7. Session : Areas, Sums and Integrals II

Riemann sums

Def. Let f be a function defined on the interval $[a,b]$.

- A partition P of $[a,b]$ is an increasing sequence

$$a = x_0 < x_1 < x_2 < \dots < a_n = b.$$

The length of the subinterval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1}$.

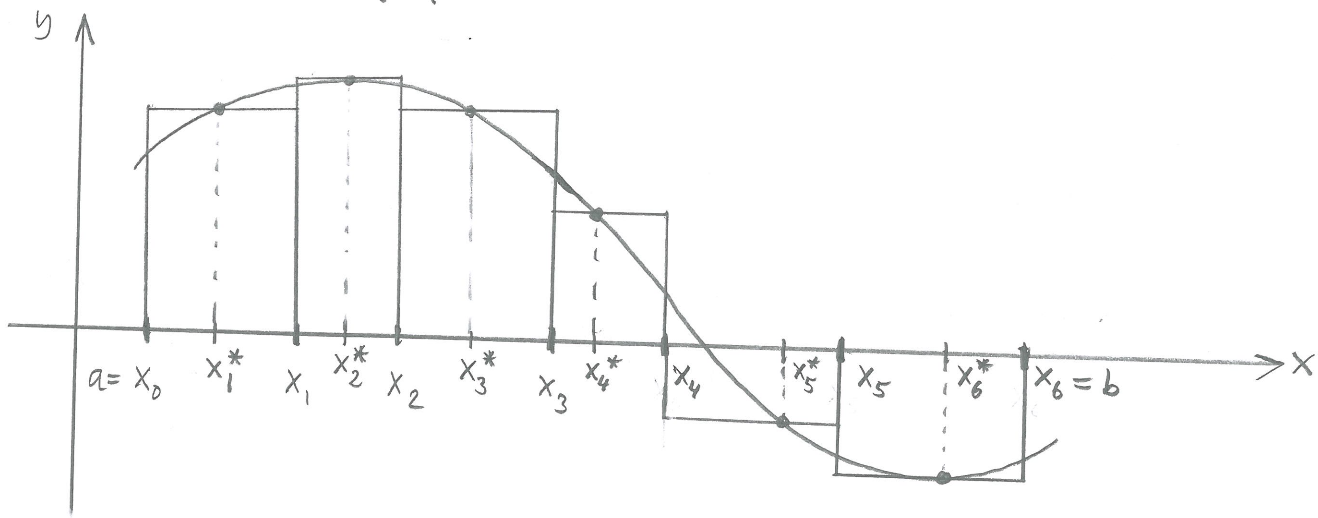
- A selection for the partition P, is a collection of points

$$x_1^*, x_2^*, \dots, x_n^*$$

with $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$.

- The Riemann sum for f determined by P and S is

$$R = \sum_{i=1}^{n} f(x_i^*) \Delta x_i .$$



The norm of the partition P is

$$|P| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}.$$

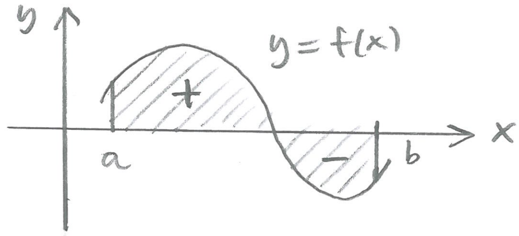
Def. The definite integral of f from a to b is the number

$$I = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

provided that this limit exists. If it does, f is said to be integrable on $[a,b]$.

Notation : $\int_a^b f(x)\,dx = I$.

! Note : $\int_a^b f(x)\,dx$ is the area from a to b under the graph $y = f(x)$ and above the x-axis minus the area below the x-axis and above the graph



Note: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

By definition,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Theorem : If the function f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Evaluation of integrals

! Def. An antiderivative of a function f is a function F such that

$$F'(x) = f(x).$$

Theorem*) : If $g'(x) = 0$, then there exists a constant C such that $g(x) = C$ for all x.

Corollary : If F and G are antiderivatives of f, then there exists a constant C such that

$$F(x) = G(x) + C$$

for all x.

Proof: $(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$ so

$F(x) - G(x) = C \iff F(x) = G(x) + C$ q.e.d.

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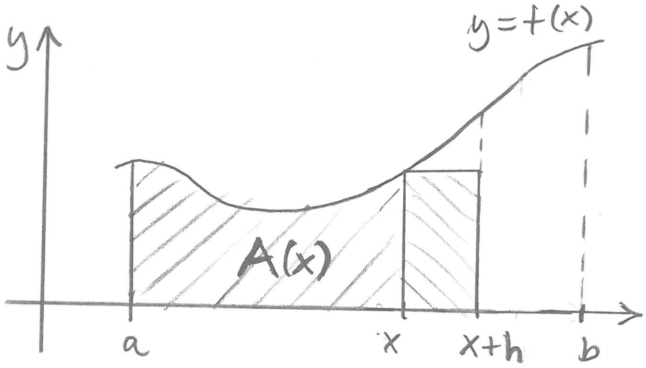
*) g is defined on an interval $]a,b[$

Theorem: If F is an antiderivative of the continuous function f on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Idea of proof



$$A(x) = \int_a^x f(t)\,dt$$

$$A(a) = 0$$

$$A(b) = \int_a^b f(t)\,dt$$

h small : $A(x+h) \approx A(x) + f(x)h \Rightarrow \dfrac{A(x+h)-A(x)}{h} \approx f(x)$.

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x).$$

Thus, A is an antiderivative of f such that $A(x) = F(x) + C$.

$$\int_a^b f(t)\,dt = A(b) - A(a) = F(b) + C - (F(a)+C) = F(b) - F(a).$$

Notation: $[F(x)]_a^b = F(b) - F(a)$.

Notation: If $F'(x) = f(x)$, then we write $\int f(x)\,dx = F(x) + C$.

Thus,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = \left[\int f(x)\,dx\right]_a^b.$$

Ex. $\int_0^2 x^5\,dx = \left[\frac{1}{6}x^6\right]_0^2 = \frac{1}{6}\cdot 2^6 - \frac{1}{6}\cdot 0^6 = \frac{64}{6} = \frac{32}{3}$

Ex. $\int_0^\pi \sin(x)\,dx = \left[-\cos(x)\right]_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2$

Ex. $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}e^{2\cdot 1} - \frac{1}{2}e^{2\cdot 0} = \frac{1}{2}(e^2-1)$

Note: One cannot compute $\int_0^1 e^{x^2}dx$ by this method.

Properties of integrals

- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- If $a < c < b$, then
  $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

- If $f(x) \le g(x)$ for $x \in [a,b]$, then
  $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$