9. Session: Dot Product and Cross Product

Def.: The dot-product of $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ is the number

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

Def.: The dot-product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is the number

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Ex.

$$(1,2,-1) \cdot (3,0,5) = 1 \cdot 3 + 2 \cdot 0 + (-1) \cdot 5 = -2 .$$

Theorem: For all vectors $\vec{a}, \vec{b}, \vec{c}$ and scalars r one has:

(1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

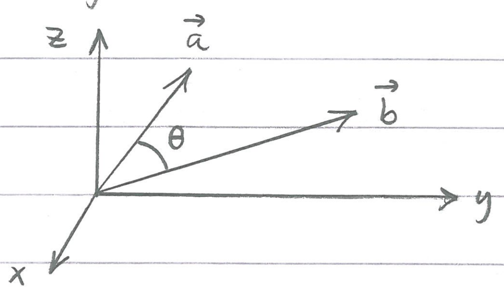
(4) $(r\vec{a}) \cdot \vec{b} = r(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (r\vec{b})$

Proof of (1):

Recall that $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 = \|\vec{a}\|^2$. q.e.d.

The angle between two vectors:



$0 \le \theta \le \pi$

$\vec{a}$ and $\vec{b}$ are

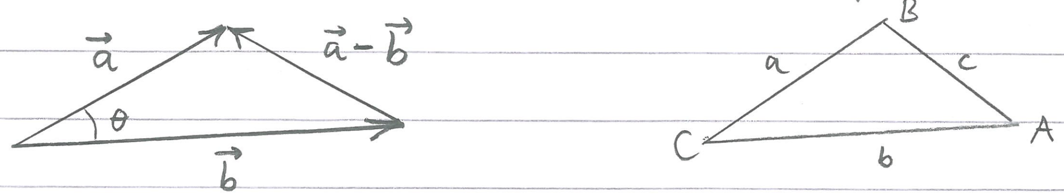
- perpendicular if $\theta = \frac{\pi}{2}$
- parallel if $\theta = 0$ or $\theta = \pi$

Convention: $\vec{0}$ is both parallel and perpendicular to any vector.

! Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos(\theta)$$

Proof for $\vec{a}$ and $\vec{b}$ non-zero and non-parallel:



By the laws of cosines ($c^2 = a^2 + b^2 - 2ab\cos(C)$) :

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta) \qquad (*)$$

By computations :

$$\|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$
$$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\,\vec{a}\cdot\vec{b} \qquad (**)$$

From $(*)$ and $(**)$ we find :

$$\|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta) = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b} \Rightarrow$$
$$\|\vec{a}\|\|\vec{b}\|\cos(\theta) = \vec{a}\cdot\vec{b}$$

q.e.d.

Remark: $\cos(\theta) = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$ for $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

Ex. Calculate the angle between $\vec{a} = (3,-3,3)$ and $\vec{b} = (-1,-2,2)$.

$\vec{a}\cdot\vec{b} = 3\cdot(-1) + (-3)\cdot(-2) + 3\cdot 2 = 9$,

$\|\vec{a}\| = \sqrt{3^2 + (-3)^2 + 3^2} = \sqrt{3^2\cdot 3} = 3\sqrt{3}$,

$\|\vec{b}\| = \sqrt{(-1)^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$.

$\cos(\theta) = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \dfrac{9}{3\sqrt{3}\cdot 3} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta \approx \underline{\underline{54,71^\circ}}$

! Theorem: $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $\vec{a}\cdot\vec{b} = 0$.

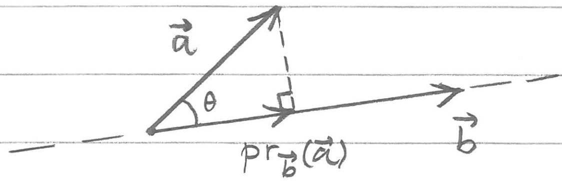
Proof: $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \|\vec{a}\|\|\vec{b}\|\cos(\theta) = 0 \Leftrightarrow \|\vec{a}\| = 0 \vee \|\vec{b}\| = 0 \vee \cos(\theta) = 0$

$\Leftrightarrow \vec{a} = \vec{0} \vee \vec{b} = \vec{0} \vee \theta = \frac{\pi}{2}$.

q.e.d.

Theorem: The perpendicular projection of a vector $\vec{a}$ onto a nonzero vector $\vec{b}$ is

$$pr_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}$$



Proof for $0 < \theta < \frac{\pi}{2}$: Put $\vec{p} = pr_{\vec{b}}(\vec{a})$. We have

$$\cos(\theta) = \frac{\|\vec{p}\|}{\|\vec{a}\|} \Rightarrow \|\vec{p}\| = \|\vec{a}\|\cos(\theta) = \frac{\|\vec{a}\|\|\vec{b}\|\cos(\theta)}{\|\vec{b}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}.$$

$$\vec{p} = \|\vec{p}\|\frac{\vec{b}}{\|\vec{b}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}\,\frac{\vec{b}}{\|\vec{b}\|} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}$$

q.e.d.

The Cross - Product

Def. : Determinant of order two :

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order three :

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

! Def. : The cross-product of two 3D-vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is the formal determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}.$$

Ex. $\vec{a} = (3, -1, 2)$, $\vec{b} = (2, 2, -1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= ((-1)\cdot(-1) - 2\cdot 2)\vec{i} - (3\cdot(-1) - 2\cdot 2)\vec{j} + (3\cdot 2 - (-1)\cdot 2)\vec{k}$$

$$= -3\vec{i} + 7\vec{j} + 8\vec{k} = (-3, 7, 8).$$

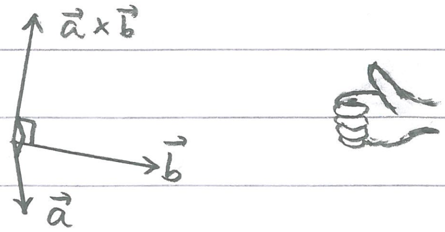
! Theorem :

(1) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

(2) If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin(\theta)$$

(3) If $\vec{a} \times \vec{b} \neq \vec{0}$, then $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ form a right handed triple



Proof (1) :

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = a_1 (a_2 b_3 - a_3 b_2) - a_2 (a_1 b_3 - a_3 b_1) + a_3 (a_1 b_2 - a_2 b_1)$$

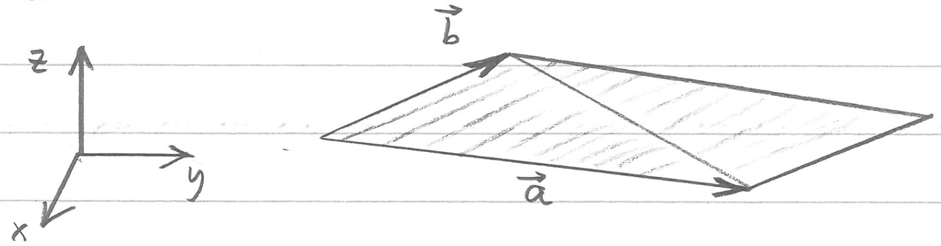
$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0.$$

Similarly, $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$. q.e.d.

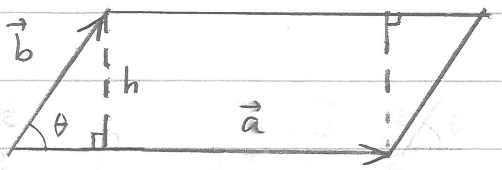
<u>Theorem</u>: $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

<u>Proof</u>: Follows from $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin(\theta)$ q.e.d.

<u>Theorem</u>: The parallelogram generated by $\vec{a}$ and $\vec{b}$ has area $\|\vec{a} \times \vec{b}\|$. The triangle generated by $\vec{a}$ and $\vec{b}$ has area $\frac{1}{2} \|\vec{a} \times \vec{b}\|$.



<u>Proof</u> for $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$, $0 < \theta < \frac{\pi}{2}$:



$$\sin(\theta) = \frac{h}{\|\vec{b}\|} \Rightarrow h = \|\vec{b}\| \sin(\theta)$$

$$A = \|\vec{a}\| h = \|\vec{a}\| \|\vec{b}\| \sin(\theta)$$

$$= \|\vec{a} \times \vec{b}\|$$

q.e.d.

<u>Algebraic properties</u>

<u>Theorem</u>:

(1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(2) $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

(3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(4) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

(5) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$