

19. Session: Gaussian Elimination

Recall: The elementary row operations

- $r_i \leftrightarrow r_j$ interchange
- $c r_j \rightarrow r_j, c \neq 0$ scaling
- $k r_i + r_j \rightarrow r_j, i \neq j$ row addition

Recall: Row echelon form (REF)

$$\left[\begin{array}{cccc|ccccccccc} 0 & 0 & \dots & 0 & * & * & \dots & * & * & \dots & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \blacksquare & * & \dots & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \blacksquare & \dots & * & * \\ & & & & & & & & & \ddots & & \end{array} \right] \quad \begin{array}{l} * \text{ any number} \\ \blacksquare \text{ nonzero number} \end{array}$$

Reduced row echelon form (RREF)

$$\left[\begin{array}{cccc|ccccccccc} 0 & 0 & \dots & 0 & 1 & * & * & \dots & * & 0 & * & * & \dots & * & 0 & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & * & * & \dots & * & 0 & * & * \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & * \\ & & & & & & & & & & & \ddots & & \end{array} \right]$$

Gaussian Elimination

Recall: A given matrix can be transformed into one and only one matrix in RREF by means of a sequence of elementary row operations.

Gaussian Elimination is an algorithm which computes the RREF of a matrix via elementary row operations.

Def. Let R be the RREF of a matrix A .

- The position that contains the first nonzero entry in a nonzero row of R (the pivot) is called a pivot position of A
- A column of A that contains some pivot position is called a pivot column.

[Link : The Row Reduction Algorithm]

Ex.

$r_1 \leftrightarrow r_2$

$-6r_1 + r_3 \rightarrow r_3$

$$\left[\begin{array}{ccccccc} 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{array} \right] \rightarrow \left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{array} \right] \rightarrow$$

$-3r_2 + r_3 \rightarrow r_3$

in REF

$\frac{1}{2}r_3 \rightarrow r_3$

$$\left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 6 & -12 & -13 & 10 & 13 \end{array} \right] \rightarrow \left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 2 & 4 & -2 \end{array} \right] \rightarrow$$

$5r_3 + r_2 \rightarrow r_2$
 $-3r_3 + r_1 \rightarrow r_1$

$\frac{1}{2}r_2 \rightarrow r_2$

$$\left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 2 & -4 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array} \right] \rightarrow$$

$r_2 + r_1 \rightarrow r_1$

in RREF

$$\left[\begin{array}{ccccccc} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccccccc} 0 & 1 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{array} \right]$$

From the RREF to the general solution

Ex.

$x_1 \ x_2 \ x_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ \underline{x_3 = 3} \end{cases}$$

Ex.

$x_1 \ x_2 \ x_3$

$$\left[\begin{array}{ccc|c} 1 & 5 & 8 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array} \right]$$

$$0x_1 + 0x_2 + 0x_3 = -1$$

no solution

$x_1 \ x_2 \ x_3 \ x_4$

$$\left[\begin{array}{cccc|c} \textcircled{1} & 0 & 7 & 5 & 8 \\ 0 & \textcircled{1} & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 + 7x_3 + 5x_4 = 8 \\ x_2 + 2x_3 - x_4 = 2 \end{cases}$$

x_1 and x_2 are basic variables (pivot below)

x_3 and x_4 are free variables

Solve the system for the basic variables in terms of

the free variables

$$\begin{cases} x_1 = 8 - 7x_3 - 5x_4 \\ x_2 = 2 - 2x_3 + x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

(2)

Procedure for solving a system of linear equations

1. Write the augmented matrix $[A | \vec{b}]$ of the system.

2. Find the RREF $[R | \vec{c}]$ of $[A | \vec{b}]$ (use eg. Gaussian elimination).

3. If $[R | \vec{c}]$ has a row of the form

$$[0 \ 0 \dots 0 | 1],$$

then the system has no solution. Otherwise, the system has at least one solution. Write the system corresponding to $[R | \vec{c}]$, and solve it for the basic variables in terms of the free variables.

Def. A system of linear equations that has one or more solutions is called consistent; otherwise, the system is called inconsistent.

Ex. Solve the system

$$3x_1 + 5x_2 - 4x_3 = 7$$

$$-3x_1 - 2x_2 + 4x_3 = -1$$

$$6x_1 + x_2 - 8x_3 = -4$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] \xrightarrow{\substack{r_1+r_2 \rightarrow r_2 \\ -2r_1+r_3 \rightarrow r_3}} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array} \right] \xrightarrow{3r_2+r_3 \rightarrow r_3}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{3}r_2 \rightarrow r_2} \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-5r_2+r_1 \rightarrow r_1}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{3}r_1 \rightarrow r_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(3)

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases}$$

$$\begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}t \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

The solution set is a line in 3D-space.