3. Session: Derivatives of Trigonometric Functions

<u>Recall:</u> The derivative $f'(x)$ of a function $f(x)$ is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

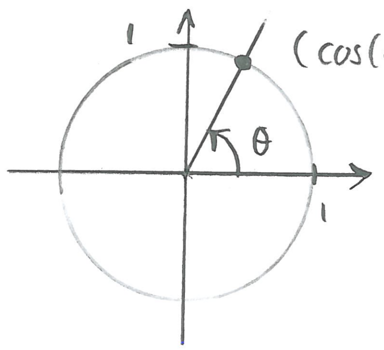
Notation: $f'(x) = \frac{d}{dx} f(x) = D_x f(x)$.

<u>Recall:</u>

$$\frac{d}{dx}(x^n) = n x^{n-1},$$

$$\frac{d}{dx}(c) = 0, \quad c \text{ constant.}$$

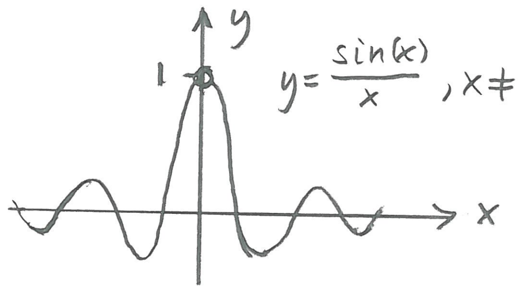
<u>Remark:</u>



$\cos(\theta) \to 1$ as $\theta \to 0$,

$\sin(\theta) \to 0$ as $\theta \to 0$.

<u>Remark:</u>



One can prove that

$$\frac{\sin(\theta)}{\theta} \to 1 \text{ as } \theta \to 0 \qquad (*)$$

<u>Lemma:</u>

$$\frac{1 - \cos(\theta)}{\theta} \to 0 \text{ as } \theta \to 0 \qquad (**)$$

<u>Proof:</u>

$$\frac{1-\cos(\theta)}{\theta} = \frac{(1-\cos(\theta))\cdot(1+\cos(\theta))}{\theta\cdot(1+\cos(\theta))} = \frac{1-\cos^2(\theta)}{\theta\cdot(1+\cos(\theta))}$$

$$= \frac{\sin^2(\theta)}{\theta\cdot(1+\cos(\theta))} = \frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{1+\cos(\theta)}$$

$$\xrightarrow{(*)} 1 \cdot \frac{0}{1+1} = 0 \text{ as } \theta \to 0 \qquad \text{q.e.d.}$$

Theorem: 1) $\frac{d}{dx}\sin(x) = \cos(x)$

2) $\frac{d}{dx}\cos(x) = -\sin(x)$.

Proof:

1) Put $f(x) = \sin(x)$. We have

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin(x)}{h} =$$

$$\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(X)}{h} =$$

$$\frac{\cos(x)\sin(h) - \sin(x)(1-\cos(h))}{h} =$$

$$\cos(x)\cdot\frac{\sin(h)}{h} - \sin(x)\cdot\frac{1-\cos(h)}{h} \xrightarrow{(*)\ (**)}$$

$\cos(x)\cdot 1 - \sin(x)\cdot 0 = \cos(x)$ as $h \to 0$.

2) Similar.

q.e.d.

Recall: Differentiation rules

- $(f(x)+g(x))' = f'(x)+g'(x)$
- $(cf(x))' = cf'(x)$
- $(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

Theorem:

1) $\frac{d}{dx}\tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)} = 1+\tan^2(x)$

2) $\frac{d}{dx}\cot(x) = -\csc^2(x)$

3) $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

4) $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Proof:

1) $\tan(x) = \frac{\sin(x)}{\cos(x)} \Rightarrow$

$$\frac{d}{dx}\tan(x) = \frac{\frac{d}{dx}(\sin(x))\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}(\cos(x))}{(\cos(x))^2}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$= 1 + \tan^2(x)$$

2)–4) are similar. q.e.d.

The Chain Rule

Theorem (The Chain Rule)

$$(f(g(x)))' = f'(g(x))\cdot g'(x)$$

Ex.

$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2)\cdot\frac{d}{dx}(x^2) = \cos(x^2)\cdot 2x$$

$$= 2x\cos(x)$$

Ex.

$$\frac{d}{dx}((3x^2+5)^{17}) = 17\cdot(3x^2+5)^{16}\cdot 6x$$

$$= 102x(3x^2+5)^{16}$$

Ex. $\frac{d}{dx}(\sin(kx)) = k\cos(kx)$,

$\frac{d}{dx}(\cos(kx)) = -k\sin(kx)$, k constant.

Ex. Let $\mathrm{sind}(x)$ and $\mathrm{cosd}(x)$ denote sine and cosine of an angle x measured in degrees. We have

$$\mathrm{sind}(x) = \sin\left(\frac{\pi}{180}x\right),$$

$$\mathrm{cosd}(x) = \cos\left(\frac{\pi}{180}x\right)$$

such that

$$\frac{d}{dx}(\mathrm{sind}(x)) = \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\mathrm{cosd}(x),$$

$$\frac{d}{dx}(\mathrm{cosd}(x)) = -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right) = -\frac{\pi}{180}\mathrm{sind}(x).$$