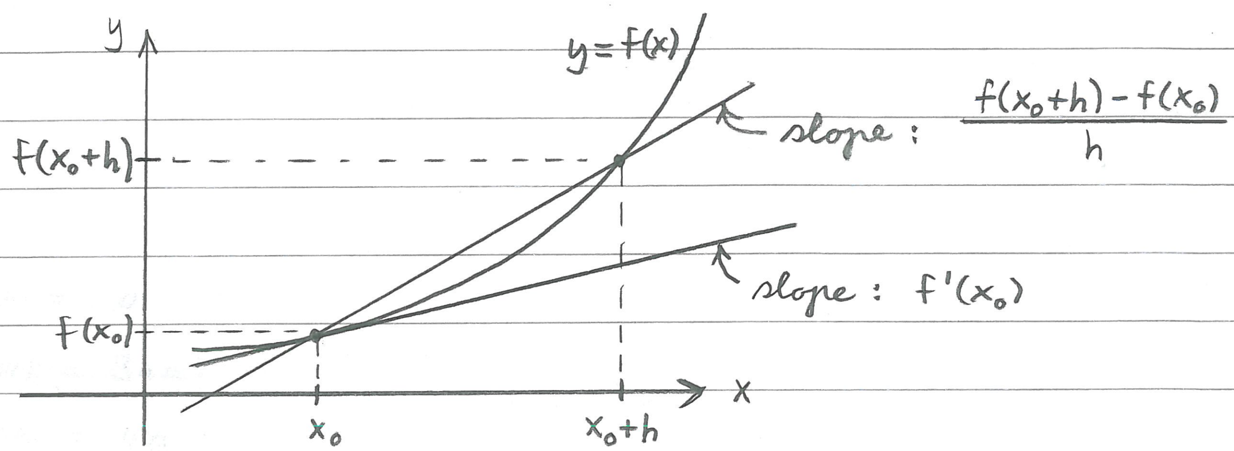
## 2. Session : The Derivative and Rates of Change

Let $f(x)$, $x \in I$ be a function defined on an open interval $I = ]a,b[$, and let $x_0 \in I$.

!Def. The derivative of $f$ at $x_0$ is the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

providing that this limit exists. If $f'(x_0)$ exists for all $x_0 \in I$, we get a new function $f'(x)$, $x \in I$ called the derivative of $f$.




!Note : $f'(x_0)$ is the slope of the tangent at $(x_0, f(x_0))$.

Notation : $f'(x) = \frac{d}{dx} f(x) = D_x f(x)$.

Ex. : $f(x) = x^2$, $x \in \mathbb{R}$

$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{(x_0+h)^2 - x_0^2}{h} = \frac{x_0^2 + h^2 + 2x_0h - x_0^2}{h} =$$

$$\frac{h^2 + 2x_0h}{h} = h + 2x_0 \to 2x_0 \text{ as } h \to 0.$$

Thus, $f'(x_0) = 2x_0$. Since this holds for all $x_0 \in \mathbb{R}$, we have that

$$f'(x) = 2x, \quad x \in \mathbb{R}.$$

Applet : Secant to Tangent , $x^2$

Applet : Tangent to derivative , 1 , $x^2$, 0

Ex.: $f(x) = x$, $x \in \mathbb{R}$

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{x_0+h-x_0}{h} = 1 \longrightarrow 1 = f'(x_0) \text{ as } h \to 0.$$

$$f'(x) = 1, \quad x \in \mathbb{R}$$

One can prove the following result:

Theorem: For any real number $n$, one has
$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Ex.
$$\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4,$$
$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x,$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1x^{1-1} = 1x^0 = 1,$$
$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Note: For any constant $c$, one has
$$\frac{d}{dx}(c) = 0.$$

Proof: $f(x) = c$, $x \in \mathbb{R}$

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{c-c}{h} = 0 \to 0 = f'(x_0) \text{ as } h \to 0.$$

$$f'(x) = 0, \quad x \in \mathbb{R}$$

Differentiation rules:

- $(f(x)+g(x))' = f'(x) + g'(x)$
- $(c\,f(x))' = c\,f'(x)$
- $(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Ex. $\frac{d}{dx}(5x^3 + 2x + 1) = 15x^2 + 2$

Ex. $\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

## Rates of change

Let $Q$ be a quantity* that varies with time $t$,

$$Q = f(t)$$

The change in $Q$ from time $t$ to time $t+\Delta t$ is

$$\Delta Q = f(t+\Delta t) - f(t)$$

The average rate of change is

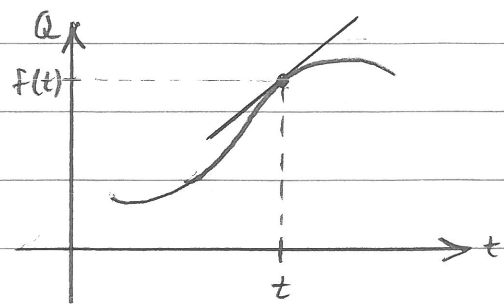
$$\frac{\Delta Q}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The instantaneous rate of change at time $t$ is

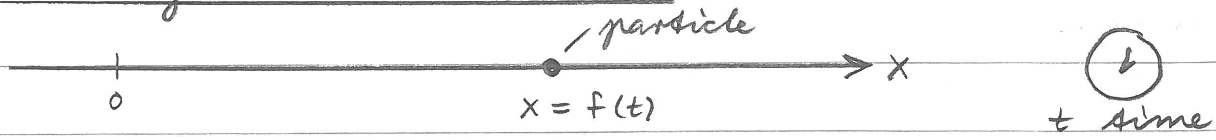
$$\lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = f'(t) = \frac{dQ}{dt}$$

Note: If $Q = f(t)$, then $f'(t)$ is

- The slope of the tangent to the curve $Q = f(t)$ at the point $(t, f(t))$.
- The instantaneous rate of change of $Q$ at time $t$.




## Velocity and acceleration




The change in $x$ from time $t$ to time $t+\Delta t$:

$$\Delta x = f(t+\Delta t) - f(t)$$

Average velocity : $\bar{v} = \frac{\Delta x}{\Delta t}$

Velocity : $v = \frac{dx}{dt} = f'(t)$

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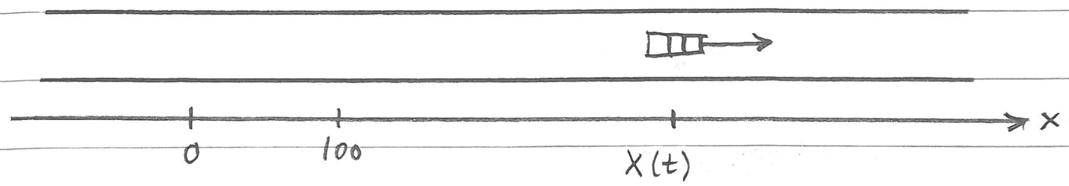
* $Q$ could be distance traveled at time $t$ or size of a population at time $t$.

Speed*: $|v|$

Acceleration: $a = \frac{dv}{dt} = f''(t)$.

Ex.




x in m
t in s

The car starts at time $t=0$. Suppose that

$$x(t) = 2t^2 + 100.$$

The velocity and acceleration at time $t$ is

$$v(t) = x'(t) = 4t,$$
$$a(t) = v'(t) = 4.$$

We have

$\left.\begin{array}{l} x(0) = 100 \\ v(0) = 0 \end{array}\right\}$: The car starts from rest at time $t=0$ at position 100 m.

$\left.\begin{array}{l} x(10) = 300 \\ v(10) = 40 \end{array}\right\}$: After 10 s the car has traveled 200 m from its start position, and its velocity is 40 m/s.

$a(t)=4$ : The acceleration is constantly 4 m/s².

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* Recall: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$