4. Session: Exponential and Logarithmic Functions

Notation:

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the set of integers

$\mathbb{R}$ the set of real numbers

$\mathbb{Z}_+$ positive integers , $\mathbb{R}_+$ positive real numbers.

Exponential functions

Def.: For $a \in \mathbb{R}$ and $n \in \mathbb{Z}_+$, we define

$$a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n}$$

For $a \neq 0$:

$$a^{-n} = \frac{1}{a^n}, \quad a^0 = 1$$

Def.: For $a \in \mathbb{R}_+$ and $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, we define

$$a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p.$$

It is possible to define a^x for any $x \in \mathbb{R}$ by rational approximation ($x \approx \frac{p}{q}$ for some p and q).
One has $a^x > 0$ for all $x \in \mathbb{R}$.

Theorem (Laws of exponents)
For all $a, b \in \mathbb{R}_+$ and $r, s \in \mathbb{R}$ one has

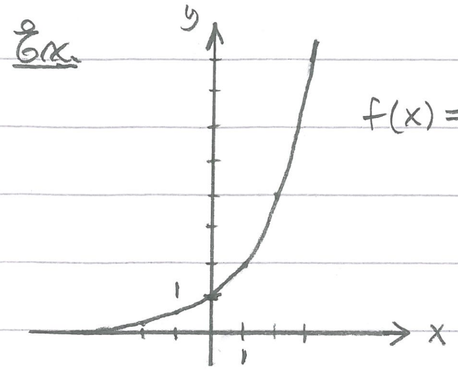
$$a^{r+s} = a^r a^s, \quad (a^r)^s = a^{rs}, \quad a^{-r} = \frac{1}{a^r},$$
$$(ab)^r = a^r b^r, \quad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}.$$

Def. An exponential function is a function of the form

$$f(x) = a^x, \quad x \in \mathbb{R}$$

where $a \in \mathbb{R}_+$. The number a is called the base.

Ex.



$f(x) = 2^x$

Remark:

$f(x) = a^x$ is
- increasing for $1 < a$
- decreasing for $0 < a < 1$

Remark: One can prove that there exists a constant $e > 1$ such that
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

h	0,1	0,01	0,001	0,0001
$\frac{2^h - 1}{h}$	0,718	0,696	0,693	0,693
$\frac{3^h - 1}{h}$	1,161	1,105	1,099	1,099

$2 < e < 3$,

$\boxed{e \approx 2{,}71828\ldots}$

! **Theorem:** $\frac{d}{dx}(e^x) = e^x$

Proof: Put $f(x) = e^x$. We have
$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h - 1)}{h} =$$
$$e^x \cdot \frac{e^h - 1}{h} \longrightarrow e^x \cdot 1 = e^x \quad \text{as } h \to 0. \qquad \text{q.e.d.}$$

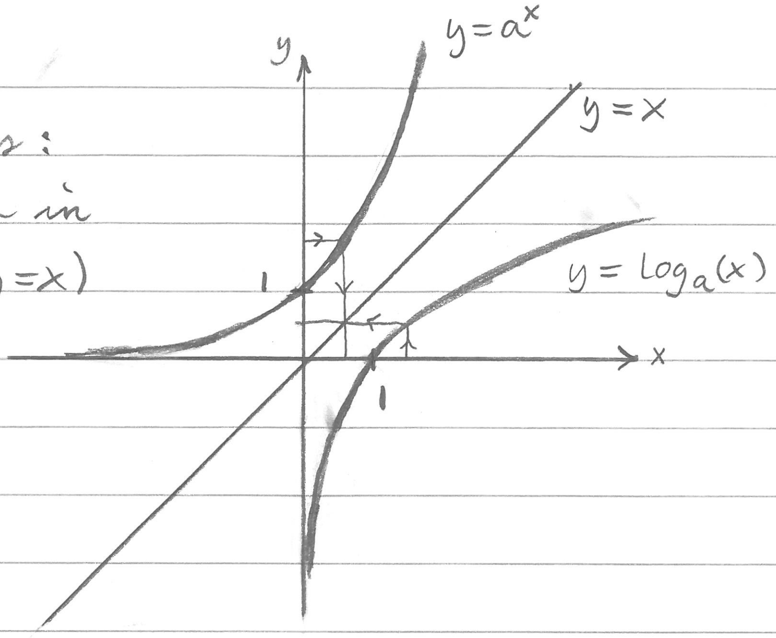
Note: $\frac{d}{dx}(e^{kx}) = k e^{kx}$ (by the chain rule)

Logarithmic functions

Since $f(x) = a^x$ is increasing, when $a > 1$, it has an inverse function.

Def. Let $a > 1$. The logarithmic function $\log_a(x)$, $x \in \mathbb{R}_+$ is the inverse of the exponential function a^x, $x \in \mathbb{R}$. That is
$$y = \log_a(x) \iff a^y = x.$$

The graphs:
(reflection in the line $y=x$)



Note: $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$.

Theorem: (Laws of logarithms)

$\log_a(xy) = \log_a(x) + \log_a(y)$, $\log_a(x^y) = y\log_a(x)$,

$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, $\log_a(\sqrt[r]{x}) = \frac{1}{r}\log_a(x)$,

$\log_a(a) = 1$

Proof for the first law: $x = a^r$, $y = a^s$ for some r, s.

$\log_a(xy) = \log_a(a^r a^s) = \log_a(a^{r+s}) = r+s$

$= \log_a(a^r) + \log_a(a^s) = \log_a(x) + \log_a(y)$. q.e.d.

Def. $\ln(x) = \log_e(x)$, $x \in \mathbb{R}_+$ is called the natural logarithm.

Theorem: $\frac{d}{dx}\ln(x) = \frac{1}{x}$, $x > 0$.

Proof: Since $\frac{d}{dx}(e^x) = e^x > 0$, we see from the graphs above, with $a = e$, that $\ln(x)$ is differentiable.

We apply the chain rule on $x = e^{\ln(x)}$:

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}(e^{\ln(x)}) = e^{\ln(x)}\frac{d}{dx}\ln(x) = x\frac{d}{dx}\ln(x)$$

$$\Rightarrow \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

q.e.d.

Theorem: 1) $a^x = e^{x\ln(a)}$

2) $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, $x > 0$

Proof

1) $e^{x\ln(a)} = e^{\ln(a)x} = (e^{\ln(a)})^x = a^x$

2) $y = \log_a(x) \Leftrightarrow a^y = x \Leftrightarrow e^{y\ln(a)} = x \Leftrightarrow$

$y\ln(a) = \ln(x) \Leftrightarrow y = \dfrac{\ln(x)}{\ln(a)}$. q.e.d.

Theorem: $\dfrac{d}{dx}(\ln(|x|)) = \dfrac{1}{x}$, $x \neq 0$

Proof:

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$x > 0$: OK

$x < 0$: $\dfrac{d}{dx}(\ln|x|) = \dfrac{d}{dx}(\ln(-x)) = \dfrac{1}{-x}\cdot\dfrac{d}{dx}(-x) = \dfrac{1}{-x}(-1) = \dfrac{1}{x}$

q.e.d.

Note: For a positive function $u(x)$, we get the following formula by the chain rule

$$\frac{d}{dx}\left(\ln(u(x))\right) = \frac{u'(x)}{u(x)}.$$

Ex. (Logarithmic differentiation)

$f(x) = x^{x+1}$, $x > 0$. Compute $f'(x)$.

$\ln f(x) = \ln(x^{x+1}) = (x+1)\ln(x) \Rightarrow$

$\dfrac{f'(x)}{f(x)} = 1\cdot\ln(x) + (x+1)\dfrac{1}{x} = \ln(x) + 1 + \dfrac{1}{x} \Rightarrow$

$f'(x) = f(x)\left(\ln(x) + 1 + \dfrac{1}{x}\right) = x^{x+1}\left(\ln(x) + 1 + \dfrac{1}{x}\right).$