14. kursusgang: Koordinatsystemer

Lad V være et underrum af $\mathbb{R}^n$ og lad
$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p\} \subseteq V$ være en basis for V dvs.
(1) $\mathcal{B}$ er lineært uafhængig
(2) $Sp(\mathcal{B}) = V$

"Baser kan bruges som koordinatsystemer":

__Sætning__: Enhver vektor $\vec{v} \in V$ kan entydigt skrives som en linearkombination af vektorerne i $\mathcal{B}$.
Dvs. der findes __entydige__ skalarer $a_1, a_2, \dots, a_p$ så

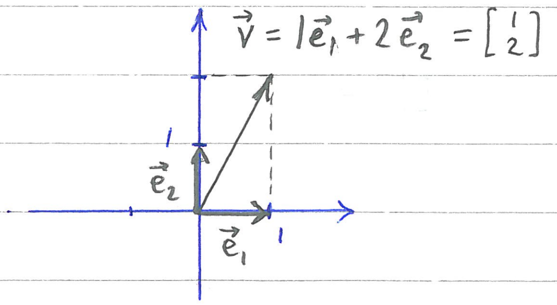
$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_p\vec{b}_p.$$

__Bevis__:
Da $\vec{v} \in V = Sp(\mathcal{B})$ findes $a_1, a_2, \dots, a_p \in \mathbb{R}$ så $\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_p\vec{b}_p$.
Entydighed: Antag vi også har $c_1, c_2, \dots, c_p \in \mathbb{R}$ så
$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_p\vec{b}_p$. Vi skal så vise at $a_i = c_i$, $i = 1, 2, \dots, p$.
Vi har

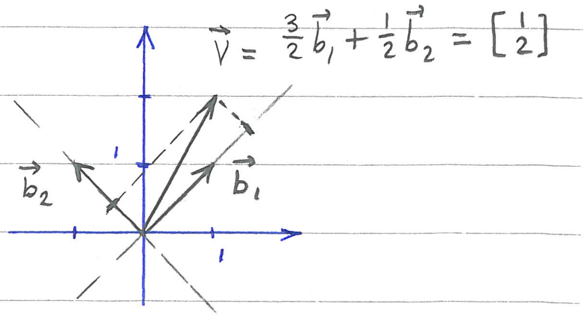
$$\vec{0} = \vec{v} - \vec{v} = (a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_p\vec{b}_p) - (c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_p\vec{b}_p)$$
$$= (a_1 - c_1)\vec{b}_1 + (a_2 - c_2)\vec{b}_2 + \dots + (a_p - c_p)\vec{b}_p.$$

Da $\mathcal{B}$ er lineært uafhængig, følger det at $a_i - c_i = 0$, hvormed $a_i = c_i$, $i = 1, 2, \dots, p$. q.e.d.

__Eks.__
$V = \mathbb{R}^2$



Standard basen $\{\vec{e}_1, \vec{e}_2\}$, hvor $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ og $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.
(Sædvanlige koordinatsystem)



Basen $\{\vec{b}_1, \vec{b}_2\}$, hvor $\vec{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

Def. Koordinatvektoren for $\vec{v} \in V$ relativt til basen $\mathcal{B}$ er givet ved

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \text{ hvor } \vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_p\vec{b}_p.$$

Bemærk: Koordinatvektoren for $\vec{v} \in V$ er den entydige løsning til ligningssystemet

$$[\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p \mid \vec{v}].$$

Specialtilfælde: Hvis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ er en basis for $\mathbb{R}^n$, så er $n \times n$-matricen $B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_n]$ invertibel, og vi har for $\vec{v} \in \mathbb{R}^n$,

$$B[\vec{v}]_{\mathcal{B}} = \vec{v} \quad \text{hvormed} \quad [\vec{v}]_{\mathcal{B}} = B^{-1}\vec{v}.$$

Eks. (fortsat) Koordinatvektoren for $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ relativt til basen $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, hvor $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ og $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, beregnes som følger:

$$[\vec{b}_1 \ \vec{b}_2 \mid \vec{v}] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} \end{array}\right] \to \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \end{array}$$

$c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$. Dvs $[\vec{v}]_{\mathcal{B}} = \underline{\underline{\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}}}$.

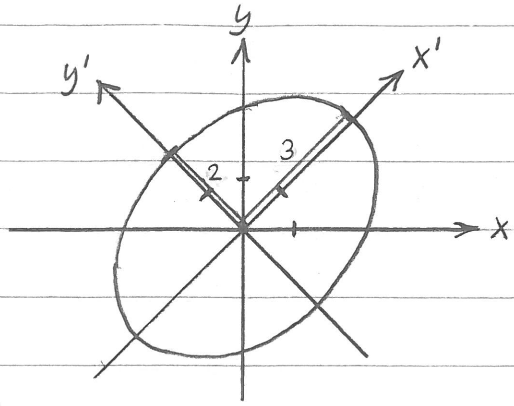
Kontrol: $\frac{3}{2}\vec{b}_1 + \frac{1}{2}\vec{b}_2 = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{v}$ OK.

Alternativt:

$$B = [\vec{b}_1 \ \vec{b}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{1\cdot 1 - (-1)\cdot 1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

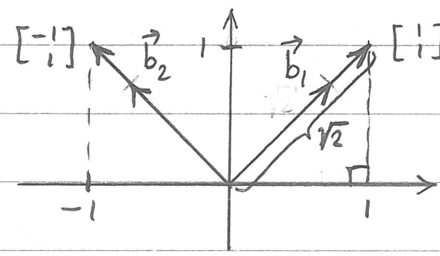
$$[\vec{v}]_{\mathcal{B}} = B^{-1}\vec{v} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}}}$$

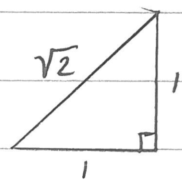
Eks.



Find ellipsens ligning i xy-koordinatsystemet.

koordinat-system	basis
xy	$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$x'y'$	$\vec{b}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$





Sæt $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. Vi har

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{og} \quad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Vi finder formler for koordinatskiftet fra (x,y) til (x',y'):

$$B = [\vec{b}_1 \;\; \vec{b}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \det(B) = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}} = 1,$$

$$B^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = [\vec{v}]_{\mathcal{B}} = B^{-1}\vec{v} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x+y \\ -x+y \end{bmatrix}$$

Dvs. $x' = \frac{1}{\sqrt{2}}(x+y)$, $y' = \frac{1}{\sqrt{2}}(-x+y)$.

Ellipsens ligning i $x'y'$-koordinatsystemet

$$\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{2}\right)^2 = 1$$

Ellipsens ligning i xy - koordinatsystemet:

$$\left(\frac{\frac{1}{\sqrt{2}}(x+y)}{3}\right)^2 + \left(\frac{\frac{1}{\sqrt{2}}(-x+y)}{2}\right)^2 = 1 \Leftrightarrow$$

$$\frac{\frac{1}{2}(x^2+y^2+2xy)}{9} + \frac{\frac{1}{2}(x^2+y^2-2xy)}{4} = 1 \Leftrightarrow$$

$$4(x^2+y^2+2xy)+9(x^2+y^2-2xy) = 72 \Leftrightarrow$$

$$\underline{\underline{13x^2 - 10xy + 13y^2 = 72}}$$