^2}$ hvor $x = t$, $y = \sqrt{t}$.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\
&= -2x\, e^{-x^2-y^2} \cdot 1 - 2y e^{-x^2-y^2} \cdot \frac{1}{2\sqrt{t}} \\
&= \left(-2x - \frac{y}{\sqrt{t}}\right) e^{-x^2-y^2} = (-2t-1)\, e^{-t^2-t} \\
&= -(2t+1)\, e^{-t^2-t}
\end{aligned}$$

Den generelle kæderegel :

$w = f(x_1, x_2, \dots, x_m)$ hvor $x_j = g_j(t_1, t_2, \dots, t_n)$ for $j = 1, 2, \dots, m$. Da er

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial w}{\partial x_m} \frac{\partial x_m}{\partial t_i}$$

for $i = 1, 2, \dots, n$.

Implicit differentiation

Eks. Antag at $z = f(x,y)$ tilfredsstiller ligningen

$$z^3 + xz - y^2 = 1.$$

Beregn $\frac{\partial z}{\partial x}$ og $\frac{\partial z}{\partial y}$.

$$0 = \frac{\partial}{\partial x}(1) = \frac{\partial}{\partial x}(z^3 + xz - y^2)$$
$$= 3z^2 \frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x}$$
$$= (3z^2 + x)\frac{\partial z}{\partial x} + z$$
$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{z}{3z^2 + x}$$

$$0 = \frac{\partial}{\partial y}(1) = \frac{\partial}{\partial y}(z^3 + xz - y^2)$$
$$= 3z^2 \frac{\partial z}{\partial y} + x\frac{\partial z}{\partial y} - 2y$$
$$= (3z^2 + x)\frac{\partial z}{\partial y} - 2y$$
$$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{2y}{3z^2 + x}$$

<u>Sætning</u> (Implicit funktionssætning)
Antag at $F(x_1, x_2, \dots, x_n, z)$ er kontinuert differentiabel i en omegn af punktet $(\vec{a}, b) = (a_1, a_2, \dots, a_n, b)$, hvor
$$F(\vec{a}, b) = 0 \quad \text{og} \quad F_z(\vec{a}, b) \neq 0.$$
Så eksisterer en kontinuert differentiabel funktion $z = g(x_1, x_2, \dots, x_n)$ således at
$g(\vec{a}) = b$ og $F(\vec{x}, g(\vec{x})) = 0$ for $\vec{x}$ i en omegn af $\vec{a}$.

Ved implicit differentiation fås
$$\frac{\partial z}{\partial x_i} = -\frac{F_{x_i}}{F_z}.$$