

## 19. lektion: Den komplekse eksponentialefunktion

Def. For  $z = x + iy$ , hvor  $x, y \in \mathbb{R}$ , defineres  $e^z$  som

$$e^z = e^x (\cos(y) + i \sin(y)).$$

Bemerk: For  $y=0$  er  $e^z = e^x (\cos(0) + i \sin(0)) = e^x$  så for  $z \in \mathbb{R}$  er  $e^z$  den sædvanlige eksponentialefunktion.

Sætning:

$$(1) \quad e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2} \quad \text{for alle } z_1, z_2 \in \mathbb{C}.$$

$$(2) \quad \frac{d}{dt} e^{it} = ie^{it} \quad \text{for } t \in \mathbb{R}.$$

Beweis:

$$\begin{aligned}
 (1) \quad e^{x_1 + iy_1 + x_2 + iy_2} &= e^{x_1 + x_2 + i(y_1 + y_2)} = \\
 e^{x_1 + x_2} (\cos(y_1 + y_2) + i \sin(y_1 + y_2)) &= \\
 e^{x_1} e^{x_2} (\cos(y_1) \cos(y_2) - \sin(y_1) \sin(y_2) + i(\sin(y_1) \cos(y_2) + \cos(y_1) \sin(y_2))) & \\
 = e^{x_1} e^{x_2} (\cos(y_1) (\cos(y_2) + i \sin(y_2)) + i \sin(y_1) (\cos(y_2) + i \sin(y_2))) & \\
 = e^{x_1} e^{x_2} ((\cos(y_1) + i \sin(y_1)) \cdot (\cos(y_2) + i \sin(y_2))) & \\
 = e^{x_1 + iy_1} e^{x_2 + iy_2} & \quad \text{OK}
 \end{aligned}$$

$$(2) \quad Vi \text{ har } e^{it} = e^{0+it} = \cos(t) + i \sin(t), \quad Vi \text{ betragter}$$

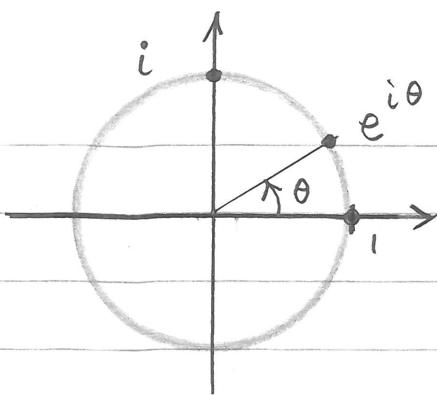
dette som en vektorfunktion, og finder

$$\begin{aligned}
 \frac{d}{dt}(e^{it}) &= \frac{d}{dt}(\cos(t)) + i \frac{d}{dt}(\sin(t)) \\
 &= -\sin(t) + i \cos(t) \\
 &= i(i \sin(t) + \cos(t)) = ie^{it} \quad \text{q.e.d.}
 \end{aligned}$$

Sætning:  $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2} \quad \text{for alle } z_1, z_2 \in \mathbb{C}.$

Bemerk: Af definitionen følger Eulers ligning

$$e^{i\theta} = \cos\theta + i \sin\theta.$$



Bemerk:

$$e^{\pi i} = -1$$

### Polar form:

Vi har  $z = r \operatorname{cis}(\theta) = r(\cos \theta + i \sin \theta) = r e^{i\theta}$ .

Dvs.

$$z = |z| e^{i \arg(z)}$$

Bemerk: For  $z_1 = r_1 e^{i\theta_1}$  og  $z_2 = r_2 e^{i\theta_2}$  har vi

- $z_1 \cdot z_2 = (r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, z_2 \neq 0$ .

Bemerk: For  $z = r e^{i\theta}$  er  $\bar{z} = r e^{-i\theta}$ .

Dette følger af Eulers ligning.

### Cosinus og sinus

$$\cos \theta = \operatorname{Re}(e^{i\theta}) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \operatorname{Im}(e^{i\theta}) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

### De Moivre's formel:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta), n=1, 2, 3, \dots$$

Bewis:

$$(e^{i\theta})^n = e^{i\theta}, e^{i\theta} \cdot \dots \cdot e^{i\theta} = e^{i\theta + i\theta + \dots + i\theta} = e^{in\theta} \text{ q.e.d.}$$

## Binomialformlen og Pascal's trekant

1	$(a+b)^1 = a+b$
1 2 1	$(a+b)^2 = a^2 + 2ab + b^2$
1 3 3 1	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
1 4 6 4 1	$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
...	...

Eks.  $\cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3 =$   
 $\cos^3\theta + 3\cos^2\theta i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 =$   
 $\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta),$

Dvs.  $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta,$   
 $\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$