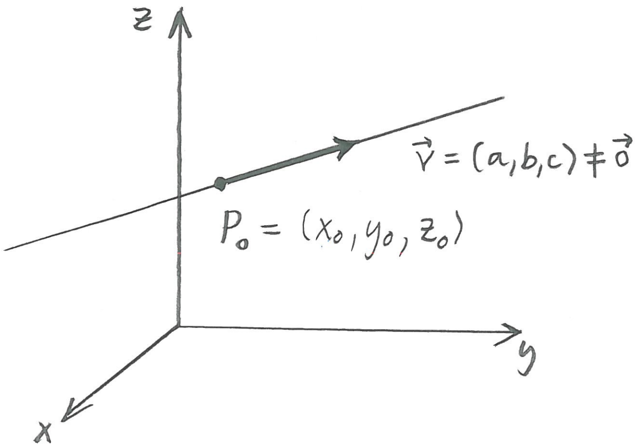
12. Session: Planes in Space

Recall:



Parametric equation:

$(x,y,z) = (x_0, y_0, z_0) + t(a,b,c)$, $t \in \mathbb{R}$

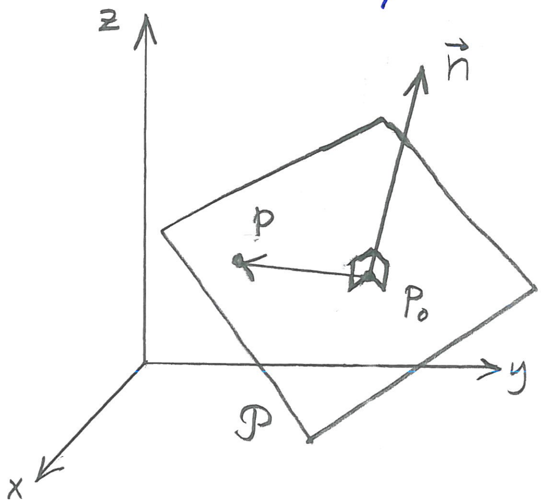
Equivalently,

$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.

Symmetric equations (for $a \neq 0$, $b \neq 0$, $c \neq 0$)

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Planes in 3D-space



Let $\mathcal{P}$ be the plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n} = (a, b, c) \neq \vec{0}$.

For any point $P = (x,y,z)$, we have

$$P \in \mathcal{P} \iff$$

$$\vec{n} \cdot \overrightarrow{P_0 P} = 0 \iff$$

$$(a,b,c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \iff$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Equation for the plane:

!
$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Ex. The plane through $P_0 = (-1, 5, 2)$ with normal vector $\vec{n} = (1, -3, 2)$ has equation

$$1 \cdot (x - (-1)) + (-3) \cdot (y - 5) + 2 \cdot (z - 2) = 0 \iff$$

$$\underline{x - 3y + 2z = -12}$$

Note: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \iff$

$ax+by+cz = ax_0+by_0+cz_0$

! Theorem: Every equation of the form

$$ax+by+cz=d$$

where $a\neq 0$, $b\neq 0$ or $c\neq 0$, is an equation for a plane with normal vector (a,b,c).

Proof: If for instance $c\neq 0$, we have

$$ax+by+cz=d \iff ax+by+c(z-\tfrac{d}{c})=0,$$

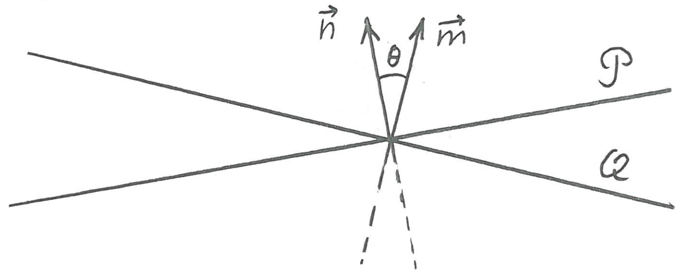
which is the equation for the plane through $(0,0,\frac{d}{c})$ with normal vector (a,b,c). q.e.d.

Angles between planes and lines of intersection

$\mathcal{P}$ plane with normal vector $\vec{n}\neq\vec{0}$,

$\mathcal{Q}$ plane with normal vector $\vec{m}\neq\vec{0}$,

θ angle between $\vec{n}$ and $\vec{m}$.



(seen from the side)

- If $\theta=0$ or $\theta=\pi$, then $\mathcal{P}$ and $\mathcal{Q}$ are parallel.
- If $\theta\neq 0$ and $\theta\neq\pi$, then the angle between $\mathcal{P}$ and $\mathcal{Q}$ is either θ or $\pi-\theta$, wichever is an acute angle. $\mathcal{P}$ and $\mathcal{Q}$ intersect in a line.

Ex. $\mathcal{P}$: $2x+3y-z=-3$

$\mathcal{Q}$: $4x+5y+z=1$

Then $\vec{n}=(2,3,-1)$, $\vec{m}=(4,5,1)$.

$$\cos(\theta) = \frac{\vec{n} \cdot \vec{m}}{\|\vec{n}\| \, \|\vec{m}\|} = \frac{2 \cdot 4 + 3 \cdot 5 + (-1) \cdot 1}{\sqrt{2^2 + 3^2 + (-1)^2} \cdot \sqrt{4^2 + 5^2 + 1^2}} = \frac{22}{\sqrt{14}\sqrt{42}}$$

$\theta \approx 24{,}87°$ acute OK, so the angle between $\mathcal{P}$ and $\mathcal{Q}$ is $\underline{\underline{24{,}87°}}$

Line of intersection:

We find two points on the line.

Put $x = 1$ (arbitrarily).

$$\left.\begin{aligned} 2 \cdot 1 + 3y - z &= -3 \\ 4 \cdot 1 + 5y + z &= 1 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} z &= 3y + 5 \\ 4 + 5y + (3y+5) &= 1 \end{aligned}\right\} \Leftrightarrow$$

$$\left.\begin{aligned} z &= 3y + 5 \\ 8y &= -8 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} z &= 2 \\ y &= -1 \end{aligned}\right\}.$$

$P_1 = (1, -1, 2)$ lies on both planes.

Put $x = 5$.

$$\left.\begin{aligned} 2 \cdot 5 + 3y - z &= -3 \\ 4 \cdot 5 + 5y + z &= 1 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} z &= 3y + 13 \\ 20 + 5y + (3y+13) &= 1 \end{aligned}\right\} \Leftrightarrow$$

$$\left.\begin{aligned} z &= 3y + 13 \\ 8y &= -32 \end{aligned}\right\} \Leftrightarrow \left.\begin{aligned} z &= 1 \\ y &= -4 \end{aligned}\right\}$$

$P_2 = (5, -4, 1)$ lies on both planes.

Direction vector:

$$\vec{v} = \overrightarrow{P_1 P_2} = P_2 - P_1 = (5,-4,1) - (1,-1,2) = (4,-3,-1)$$

Parametric equation for the line of intersection:

$$\underline{\underline{(x,y,z) = (1,-1,2) + t(4,-3,-1), \quad t \in \mathbb{R}}}$$