Self-study 1: Trigonometric functions and sound

A sinusoid is a function of the form

$$y(t) = A\sin(\omega t + \phi), \tag{1}$$

where the independent variable t denotes time (measured in seconds s) and the three constants A, ω, ϕ are given as follows: A is the peak amplitude (measured in eg. volt V), ω the angular frequency (in rad/s) and ϕ the initial phase (in rad).

The frequency f of a sinusoid is by definition the number of cycels per time unit. Frequency is measured in Hertz ($1\ Hz = 1\ s^{-1}$). From the 2π-periodicity of the sine function, one can deduce that

$$f = \frac{\omega}{2\pi}. \tag{2}$$

Exercise 1

1. Let $\omega = 2\pi$ and $\phi = 0$. Sketch the graph of $y(t)$, $-1 \leq t \leq 1$ for $A = 1$, $A = 2$ and $A = 3$ in the same (t, y)-coordinate system. What are the frequencies of these sinusoids?
2. Sketch the graph of $y(t)$ for $A = 1$, $\omega = 2\pi$ and $\phi = \pi/3$. Where does the graph intersect the y-axis? What is the point of intersection for a general sinusoid?
3. Now fix $A = 1$ and $\phi = 0$. Sketch the graphs of $y(t)$ for $\omega = 2\pi$ and $\omega = 4\pi$ in the same coordinate system. Sketch the graph of $y(t)$ for $\omega = 8\pi$. What are the frequencies of these three sinusoids?
4. By forming sums of sinusoids, one can create other periodic waveforms. In fact, any periodic waveform can be obtained as a sum of a number of sinusoids, according to a branch of mathematics called Fourier Analysis. As an example, use a graphing calculator or a computer to plot the graph of
$$y(t) = \sin(2\pi t) + \sin(4\pi t).$$

Exercise 2

In this exercise, we will examine what happens when one adds sinusoids of the *same* frequency.

1. Using the trigonometric addition formulas, show that the sinusoid (1) can be written as
$$y(t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi).$$
2. Assume that we have two sinusoids with the same angular frequency ω as follows:
$$y_1(t) = A_1\sin(\omega t + \phi_1),$$
$$y_2(t) = A_2\sin(\omega t + \phi_2).$$

Rewrite these as above, and show that

$$\begin{aligned} y_1(t) + y_2(t) &= \sin(\omega t)(A_1 \cos(\phi_1) + A_2 \cos(\phi_2)) \\ &\quad + \cos(\omega t)(A_1 \sin(\phi_1) + A_2 \sin(\phi_2)). \end{aligned}$$

3. Show that if we can find A and ϕ such that

$$A \cos(\phi) = A_1 \cos(\phi_1) + A_2 \cos(\phi_2) \tag{3}$$

$$A \sin(\phi) = A_1 \sin(\phi_1) + A_2 \sin(\phi_2) \tag{4}$$

then $y_1(t) + y_2(t) = y(t)$.

4. Show that there are always an A and a ϕ which solve the two equations (3) and (4).

5. Conclude that the sum of two sinusoids of the *same* frequency is again a sinusoid of that frequency.