16. Session: Matrices and Vectors

Def.: An <u>$m \times n$-matrix</u> A is a rectangular array of real numbers with m <u>rows</u> and n <u>columns</u>.

Ex.

$$A = \begin{bmatrix} 7 & 2 \\ 5 & 0 \\ -1 & 11 \end{bmatrix} \qquad 3 \times 2\text{-matrix}$$

$$\begin{bmatrix} 8 & 0 & 7 \\ 5 & -1 & 10 \\ 2 & \sqrt{5} & 7 \end{bmatrix} \qquad 3 \times 3\text{-matrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad 2 \times 1\text{-matrix}$$

Def.: The number in the i^{th} row and j^{th} column of A is called the <u>(i,j)-entry</u> and denoted a_{ij} or $[A]_{ij}$.

Ex.

$$A = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 7 & 2 \\ 5 & 0 \\ -1 & 11 \end{bmatrix} \end{array}$$

$[A]_{11} = 7$, $[A]_{12} = 2$
$[A]_{21} = 5$, $[A]_{22} = 0$
$[A]_{31} = -1$, $[A]_{32} = 11$

Def.: Let A and B be $m \times n$-matrices and let c be a scalar.* Then $A+B$, $A-B$ and cA are the $m \times n$-matrices with entries as follows:

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij},$$

$$[A-B]_{ij} = [A]_{ij} - [B]_{ij},$$

$$[cA]_{ij} = c[A]_{ij},$$

for $i = 1,2,3,\dots,m$ and $j = 1,2,3,\dots,n$. The zero-matrix O is the $m \times n$-matrix with $[O]_{ij} = 0$ for every i and j.

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* "Scalar" means real number

Ex.

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 3+(-4) & 4+1 & 2+0 \\ 2+5 & -3+(-6) & 0+1 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 2 \\ 7 & -9 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 0 \\ 5 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 3-(-4) & 4-1 & 2-0 \\ 2-5 & -3-(-6) & 0-1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 2 \\ -3 & 3 & -1 \end{bmatrix},$$

$$3 \cdot \begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 & 3\cdot 4 & 3\cdot 2 \\ 3\cdot 2 & 3\cdot(-3) & 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 6 \\ 6 & -9 & 0 \end{bmatrix}.$$

Theorem: Let A, B, C be $m\times n$-matrices and let s, t be scalars. Then

(1) $A + B = B + A$ (2) $(A+B)+C = A+(B+C)$

(3) $A + 0 = A$ (4) $A - A = 0$

(5) $(st)A = s(tA)$ (6) $s(A+B) = sA + sB$

(7) $(s+t)A = sA + tA$.

Proof of (1): Both sides are $m\times n$-matrices, and
$[A+B]_{ij} = [A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij} = [B+A]_{ij}$. q.e.d.

Def.: Let A be an $m\times n$-matrix. The transpose of A is the $n\times m$-matrix A^T with

$$[A^T]_{ij} = [A]_{ji}$$

for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Note: The columns of A^T are the rows of A and vice versa.

Ex.

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & -3 & 0 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 \\ 4 & -3 \\ 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 7 \\ -2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 7 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix}^T = \begin{bmatrix} 5 & -3 \end{bmatrix}.$$

Theorem: Let A and B be $m \times n$-matrices, and let s be a scalar. Then

(1) $(A+B)^T = A^T + B^T$

(2) $(sA)^T = s(A^T)$

(3) $(A^T)^T = A$.

Def. An $1 \times n$-matrix is called a row vector. An $m \times 1$-matrix is called a column vector.

Ex. $\begin{bmatrix} 5 & 0 & -1 \end{bmatrix}$ row vector

$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ column vector

Note: We usually work with column vectors.

Ex.

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+(-1) \\ 7+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix},$$

$$5 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 \\ 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}.$$

Def. A linear combination of the $m \times 1$-vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ is a vector of the form

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + \ldots + c_k \vec{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars. These scalars are called the coefficients or weights of the linear combination.

Ex.

$$1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

Thus, $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Ex. Is $\begin{bmatrix}4\\-1\end{bmatrix}$ a linear combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$?

$$x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix} \Leftrightarrow \begin{bmatrix}2x+3y\\3x+y\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix} \Leftrightarrow \left.\begin{matrix}2x+3y=4\\3x+y=-1\end{matrix}\right\} \Leftrightarrow$$

$$\left.\begin{matrix}2x+3y=4\\y=-3x-1\end{matrix}\right\} \Leftrightarrow \left.\begin{matrix}2x+3(-3x-1)=4\\y=-3x-1\end{matrix}\right\} \Leftrightarrow \left.\begin{matrix}-7x=7\\y=-3x-1\end{matrix}\right\} \Leftrightarrow \left.\begin{matrix}x=-1\\y=2\end{matrix}\right\}.$$

Yes, $(-1)\begin{bmatrix}2\\3\end{bmatrix}+2\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}4\\-1\end{bmatrix}$.

<u>Notation:</u> The set of column vectors with m coordinates is denoted $\mathbb{R}^m$. Thus, a vector $\vec{v}$ in $\mathbb{R}^m$ has the form

$$\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_m\end{bmatrix}.$$

<u>Def.</u> The <u>standard basis vectors</u> in $\mathbb{R}^m$ are

$$\vec{e}_1=\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix},\quad \vec{e}_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\quad \dots,\quad \vec{e}_m=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$$

<u>Ex.</u> Standard basis vectors in $\mathbb{R}^2$:

$$\vec{e}_1=\begin{bmatrix}1\\0\end{bmatrix}=\vec{i},\quad \vec{e}_2=\begin{bmatrix}0\\1\end{bmatrix}=\vec{j}.$$

<u>Ex.</u> Standard basis vectors in $\mathbb{R}^3$:

$$\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}=\vec{i},\quad \vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}=\vec{j},\quad \vec{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}=\vec{k}.$$

Note that

$$\begin{bmatrix}x\\y\end{bmatrix}=x\vec{i}+y\vec{j},$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=x\vec{i}+y\vec{j}+z\vec{k},$$

$$\vec{v}=v_1\vec{e}_1+v_2\vec{e}_2+\dots+v_m\vec{e}_m.$$