Alternative route: from van Schooten's Problem to Ptolemy's Theorem

ABC

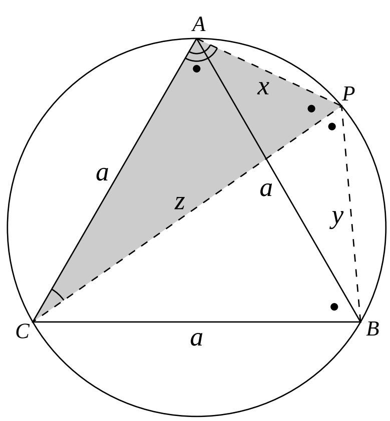
1 The equilateral triangle

Frans van Schooten (1615–1660) was influential in the mathematical circle of his day in several ways, but not least in the selection or formulation of choice problems. One of van Schooten's problems brings to light a curious property of equilateral triangles (see [8, 6]). Let $\triangle ABC$ be an equilateral triangle. If P is a point on the arc AB of the circumcircle of $\triangle ABC$ opposite C, as in Figure 1(i), then van Schooten asks us to show that

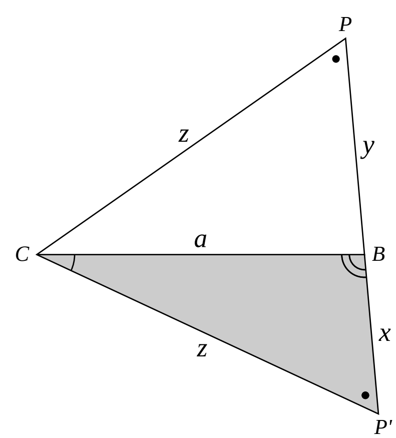
$$CP = AP + BP. \tag{1}$$

Clearly, the fact that $APBC$ is a cyclic quadrilateral is something with which to set to work. For example, we might recall a staple in school geometry textbooks in years gone by (see, for example, [7, 4, 5]; and compare [3]), the theorem of Ptolemy (c. 85–c. 165) that once served as the foundation of trigonometric computation: the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the rectangles contained by the pairs of opposite sides. So, for the cyclic quadrilateral $APBC$, we have

$$AB.CP = BC.AP + AC.BP,$$



(i)



(ii)

Figure 1: Van Schooten's Problem

or, in the notation in Figure 1(i),

$$az = ax + ay.$$

Hence,

$$z = x + y;$$

that is, (1) follows quickly as an easy exercise on Ptolemy's Theorem. Thus, depending on our prior knowledge, van Schooten's problem might not detain us long.

Still, as the equilateral triangle is rather a special case, further thought might be in order; perhaps ingenuity will come to our aid, if knowledge is deficient. When we introduce P on the circumcircle of $\triangle ABC$, the cyclic quadrilateral $APBC$ is cut into two pieces by CP. These pieces, $\triangle ACP$ and $\triangle BCP$, clearly fit together along CP. But, since the sides of $\triangle ABC$ are equal, it *looks* as though we can rotate the triangle $\triangle ACP$ about C so that AC matches up with BC. Indeed, as suggested in Figure 1(ii), this seems to produce a new equilateral triangle $\triangle PP'C$. If so, then (1) will simply express the equality of the sides CP and PP' of this triangle.

We can achieve the effect of rotation by cutting off the triangle $\triangle ACP$ from $APBC$ and juxtaposing a triangle $\triangle BCP'$ congruent with it, thus conserving the angle at C:

$$\angle PCP' = \angle ACB = \pi/3.$$

Now, *Elements III.22* tells us that the opposite angles of a cyclic quadrilateral add to two right angles. In particular, then

$$\angle CBP' + \angle CBP = \angle PAC + \angle CBP = \pi.$$

So, this property in Figure 1(i) translates into the collinearity of P, B and P' in Figure 1(ii), ensuring that our surgery on the cyclic quadrilateral $APBC$ does result in a triangle, *viz* $\triangle PP'C$. We also have equal angles abounding, in view of *Elements III.21*, that angles subtended by a chord in the same arc of a circle are equal, and the fact that $\triangle ABC$ is equiangular:

$$\angle P'PC = \angle BPC = \angle BAC = \pi/3$$

and

$$\angle PP'C = \angle BP'C = \angle APC = \angle ABC = \pi/3.$$

Hence, our new triangle $\triangle PP'C$ is indeed equiangular and so equilateral.

Thus, our sense that dissecting $APBC$ into two pieces and interchanging them relative to one another to form another equilateral triangle proves well-founded, and we can push through to answer van Schooten's question. Put another way, the dissection demonstration *verifies* Ptolemy's Theorem in the special case of a cyclic quadrilateral $APBC$ where $\triangle ABC$ is equilateral.

2 The isosceles triangle

It is clear that, with results such as Ptolemy's Theorem, or even *Elements III.21* or *III.22*, we needs must know them in order to be in a position to use them; knowledge

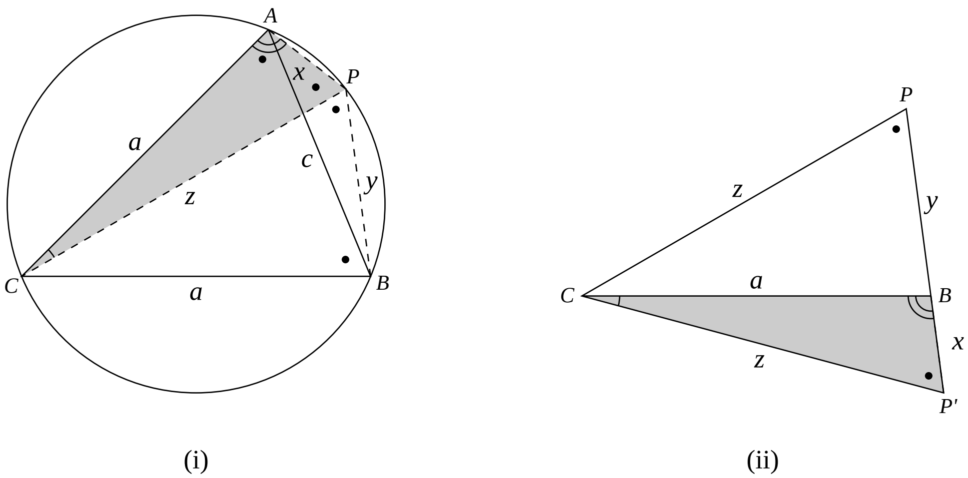


Figure 2: Isosceles triangle

is a common hurdle that everyone has to surmount. But what can be a source of insight for one person can be an impediment to progress for another. After all, acquaintance with, say, Ptolemy's Theorem may not be lacking, but we still have to have some intuition to employ it in a given instance — sometimes knowing too much leaves us uncertain where to start.

What makes the dissection argument work so well with the equilateral triangle? It is important to understand the mechanics involved, if our intuition is not to become yet another stumbling block for those anxious about thinking through an answer to van Schooten's problem for themselves. Rather, approached in a spirit of critical research, the dissection demonstration might perhaps serve as a building block encouraging further investigation.

The crucial step in the dissection argument in the previous section is the initial one. Once we are assured that the sides AC and BC of $\triangle ABC$ are equal, everything else falls neatly into place, on account of standard results in circle geometry. But this assessment means that the demonstration will also work when $\triangle ABC$ is *isosceles* with equal sides AC and BC and so with equal base angles $\angle BAC$ and $\angle ABC$ (see Figure 2). For, the angle at C remains conserved; P, B and P' continue to be collinear; and we still have plenty of equal angles:

$$\angle P'PC = \angle BPC = \angle BAC; \quad \angle PP'C = \angle BP'C = \angle APC = \angle ABC. \tag{2}$$

Since $\angle ABC = \angle BAC$, we also have $\angle P'PC = \angle PP'C$. The upshot is that the transposition of the two pieces in our dissection produces another *isosceles* triangle $\triangle PP'C$ with equal sides PC and $P'C$. Moreover, in view of (2), the base angles in the isosceles triangle $\triangle ABC$ are equal to the base angles in the isosceles triangle $\triangle PP'C$, ensuring that these triangles are similar.

It follows that

$$AB : BC = PP' : PC; \tag{3}$$

that is, in the notation of Figure 2,

$$\frac{c}{a} = \frac{x+y}{z}. \tag{4}$$

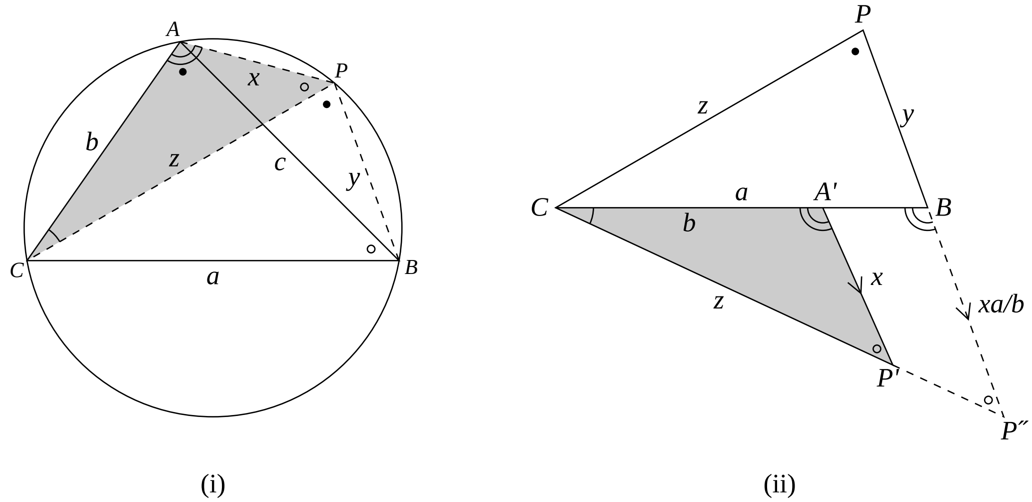


Figure 3: Ptolemy's Theorem

Hence,

$$cz = ax + ay,$$

thereby verifying Ptolemy's Theorem in the case of a cyclic quadrilateral $APBC$ in which $\triangle ABC$ is isosceles with $AC = BC$.

We might note, as an instance of the isosceles case, the *right kite*, where in addition CP is a diameter of the circumcircle of $\triangle ABC$ so that $\angle CAP$ and $\angle CBP$ are both right angles. In this instance, we can also verify Ptolemy's Theorem by computing the area of the right kite in two ways, on the one hand because it consists of two congruent right triangles and on the other because the diagonals are at right angles.

The isosceles case is, in fact, something of an historical curiosity. In one formulation, it predates Ptolemy's Theorem as such. But it reappears in another formulation in the apogee of late Arabic astronomical computation. However, we leave further discussion until Section 5, in order not to be distracted from investigating the dissection argument by digression at this stage.

3 The general triangle

Encouraged by success in extending the dissection argument to the case of an isosceles triangle, it is natural to wonder what happens if conditions are relaxed completely and $\triangle ABC$ is an *arbitrary* triangle? Naturally, there is no problem in carrying out the surgery on the cyclic quadrilateral $APBC$, deleting $\triangle ACP$ and adjoining a triangle $\triangle A'CP'$ congruent with it so that $A'C$ lies along BC, as in Figure 3. It is just that now there is no guarantee that A' will be coincident with B, as was, in effect, the case for equilateral and then isosceles triangles. But, as before, the angle at C is conserved; and at least we know that $A'P'$ is parallel to PB, because opposite angles of a cyclic quadrilateral add to two right angles. Thus, if we rescale $\triangle A'CP'$ by a factor of a/b to produce a similar triangle $\triangle BCP''$, then P, B and P'' will be collinear (see Figure 3(ii)). With this extra step, we have once more created a new triangle, $\triangle PP''C$.

Now, the significant feature of $\triangle PP''C$ is that it inherits the angles of $\triangle ABC$, extending our findings in the special cases of equilateral triangles and then isosceles triangles. Not only is the angle at C in the former the same as that in the latter by construction, but also, corresponding to (2), we have

$$\angle P''PC = \angle BPC = \angle BAC$$

and

$$\angle PP''C = \angle BP''C = \angle A'P'C = \angle APC = \angle ABC.$$

Thus, $\triangle PP''C$ and $\triangle ABC$ are similar, since the three angles of one are pairwise equal to the three angles of the other.

In this more general case, instead of (3), we have

$$AB : BC = PP'' : PC.$$

Adopting the notation of Figure 3, this gives

$$\frac{c}{b} = (\frac{ax}{b} + y)/z,$$

which, rewritten as

$$cz = ax + by, \tag{5}$$

can then be recognised as a restatement of Ptolemy's Theorem. So, at this point, our discussion tips over, from an exploration of special cases, into an alternative proof of Ptolemy's Theorem, since now the cyclic quadrilateral $APBC$ is perfectly general. But had we known Ptolemy's Theorem, and been content in that knowledge, we might never have entered upon this different avenue of approach.

Ptolemy presented his theorem on cyclic quadrilaterals in *Almagest I.10*. The instances where a side or a diagonal is a diameter of the circumcircle were instrumental in trigonometic computations for astronomical purposes. The classical proof in terms of similar triangles is that rehearsed in [7, 4]. But the Law of Cosines is called in aid in [5] to develop algebraic expressions for the squares of the diagonals in terms of the sides that are themselves of considerable antiquity and from which Ptolemy's Theorem can be deduced as an immediate consequence. Ptolemy's Theorem is a handy tool in studying *rational* cyclic quadrilaterals, as an extension of right triangles with rational sides (see [10]).

4 The right triangle

Of course, Ptolemy's Theorem contains within it Pythagoras' Theorem that, for a right triangle, the square on the hypotenuse is equal in area to the sum of the squares on the legs, as can be seen by restricting the cyclic quadrilateral to be a rectangle. As in Figure 4(i), we take $\triangle ABC$ to be a right triangle with right angle at C and CP a diameter of the circumcircle so that $APBC$ is a rectangle. Since this allows us to identify x with a, y with b and z with c, (5) becomes

$$c^2 = a^2 + b^2.$$

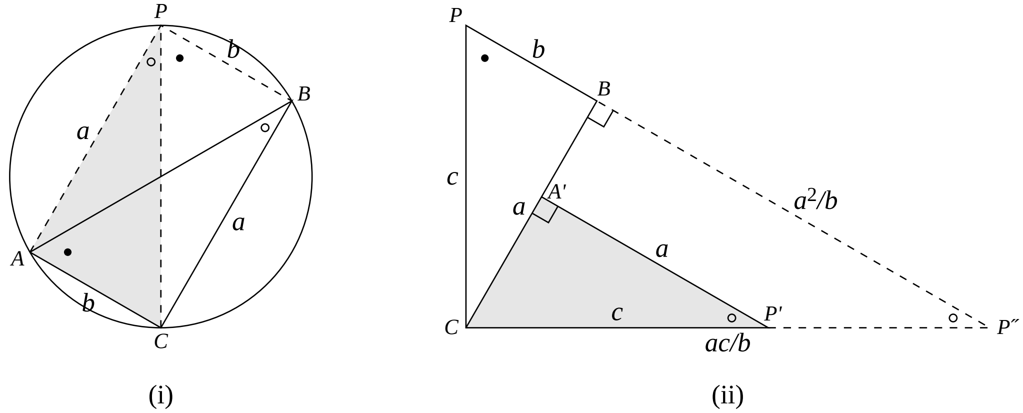


Figure 4: Pythagorean Proposition

But it is also interesting to see what our alternative proof in Section 3 looks like in this case. The result, shown in Figure 4(ii), is that $\triangle PP''C$ is a right triangle similar to $\triangle ABC$, with BC the altitude at C dividing $\triangle PP''C$ into two further similar right triangles. This arrangement of similar right triangles within a right triangle is familiar from Euclid's proof of *Elements VI.31* extending Pythagoras' Theorem from squares to similar figures similarly situated on the edges of a right triangle. At least theoretically, we could start with the figure for *VI.31* and imagine ourselves reversing the steps to devise the alternative proof of Ptolemy's Theorem by dissection and rescaling. However, the right triangle seems a less plausible starting point for this than van Schooten's equilateral triangle, perhaps because it requires a greater sense of what to do next at each stage. Some doors are trap doors.

5 Historical anomalies

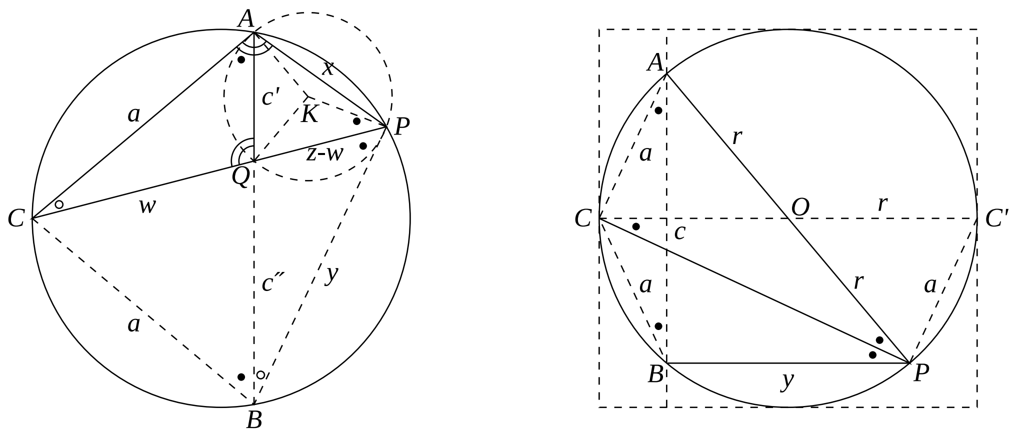
In Section 2, we considered the isosceles case only as an initial extension of van Schooten's problem. But it has an historical significance in its own right to which we now turn.

Euclid (c. $\overline{325}$–c. $\overline{265}$) presents an equivalent version of this special case of Ptolemy's Theorem in *Data 93*, the penultimate proposition of *Data*. For, in a cyclic quadrilateral $APBC$, $AC = BC$ if and only if CP is the angle bisector of $\angle APB$. Thus, we find at *Data 93*, in this notation, the assertion that, if the chord CP of the circumcircle of $\triangle ABP$ bisects $\angle APB$, then the ratio $AP + BP : CP$ is, as Euclid puts it, *given*. Indeed,

$$AP + BP : CP = AB : BC,$$

in agreement with (4).

Now, Euclid observes further in *Data 93* that the area of the rectangle with sides $AP + BP$ and CQ is another *given*, where Q is the intersection of AB and CP, the diagonals of our cyclic quadrilateral $APBC$, as in Figure 5(i). Why this might be considered of interest clearly depends on how this area is expressed. With reference to Figure 5(i), we see that the pairs of triangles, $\triangle ACP$ with $\triangle QCA$ and $\triangle BCP$ with $\triangle QCB$, have matching angles, and so are similar. Hence, in the notation of



(i) Thâbit ibn Qurra (ii) Jamshīd al-Kāshī

Figure 5: Historical instances

Figure 5(i),

$$\frac{x}{a} = \frac{c'}{w}, \quad \frac{y}{a} = \frac{c''}{w}.$$

Combining these equations and noting that $c' + c'' = c$, we have

$$(x + y)w = ac. \tag{6}$$

Comparison of (4) and (6) invites speculation that Euclid's attention may have been caught by the fact that in one we have the ratio of a and c whereas in the other we have the product (in rejoinder to [12]).

But (4) and (6) can themselves be combined to give

$$a^2 = wz, \tag{7}$$

that is, the rectangle contained by CP and CQ is equal to the square on CA. In the context of Euclid's *Elements*, we may recognise (7) as an instance of *Elements III.36*, since the disposition of the angles in Figure 5(i) ensures that CA is tangent at A to the circumcircle of $\triangle APQ$ — Euclid also takes the trouble to provide a converse to *III.36* in *III.37*. Turning the argument around, we see that either part of *Data 93* follows from the other, on appeal to *Elements III.36*. However, looking further afield, (7) is in effect the extension of the Pythagorean Proposition, *Elements I.47*, attributed in [11] to Al-Sabi Thâbit ibn Qurra al-Harrani (836–901):

Theorem (Thâbit) *Given $\triangle ACP$, let Q on CP be such that $\angle AQC = \angle PAC$. Then $AC^2 = CP.CQ$.*

For, if in the circumstances envisaged by Thâbit we produce AQ to meet the circumcircle of $\triangle ACP$ in B, we recover Figure 5(i). But discussions of Thâbit's generalisation of *I.47* are apt not to mention *III.36*, still less cyclic quadrilaterals and *Data 93*, perhaps because its arresting statement deflects attention from such details.

Naturally, it is also possible that *Data 93* was put to use by some early trigonometric tabulator (compare [12]). But curiously enough this case of a cyclic quadrilateral $APBC$ where CP bisects $\angle APB$ resurfaces as the preferred means to greater accuracy in the trigonometric computations of Ghiyath al-Din Jamshīd Mas'ud al-Kāshī(1390–1450), at the culmination of the Arabic period of table-making (see [2, 1]):

Theorem (al-Kāshī) *Consider a circle on AP as diameter, with centre O and radius r. If C is the midpoint of the arc AB, then $r(2r + BP) = CP^2$.*

Although al-Kāshī was working in the tradition of Ptolemy, he derives his result *ab initio* from propositions in Euclid's *Elements*, without reference either to the general statement of Ptolemy's Theorem or the special case of relevance here already anticipated by Euclid in *Data 93*. But we can see al-Kāshī's Theorem as an immediate consequence of that latter result, namely (4), by reference to Figure 5(ii). For, $\triangle ABC$ and $\triangle CPO$ are similar isosceles triangles, so that

$$\frac{c}{a} = \frac{z}{r}. \tag{8}$$

Since $x = 2r$, combination of (4) and (8) yields al-Kāshī's Theorem in the form

$$z^2 = r(2r + y).$$

For that matter, we can give a proof based only on Ptolemy's Theorem if we consider two other cyclic quadrilaterals in Figure 5(ii). To this end, let CC' be the diameter of the circle parallel to BP, so that $AC'PC$ is a (cyclic) rectangle and $CC'PB$ is a cyclic trapezium. Thus, application of Ptolemy's Theorem to these special cyclic quadrilaterals gives

$$\begin{aligned} (2r)^2 &= z^2 + a^2, \\ a^2 + 2ry &= z^2. \end{aligned}$$

Now, we have only to add, eliminate the term a^2 from both sides and halve, for al-Kāshī's Theorem to drop out. Lively experimentation of this sort was still in play when John Ward (c. 1658–c. 1729) was attempting to express the chord of an angle $n\theta$ in terms of that of θ in his *A Compendium of Algebra* [13], first published in 1695.

On the other hand, taking the difference, rather than the sum, of the last two displayed equations leads to

$$(2r)^2 - 2ry = 2a^2. \tag{9}$$

We may recognise $(2r)^2$ easily enough as the area of the square with sides parallel and perpendicular to CC' circumscribing the circle in Figure 5(ii). Then the term $2ry$ in (9) may be interpreted as the area of the rectangle within this circumscribing square cut off by the lines AB produced and line through P perpendicular to CC'. So (9) tells us that this complement has area $2a^2$. By symmetry, AB produced cuts off an area a^2 from the circumscribing square. But now this is exactly the division of the square on the hypotenuse CC' of the right triangle $\triangle CC'A$ envisioned in Euclid's proof of his first version of the Pythagorean Proposition, *Elements I.47*, if achieved by different means.

References

[1] M. K. Azarian, Al-Kāshī's Fundamental Theorem, *Int. J. Pure Appl. Math.*, **14** (2004), 499–509.

[2] G. van Brummelen, Jamshīd al-Kāshī: calculating genius, *Mathematics in School*, **27** (1998) No. 4, 40–44; extract reprinted in [9, pp. 130–135].

[3] A. J. Crilly and C. R. Fletcher, Ptolemy's Theorem, its parent and offspring, in [9, pp. 42–49].

[4] C. V. Durell, (a) *A Course of Plane Geometry for Advanced Students, Pts. 1, 2* (Macmillan, London, UK, Pt. 1, 1909; Pt. 2, 1910); Pt. 1 revised as (b) *Modern Geometry: the straight line and circle* (Macmillan, London, UK, 1920), esp. pp. 17–18.

[5] C. V. Durell and A. Robson, *Advanced Trigonometry* (G. Bell and Sons, London, UK, 1939), esp. pp. 25, 27; preface available at ⟨`http://turnbull.mcs.st-and.ac.uk/~history/Extras`⟩.

[6] D. W. French, Van Schooten's Theorem, *in* [9, pp. 184–186].

[7] C. Godfrey and A. W. Siddons, *Modern Geometry* (Cambridge University Press, Cambridge, UK, 1908), esp. pp. 80–82; available at ⟨`http://quod.lib.umich.edu/u/umhistmath`⟩.

[8] R. Honsberger, *Mathematical Morsels* (Math. Assoc. Amer., Washington, DC, 1979), esp. p. 172.

[9] C. Pritchard, *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching* (Cambridge University Press, Cambridge, UK, 2003).

[10] A. P. Rollett, Rational cyclic quadrilaterals, *Math. Gaz*, **24** (1940), 329–333.

[11] A. Sayili, Thâbit ibn Qurra's generalization of the Pythagorean theorem, *Isis*, **51** (1960), 35–37.

[12] C. M. Taisbak, *Eucild's Data; Or the Importance of Being Given* (Museum Tusculanum Press, Univ. of Copenhagen, Copenhagen, 2003) esp. 232–233.

[13] J. Ward, *A Compendium of Algebra: consisting of plain, easie and concise rules for the speedy attaining to that art, exemplified by various problems, with the solution of the aequations in numbers, by a new and general method of resolving all kind of aequations, both pure and adfected, with great ease and expedition, very different from all others yet extant, applied to squaring the circle, making of sines, tangents, secants and logarithms with great facility: also an appendix concerning compound-interest and annuities* (Printed for the author, and sold by him at the Black-Boy Coffee-House in Goodmans-field's; by J. Taylor bookseller at the Ship in St. Paul's Churchyard; by J. Worgan and T. Cooke mathematical instrument-makers under St. Dunstans-Church in Fleet-street and in Threadneedle-street, London, 1695; 2nd ed., 1698; 2nd ed. corr., 1724); available at ⟨`http://eebo.chadwyck.com/home`⟩.