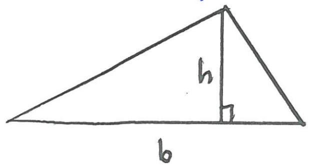
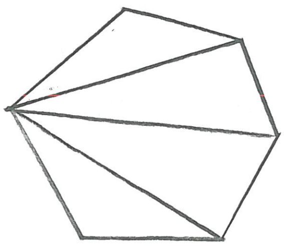
6. Session : Areas, Sums and Integrals I

The Consept of Area



Area of a triangle

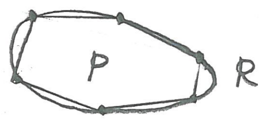
$$A = \frac{1}{2} bh$$

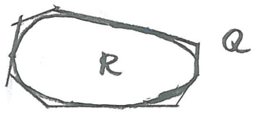


Area of a polygonal figure :

- Divide into nonoverlapping triangles
- Sum the areas of these triangles

Area of a curvilinear figure





P inscribed polygon   Q surcumscribed polygon

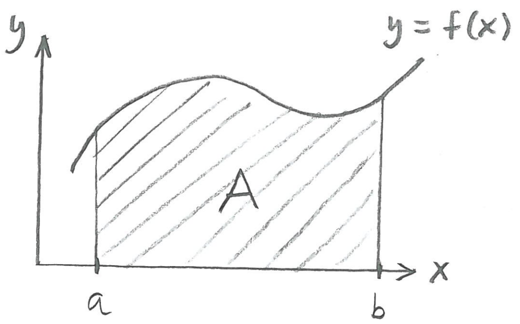
$a(P) \leq a(R) \leq a(Q)$.

If P and Q have many sides, all short, then $a(p)$ and $a(Q)$ closely approximate $a(R)$.

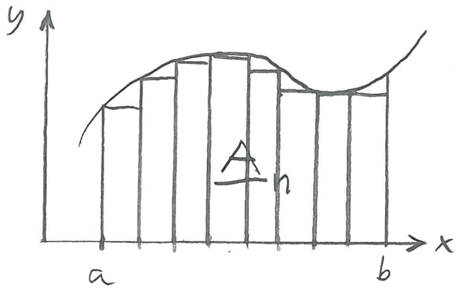
Area estimate : $\quad a(R) \approx \dfrac{a(P)+a(Q)}{2}$.

Areas under graphs

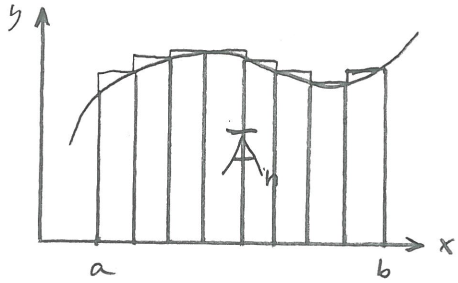
Let $f(x)$ be a continuous non-negative function defined on the interval $[a, b]$.



Divide $[a, b]$ into n subintervals of equal length



$\underline{A}_n$ area of the inscribed rectangles



$\overline{A}_n$ area of the circumscribed rectangles

$$\underbrace{\underline{A}_n}_{\text{underestimate}} \leq A \leq \underbrace{\overline{A}_n}_{\text{overestimate}} \qquad A \approx \frac{\underline{A}_n + \overline{A}_n}{2}$$

Summation notation

! Def.: Let $a_1, a_2, \dots, a_n$ be real numbers. Then,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \dots + a_n .$$

Note: $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$.

Ex.: $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

Ex.: $\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$

Theorem (Rules of summation)

(1) $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

(2) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(3) $\sum_{i=1}^{n} 1 = n$

Proof:

(1) $\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c\sum_{i=1}^{n} a_i$.

(2) $\sum_{i=1}^{n} (a_i + b_i) = a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n$

$= a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

(3) $\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \dots + 1}_{n} = n$

q.e.d.

Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

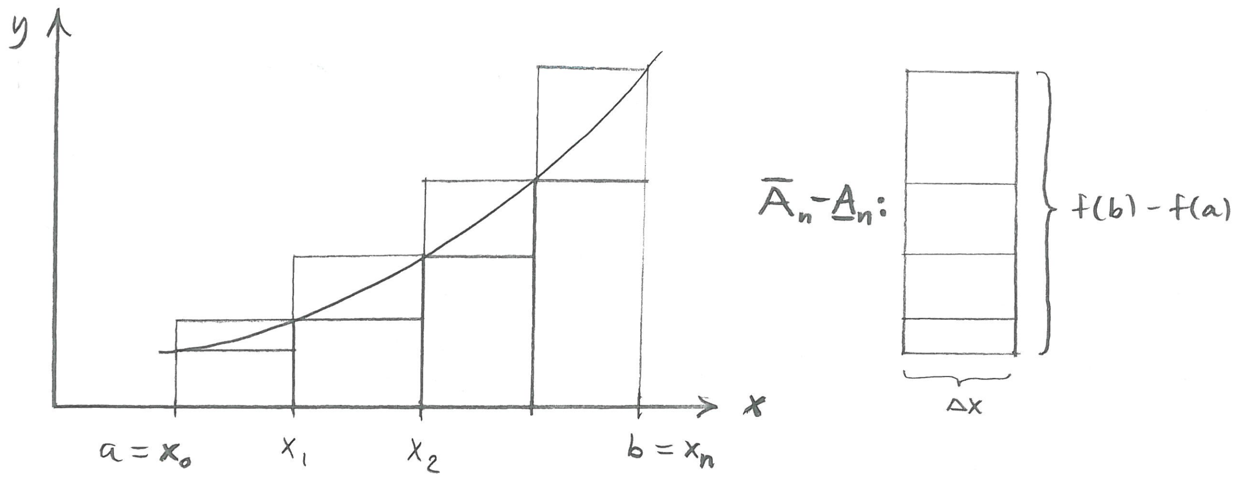
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{2}$$ (no proof)

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Ex. $\sum_{i=1}^{100} i = \frac{100 \cdot 101}{2} = 5050$

Area sums

Let $f(x)$ be a continous, non-negative and increasing function defined on the interval $[a,b]$.



Divide $[a,b]$ into n nonoverlapping subintervals $[x_i, x_{i+1}]$ of the same length $\Delta x = \frac{b-a}{n}$. We have

$$x_i = a + i\,\Delta x \quad \text{for } i = 0, 1, 2, \dots, n.$$

Underestimate : $\underline{A}_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

Overestimate : $\overline{A}_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$

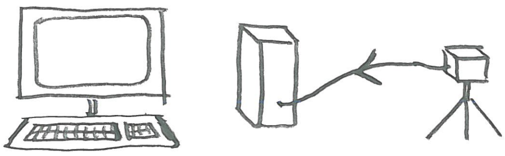
Note that $\Delta x = \dfrac{b-a}{n} \to 0$ as $n \to \infty$.

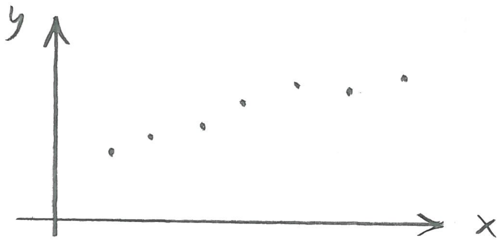
The area under the graph is

$$A = \lim_{n\to\infty} \underline{A}_n = \lim_{n\to\infty} \overline{A}_n .$$

This result also holds for $f(x)$ decreasing.

Why are area sums relevant ?





Points on a graph,

not $f(x) = 5\sin x + x^3$.