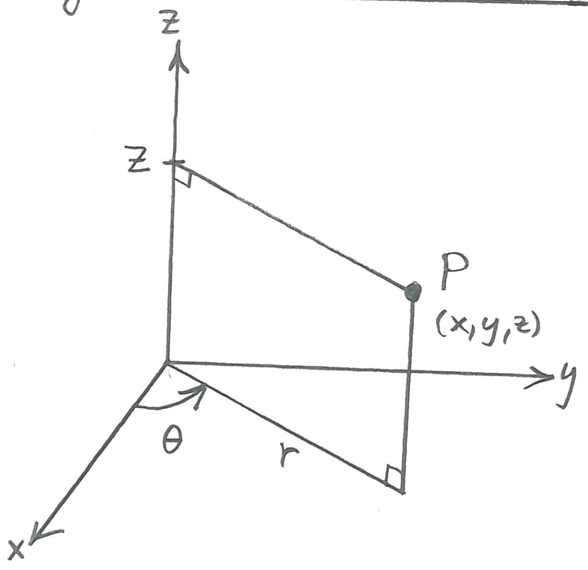
17. kursusgang: Rumintegralet i cylinder- og sfæriske koordinater

Cylinder-koordinater



Def. P har cylinder-koordinaterne (r, θ, z)

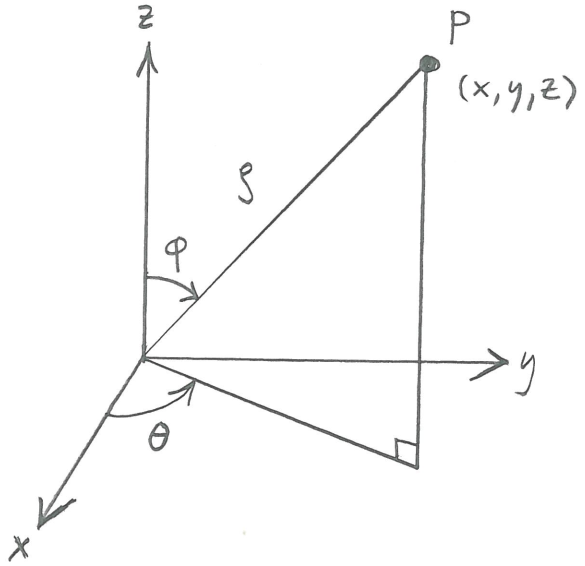
Bemærk: (r, θ) er polære koordinater for ortogonalprojektionen af P på xy-planen.

Specielt er $x = r\cos\theta$, $y = r\sin\theta$.

Omregning mellem cylinder- og rektangulære koordinater:

cylinder → rektangulære	rektangulære → cylinder
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \arctan\left(\frac{y}{x}\right)$, $x>0$
$z = z$	$\theta = \arctan\left(\frac{y}{x}\right)+\pi$, $x<0$
	$z = z$

Sfæriske-koordinater



Def. P har sfæriske-koordinater (ρ, φ, θ)

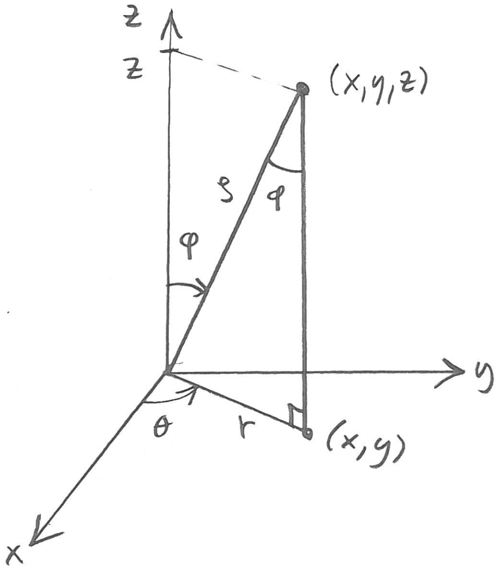
Omregning:

sfæriske → rektangulære

$$\boxed{\begin{aligned} x &= \rho\sin(\varphi)\cos(\theta) \\ y &= \rho\sin(\varphi)\sin(\theta) \\ z &= \rho\cos(\varphi) \end{aligned}}$$

Bemærk også at $\rho^2 = x^2+y^2+z^2$

Begrundelse for omregningsformlerne:



$$\cos\varphi = \frac{z}{\varrho} \Rightarrow z = \varrho\cos\varphi$$

$$\sin\varphi = \frac{r}{\varrho} \Rightarrow r = \varrho\sin\varphi$$

$$x = r\cos\theta = \varrho\sin\varphi\cos\theta$$

$$y = r\sin\theta = \varrho\sin\varphi\sin\theta$$

OK

Rumintegralet i cylinder-koordinater

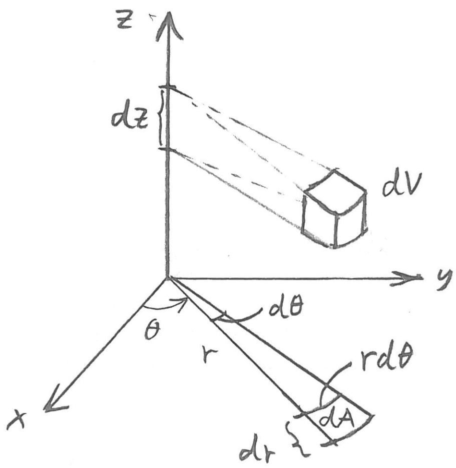
Simpelt område mht. cylinder koordinater:

$$T = \{(r,\theta,z) \mid \theta_1 \le \theta \le \theta_2,\ r_1(\theta) \le r \le r_2(\theta),\ z_1(r,\theta) \le z \le z_2(r,\theta)\}$$

For et sådant område gælder

$$\boxed{\iiint_T f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta}$$

Volumenelementet dV:

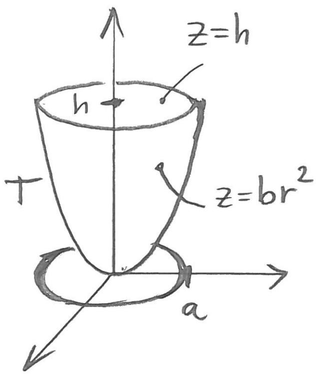


$dA = r\,dr\,d\theta$ og $dV = dz\,dA$ så

$dV = r\,dz\,dr\,d\theta$

Husk det ekstra r!

Eks. Beregn volumen af området T begrænset af omdrejningsparaboloiden $z = b(x^2+y^2)$ og planen $z=h$, hvor $b>0$ og $h>0$.



$$T = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le a,\ br^2 \le z \le h\}$$

Vi har $h = ba^2$ så

$$a = \sqrt{\frac{h}{b}} \quad \text{og} \quad b = \frac{h}{a^2}$$

$$V = \iiint_T 1\,dV = \int_0^{2\pi}\int_0^a \int_{\frac{hr^2}{a^2}}^{h} r\,dz\,dr\,d\theta =$$

$$\int_0^a \int_{\frac{hr^2}{a^2}}^{h} r\,dz\,dr \cdot \int_0^{2\pi} 1\,d\theta = \int_0^a [rz]_{z=\frac{hr^2}{a^2}}^{z=h}\,dr \cdot 2\pi =$$

$$2\pi \int_0^a \left(rh - \frac{hr^3}{a^2}\right)dr = 2\pi h \int_0^a \left(r - \frac{r^3}{a^2}\right)dr =$$

$$2\pi h \left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a = 2\pi h\left(\frac{a^2}{2} - \frac{a^4}{4a^2}\right) = 2\pi h \frac{a^2}{4} =$$

$$\frac{1}{2}\pi a^2 h.$$

Rumintegralet i sfæriske-koordinater

Sfærisk simpelt område

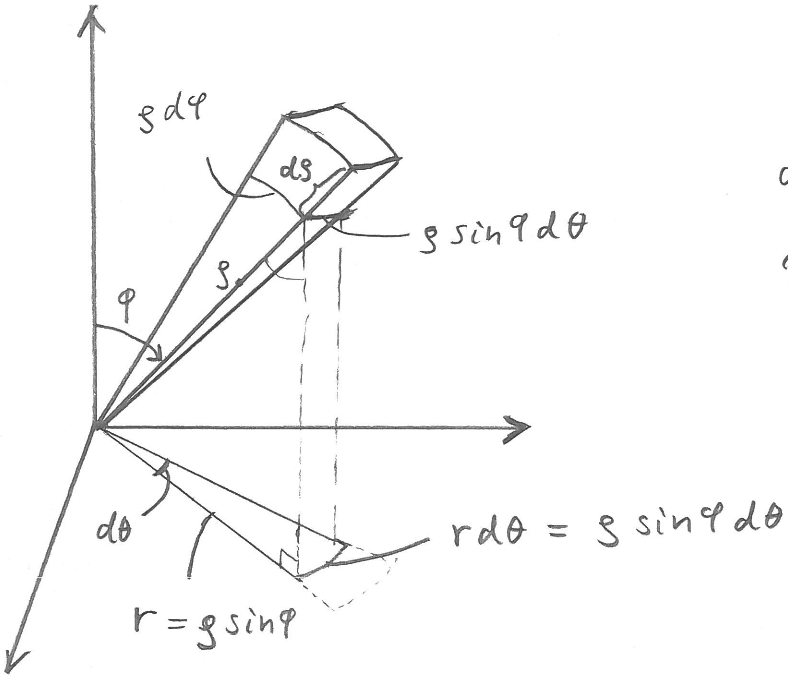
$$T = \{(\rho,\varphi,\theta) \mid \theta_1 \le \theta \le \theta_2,\ \varphi_1 \le \varphi \le \varphi_2,\ \rho_1(\varphi,\theta) \le \rho \le \rho_2(\varphi,\theta)\}$$

For et sådant område gælder

$$\iiint_T f(x,y,z)\,dV =$$

$$\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Volumenelementet dV :



$$dV = \varrho^2 \sin\varphi \, d\varrho \, d\varphi \, d\theta$$

Husk $\varrho^2 \sin\varphi$. !