

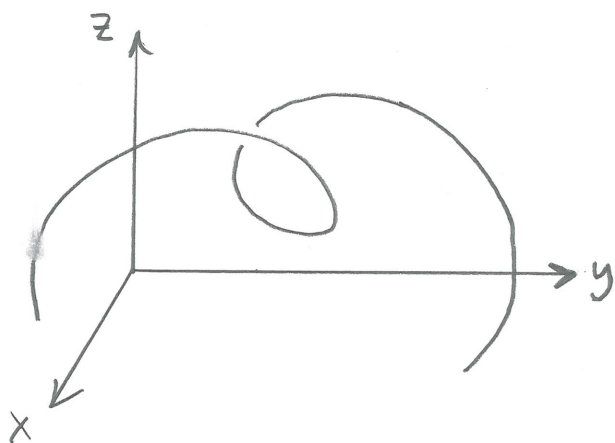
13. Session: Curves and Motion in Space I

Def. A parametric curve in 3D-space consists of three equations

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

where $f(t)$, $g(t)$ and $h(t)$ are continuous functions defined on an interval I . The corresponding curve is the set of points

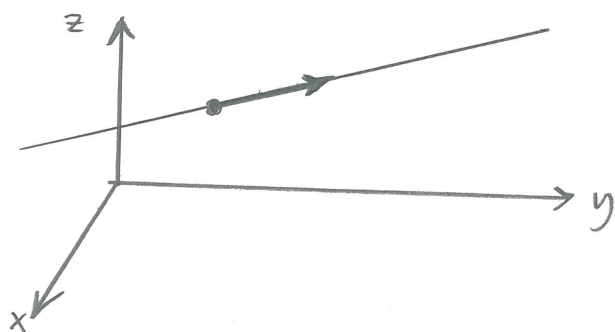
$$\{ (f(t), g(t), h(t)) \mid t \in I \}.$$



Curve in 3D-space

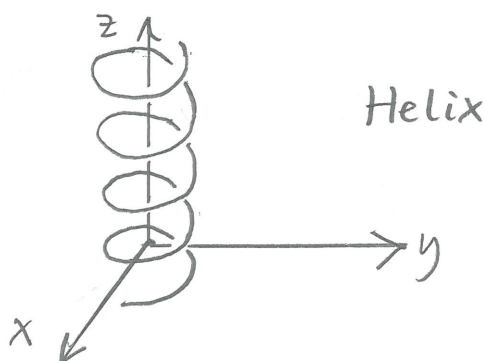
Ex. Parametric equations for a line

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

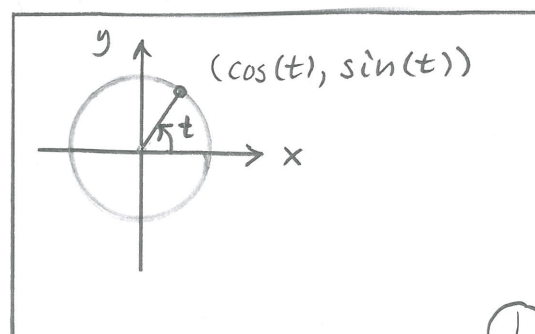


Line in 3D-space
($(a, b, c) \neq \vec{0}$)

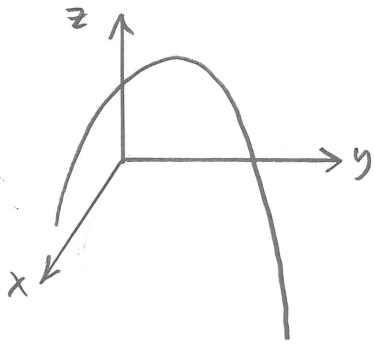
Ex. $x = \cos(t), \quad y = \sin(t), \quad z = t$



Helix

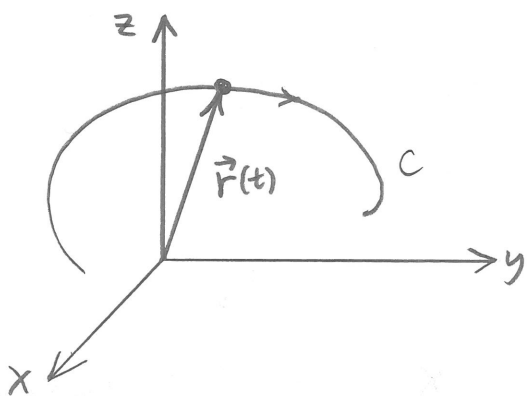


Ex. $x = 2t+1$, $y = 3t$, $z = -t^2 + 5t$



! Def. $\vec{r}(t) = (f(t), g(t), h(t))$, $t \in I$ is called a vector function

Geometric interpretation



t is the time.

$\vec{r}(t)$ is the position vector for a point moving along a curve C .

Differentiation of vector functions

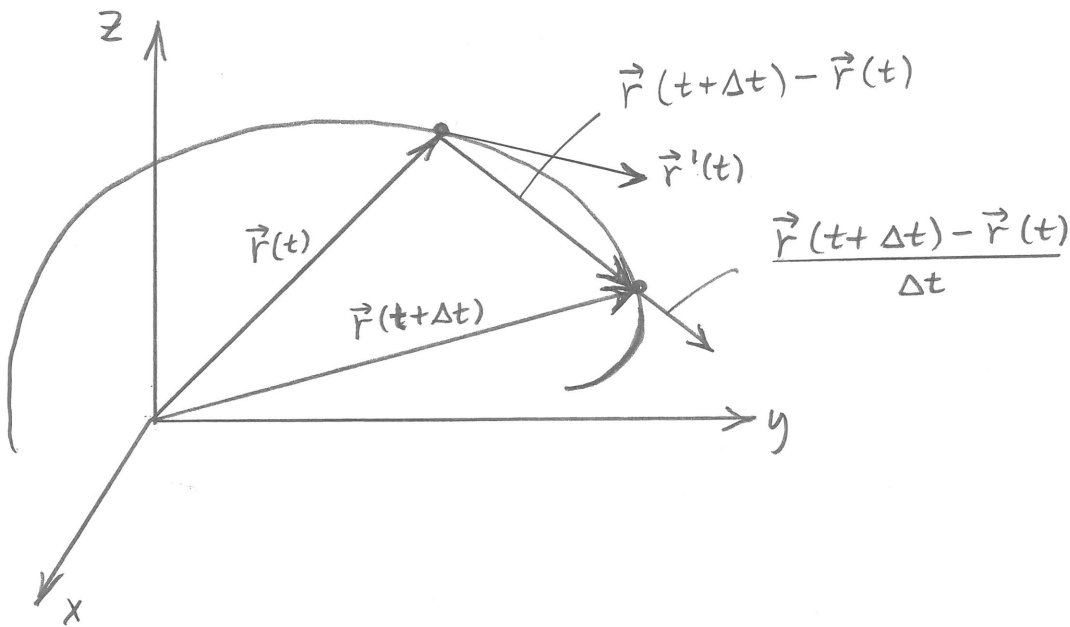
Def. $\lim_{t \rightarrow a} \vec{r}(t) = (\lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t))$

Def. The derivative $\vec{r}'(t)$ of the vector function $\vec{r}(t)$ is the limit

$$\vec{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

provided that this limit exists.

! Remark: If $\vec{r}'(t) \neq \vec{0}$, then $\vec{r}'(t)$ is a tangent vector to the curve C at the point with position vector $\vec{r}(t)$.



! Theorem: If the coordinate functions of $\vec{r}(t) = (f(t), g(t), h(t))$ are differentiable, then $\vec{r}'(t) = (f'(t), g'(t), h'(t))$.

Proof:

$$\begin{aligned} \vec{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{(f(t + \Delta t), g(t + \Delta t), h(t + \Delta t)) - (f(t), g(t), h(t))}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left(\frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right) \\ &= \left(\lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right) \\ &= (f'(t), g'(t), h'(t)). \quad \text{q.e.d.} \end{aligned}$$

Ex.

$$\begin{aligned} \vec{r}(t) &= (2t + 1, 3t, -t^2 + 5t) \Rightarrow \\ \vec{r}'(t) &= (2, 3, -2t + 5) \end{aligned}$$

Ex. $\vec{r}(t) = (\cos(t), \sin(t), t) \Rightarrow$

$$\vec{r}'(t) = (-\sin(t), \cos(t), 1).$$

For $t=0$ we have

$$\vec{r}(0) = (\cos(0), \sin(0), 0) = (1, 0, 0)$$

$$\vec{r}'(0) = (-\sin(0), \cos(0), 1) = (0, 1, 1)$$

Thus, $(0, 1, 1)$ is a tangent vector to the helix at the point $(1, 0, 0)$.

Theorem: Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector functions, $f(t)$ a differentiable function and c a scalar. Then,

$$(1) \quad (\vec{u}(t) + \vec{v}(t))' = \vec{u}'(t) + \vec{v}'(t)$$

$$(2) \quad (c\vec{u}(t))' = c\vec{u}'(t)$$

$$(3) \quad (f(t)\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$(4) \quad (\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$(5) \quad (\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Proof of (4):

$$\vec{u}(t) = (u_1(t), u_2(t), u_3(t)), \quad \vec{v}(t) = (v_1(t), v_2(t), v_3(t)).$$

$$\vec{u}(t) \cdot \vec{v}(t) = \sum_{i=1}^3 u_i(t) v_i(t) \Rightarrow$$

$$(\vec{u}(t) \cdot \vec{v}(t))' = \sum_{i=1}^3 (u_i'(t) v_i(t) + u_i(t) v_i'(t))$$

$$= \sum_{i=1}^3 u_i'(t) v_i(t) + \sum_{i=1}^3 u_i(t) v_i'(t)$$

$$= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

q.e.d.