

2.1. Session: Matrix Multiplication

Recall: (Matrix-vector product)

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n] \quad m \times \underline{n} \text{-matrix}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \underline{n} \times 1 \text{-vector}$$

$$A \cdot \vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + \dots + v_n \vec{a}_n \quad m \times 1 \text{-vector.}$$

$$\text{Ex.} \quad \begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + (-2) \cdot 7 + 4 \cdot (-1) \\ 1 \cdot 0 + (-2) \cdot 2 + 4 \cdot 0 \\ 1 \cdot 3 + (-2) \cdot 0 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} -17 \\ -4 \\ 7 \end{bmatrix}$$

Def.: (Matrix multiplication)

A $m \times \underline{n}$ -matrix

$$B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p] \quad \underline{n} \times p \text{-matrix}$$

The product matrix $A \cdot B$ is defined by

$$A \cdot B = [A \cdot \vec{b}_1 \ A \cdot \vec{b}_2 \ \dots \ A \cdot \vec{b}_p] \quad m \times p \text{-matrix.}$$

$$\text{Ex.} \quad \begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right] =$$
$$\begin{bmatrix} -17 & 2 \\ -4 & 0 \\ 7 & 10 \end{bmatrix} \quad \text{"(3 \times 3) \cdot (3 \times 2) = (3 \times 2)"}$$

Note: " $(m \times n) \cdot (q \times p) = (m \times p)$ "
 $\uparrow \quad \uparrow$
 $n = q$

The product is not defined if $n \neq q$.

Theorem: (The row-column rule)

Let A be an $m \times n$ -matrix and B an $n \times p$ -matrix.

Then the (i, j) -entry in $A \cdot B$ is

$$[A \cdot B]_{ij} = \sum_{k=1}^n [A]_{ik} \cdot [B]_{kj}$$

! Ex.

$$\begin{array}{c} \text{1. row} \\ \left[\begin{array}{ccc} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{array} \right] \cdot \begin{array}{c} \text{2. column} \\ \left[\begin{array}{c} 3 \\ 0 \\ 1 \end{array} \right] \end{array} = \begin{array}{c} \text{(1,2)-entry:} \\ \left[\begin{array}{cc} -17 & 2 \\ -4 & 0 \\ 7 & 10 \end{array} \right] \end{array} \quad 1 \cdot 3 + 7 \cdot 0 + (-1) \cdot 1 = 2 \\ (3 \times 3) \cdot (3 \times 2) \qquad \qquad \qquad (3 \times 2)\end{array}$$

$$\begin{array}{c} \text{Ex.} \\ \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \right] \left[\begin{array}{cc} -1 & 1 \\ 3 & 2 \end{array} \right] = \left[\begin{array}{cc} 1 \cdot (-1) + 2 \cdot 3 & 1 \cdot 1 + 2 \cdot 2 \\ 3 \cdot (-1) + 4 \cdot 3 & 3 \cdot 1 + 4 \cdot 2 \\ 5 \cdot (-1) + 6 \cdot 3 & 5 \cdot 1 + 6 \cdot 2 \end{array} \right] = \left[\begin{array}{cc} 5 & 5 \\ 9 & 11 \\ 13 & 17 \end{array} \right] \\ (3 \times 2) \cdot (2 \times 2) \qquad \qquad \qquad (3 \times 2)\end{array}$$

Theorem: For any $m \times n$ -matrix A , $n \times p$ -matrix B and $p \times 1$ -vector \vec{v} , one has

$$(A \cdot B) \cdot \vec{v} = A \cdot (B \cdot \vec{v}).$$

Proof: Recall that

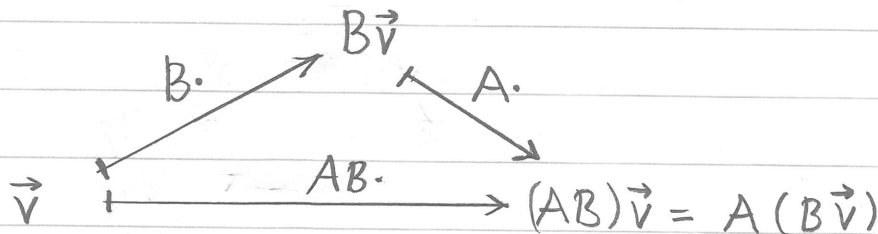
$$A \cdot (\vec{u} + \vec{w}) = A\vec{u} + A\vec{w} \quad \text{and} \quad A \cdot (c\vec{u}) = c(A\vec{u}).$$

For $B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p]$ and $\vec{v} = [v_1 \ v_2 \ \dots \ v_p]^T$, we have

$$B\vec{v} = v_1\vec{b}_1 + v_2\vec{b}_2 + \dots + v_p\vec{b}_p. \quad \text{Thus,}$$

$$\begin{aligned} A \cdot (B\vec{v}) &= A \cdot (v_1\vec{b}_1 + v_2\vec{b}_2 + \dots + v_p\vec{b}_p) \\ &= A \cdot (v_1\vec{b}_1) + A \cdot (v_2\vec{b}_2) + \dots + A \cdot (v_p\vec{b}_p) \\ &= v_1(A\vec{b}_1) + v_2(A\vec{b}_2) + \dots + v_p(A\vec{b}_p) \\ &= [A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_p] \cdot \vec{v} \\ &= (AB) \cdot \vec{v} \end{aligned}$$

q.e.d.



! Remark: In general, $A \cdot B \neq B \cdot A$

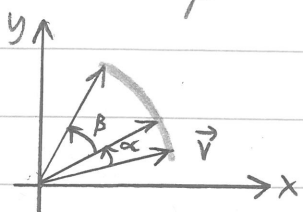
If $AB = BA$, we say that A and B commute.

Ex. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Ex. Recall: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, 2x2-rotation matrix.

$(R_\alpha R_\beta) \cdot \vec{v} = R_{\alpha+\beta} \cdot \vec{v} = R_{\beta+\alpha} \cdot \vec{v} = (R_\beta R_\alpha) \vec{v}$.



The identity holds for $\vec{v} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\vec{v} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so we have

$R_\alpha R_\beta = R_\beta R_\alpha$.

Thus R_α and R_β commute.

Remark: $A \cdot B = 0$ does not imply that $A = 0$ or $B = 0$.

Ex. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Recall: The $n \times n$ -identity matrix I_n :

$I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

Theorem: For matrices A, B, C, P, Q of appropriate sizes, one has:

(1) $s(AB) = (sA)B = A(sB)$ for any scalar s

(2) $A(CP) = (AC)P$

(3) $(A+B)C = AC + BC$

(4) $C(P+Q) = CP + CQ$

(5) $I_k A = A = A I_m$

(6) $(AC)^T = C^T A^T$.

Proof of (6): If A is a $k \times m$ -matrix and C an $m \times n$ -matrix, then $(AC)^T$ and $C^T A^T$ are both $n \times k$ -matrices OK. We check that they also have the same entries:

$$\begin{aligned} [(AC)^T]_{ij} &= [AC]_{ji} = \sum_{l=1}^m [A]_{jl} \cdot [C]_{li} \\ &= \sum_{l=1}^m [C]_{li} \cdot [A]_{jl} = \sum_{l=1}^m [C^T]_{il} [A^T]_{lj} \\ &= [C^T A^T]_{ij} \quad \text{OK} \quad \text{q.e.d.} \end{aligned}$$

Special 2x2-matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{reflection through } x\text{-axis}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \quad \text{reflection through } y\text{-axis}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{reflection through the line } y=x.$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix} \quad \text{reflection through the line } y=-x$$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix} \quad \text{scaling}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix} \quad \text{horizontal shear}$$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx+y \end{bmatrix} \quad \text{vertical shear}$$

and the rotation matrix

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$