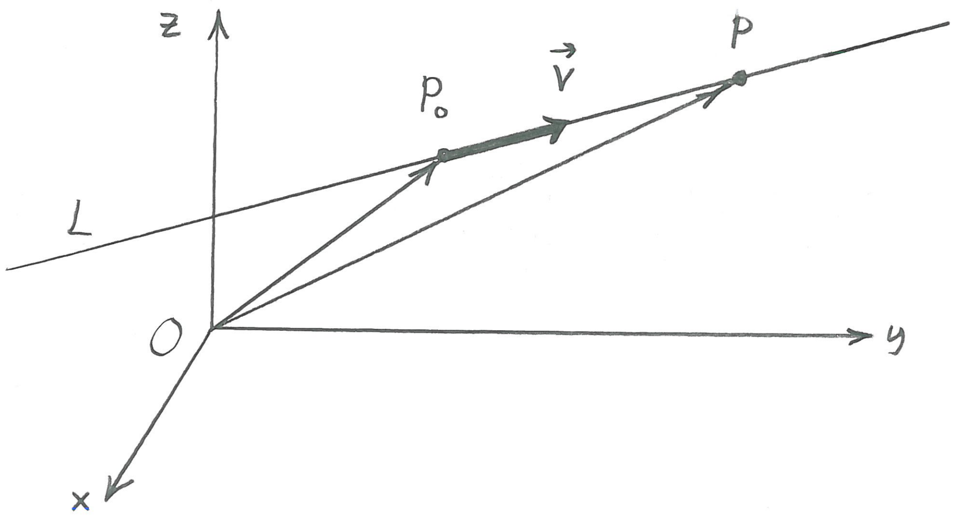
11. Session: Lines in 3D-space

Notation: $\mathbb{R}$ denotes the set of real numbers

$\in$ means "belongs to"

Parametric equations for a line

Assume that we have a point $P_0 = (x_0, y_0, z_0)$ and a vector $\vec{v} = (a, b, c) \neq \vec{0}$ in 3D-space. We want to find a description of the line L through P_0 and parallel to $\vec{v}$.



($\vec{v}$ is called a direction vector for the line.)

Let $P = (x, y, z)$ be any point in 3D-space. We have

$$P \in L \iff$$
$$\overrightarrow{P_0P} = t\vec{v} \text{ for some } t \in \mathbb{R} \iff$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{v} \text{ for some } t \in \mathbb{R} \iff$$
$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{v} \text{ for some } t \in \mathbb{R} \iff$$
$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \text{ for some } t \in \mathbb{R}.$$

Parametric equation for the line L:

!
$$\boxed{(x, y, z) = (x_0, y_0, z_0) + t(a, b, c), \quad t \in \mathbb{R}}$$

(t is called the parameter). Equivalently,

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

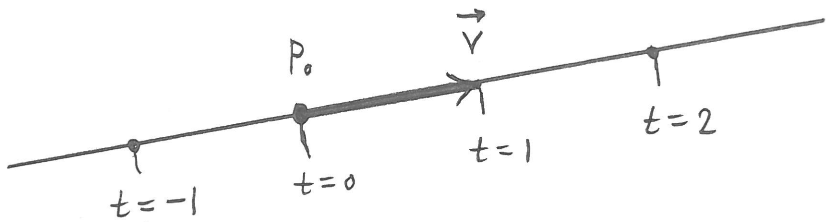
Ex. $(x,y,z) = (1,3,-1) + t(1,0,2)$, $t \in \mathbb{R}$

Line through $(1,3,-1)$ parallel to $\vec{v} = (1,0,2)$.

Some points on the line:

t	$(x,y,z) = (1,3,-1) + t(1,0,2)$
0	$(1,3,-1) + 0 \cdot (1,0,2) = (1,3,-1) = P_0$
1	$(1,3,-1) + 1 \cdot (1,0,2) = (2,3,1)$
-1	$(1,3,-1) + (-1)(1,0,2) = (0,3,-3)$

!



Ex. Find a parametric equation for the line through $A = (1,2,2)$ and $B = (3,-1,3)$.

Direction vector:

$\vec{v} = \overrightarrow{AB} = B - A = (3,-1,3) - (1,2,2) = (2,-3,1)$

Point on the line: $P_0 = A = (1,2,2)$.

Parametric equation

$$\underline{\underline{(x,y,z) = (1,2,2) + t(2,-3,1), \quad t \in \mathbb{R}}}$$

Another possibility:

$\vec{v} = 2 \cdot (2,-3,1) = (4,-6,2)$

$P_0 = B = (3,-1,3)$

Parametric equation

$$\underline{\underline{(x,y,z) = (3,-1,3) + t(4,-6,2), \quad t \in \mathbb{R}}}$$

! The parametric equation of a line is not unique.

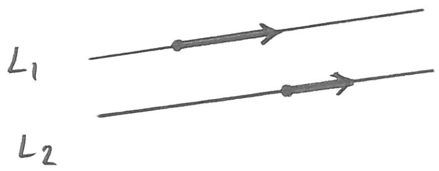
Intersections of Lines in 3D-space

$L_1: \quad (x,y,z) = (x_1,y_1,z_1) + t(a_1,b_1,c_1), \quad t \in \mathbb{R}$

$L_2: \quad (x,y,z) = (x_2,y_2,z_2) + s(a_2,b_2,c_2), \quad s \in \mathbb{R}$

Note that L_1 and L_2 are parallel if and only if

$$(a_1,b_1,c_1) = k(a_2,b_2,c_2) \quad \text{for some } k \in \mathbb{R}.$$



If L_1 and L_2 are not parallel, they can both be intersecting and nonintersecting:

Intersecting — Skew lines

Procedure:

(1) Solve the system

$$\left.\begin{aligned} x_1 + a_1 t &= x_2 + a_2 s \\ y_1 + b_1 t &= y_2 + b_2 s \end{aligned}\right\}$$

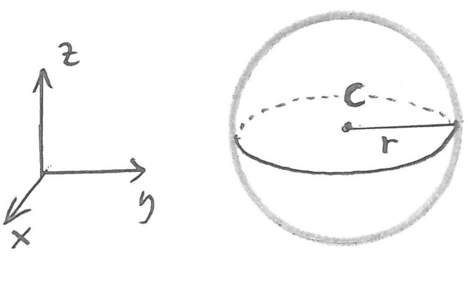
(2) If there is a solution for s and t, which also satisfy $z_1 + c_1 t = z_2 + c_2 s$, then one has a point of intersection.

Spheres

Recall: The distance between $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Equation for the sphere with radius $r > 0$ and center $C = (h,k,l)$:

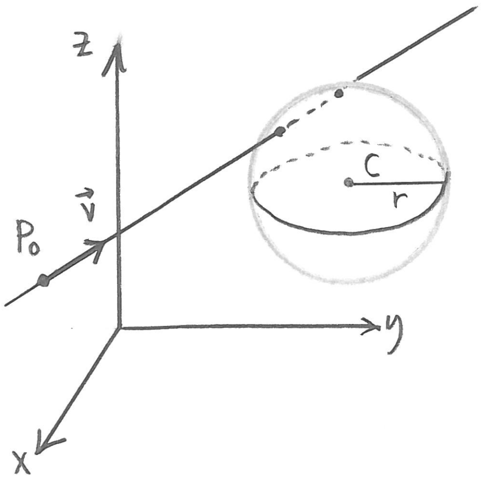


! $\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$

Proof: The sphere is the set of points $P = (x,y,z)$ such that $|CP| = r$ or equivalently $|CP|^2 = r^2$.

q.e.d.

Line - sphere intersections



Sphere with center $C=(h,k,\ell)$ and radius $r>0$:

$$(x-h)^2+(y-k)^2+(z-\ell)^2=r^2$$

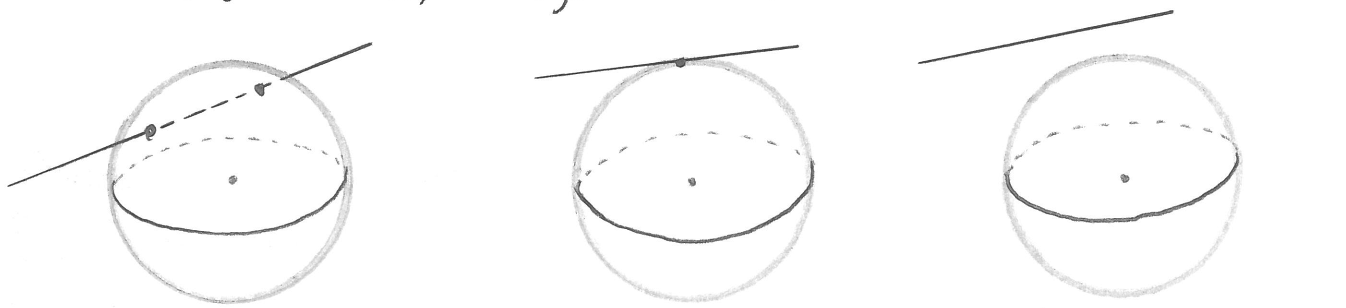
Line through $P_0=(x_0,y_0,z_0)$ and parallel to $\vec{v}=(a,b,c)\neq\vec{0}$:

$$(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)\ ,\ t\in\mathbb{R}.$$

$$x=x_0+at,\quad y=y_0+bt,\quad z=z_0+ct.$$

Solve the following quadratic equation, where t is the unknown:

$$(x_0+at-h)^2+(y_0+bt-k)^2+(z_0+ct-\ell)^2=r^2.$$



Two solutions   One solution   No solution.

Applications in computer games

Symmetric equations

If a, b and c in $\circledast$ are non-zero, we can eliminate the parameter t:

$$x=x_0+at\ ,\ y=y_0+bt\ ,\ z=z_0+ct \iff$$

$$\frac{x-x_0}{a}=t\ ,\ \frac{y-y_0}{b}=t\ ,\ \frac{z-z_0}{c}=t \iff$$

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}=t.$$

Symmetric equations for the line: $\boxed{\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}}$