

## 17. Session: Linear Combinations, Matrix-Vector Products, and Special Matrices

Recall: Matrices, entries

$$A = \begin{bmatrix} 1 & 7 & 0 & 2 \\ 5 & 0 & -1 & 11 \\ -3 & 10 & 7 & 9 \\ -1 & 8 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \quad \begin{array}{l} 5 \times 4 \text{ - matrix} \\ \uparrow \quad \quad \uparrow \\ 5 \text{ rows} \quad 4 \text{ columns} \end{array}$$

$$(3,2)\text{-entry} : a_{32} = [A]_{32} = 10$$

Recall: +, -, multiplication by a scalar

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+0 \\ -3+1 & 0+4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-0 \\ -3-1 & 0-4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -4 & -4 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot (-3) & 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -15 & 0 \end{bmatrix}$$

Recall: Transpose

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 0 & -1 \end{bmatrix}$$

Recall: Vectors,  $\mathbb{R}^n$ , standard basis vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{Standard basis for } \mathbb{R}^3$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

Recall: Linear combinations of vectors

$$2 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

↑ weights or coefficients

## Matrix-vector products

Def. Let  $A$  be an  $m \times n$ -matrix with column vectors  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ .

Let  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$  be an  $n \times 1$ -vector.

The product  $A \cdot \vec{v}$  is the  $m \times 1$ -vector

$$A \cdot \vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + \dots + v_n \vec{a}_n.$$

Ex.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \end{bmatrix} = 7 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \cdot 1 + 8 \cdot 2 \\ 7 \cdot 3 + 8 \cdot 4 \\ 7 \cdot 5 + 8 \cdot 6 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

Faster:

$$\begin{bmatrix} \boxed{1} & \boxed{2} \\ \boxed{3} & \boxed{4} \\ \boxed{5} & \boxed{6} \end{bmatrix} \cdot \begin{bmatrix} \boxed{7} \\ \boxed{8} \end{bmatrix} = \begin{bmatrix} 7 \cdot 1 + 8 \cdot 2 \\ 7 \cdot 3 + 8 \cdot 4 \\ 7 \cdot 5 + 8 \cdot 6 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-1) \cdot 2 + 1 \cdot 3 + 3 \cdot 1 \\ (-1) \cdot 1 + 1 \cdot (-2) + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

## Identity matrices

Def. The  $n \times n$ -identity matrix  $I_n$  is the  $n \times n$ -matrix with columns  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ .

Ex.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \dots$$

Theorem:  $I_n \vec{v} = \vec{v}$  for every  $n \times 1$ -vector  $\vec{v}$ .

Proof:

$$I_n \vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n = \vec{v} \quad \text{q.e.d.}$$

Ex.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \cdot 1 + b \cdot 0 \\ a \cdot 0 + b \cdot 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$

## Rotation matrices

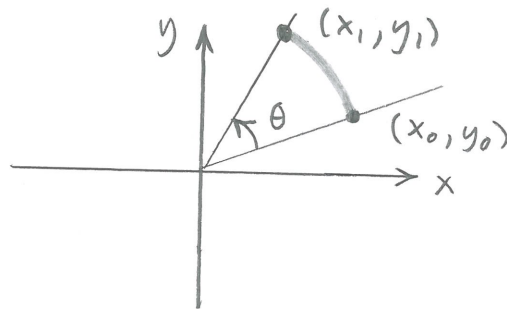
! Def. For an angle  $\theta$ , the  $2 \times 2$ -rotation matrix is defined as

$$R_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

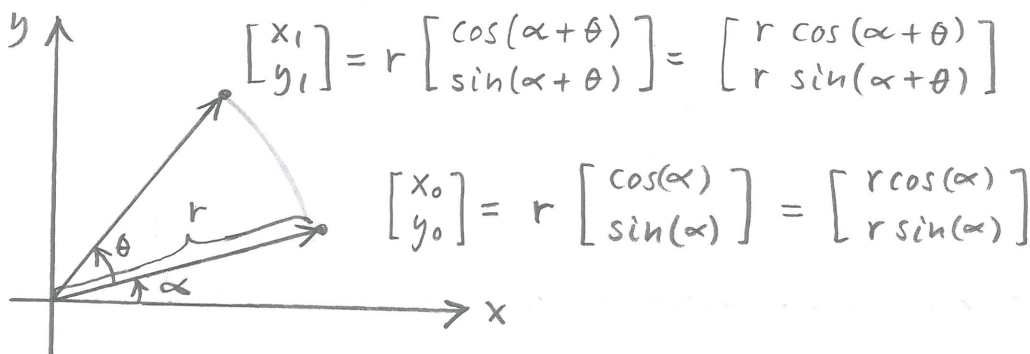
Theorem: Let  $(x_0, y_0)$  be a point in the plane.

The rotation  $(x_1, y_1)$  of  $(x_0, y_0)$  through the angle  $\theta$  about the origin, is given by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



Proof.



$$\begin{aligned} R_{\theta} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r \cos(\alpha) \\ r \sin(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} r \cos(\alpha) \cos(\theta) - r \sin(\alpha) \sin(\theta) \\ r \cos(\alpha) \sin(\theta) + r \sin(\alpha) \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \end{aligned}$$

Addition  
formulas

q.e.d.

Theorem (Properties of matrix-vector products)

Let  $A$  be an  $m \times n$ -matrix,  $\vec{u}$  and  $\vec{v}$  vectors in  $\mathbb{R}^n$ , and  $c$  a scalar. Then

$$(1) \quad A \cdot (\vec{u} + \vec{v}) = A \cdot \vec{u} + A \cdot \vec{v}$$

$$(2) \quad A \cdot (c\vec{u}) = c(A \cdot \vec{u})$$

$$(3) \quad A \cdot \vec{0} = \vec{0}$$

Proof of (1)

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n].$$

$$\begin{aligned} A \cdot (\vec{u} + \vec{v}) &= (u_1 + v_1)\vec{a}_1 + (u_2 + v_2)\vec{a}_2 + \dots + (u_n + v_n)\vec{a}_n \\ &= u_1\vec{a}_1 + u_2\vec{a}_2 + \dots + u_n\vec{a}_n + v_1\vec{a}_1 + v_2\vec{a}_2 + \dots + v_n\vec{a}_n \\ &= A \cdot \vec{u} + A \cdot \vec{v} \end{aligned}$$

q.e.d.

Ex.

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$