Reduction and Approximation of Multidimensional Persistent Homology

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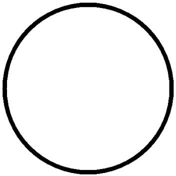
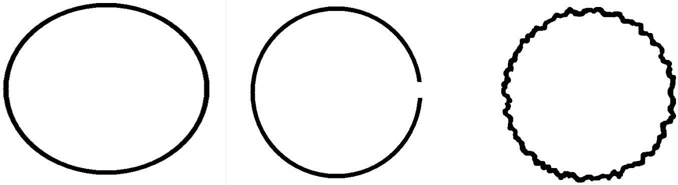
Outline

1. Shape
2. Persistent topology
3. Distances
4. Multidimensional persistent homology
5. One-dimensional reduction
6. Ball coverings
7. Combinatorial representation
8. Conclusions

Which object has the same shape as the circle?





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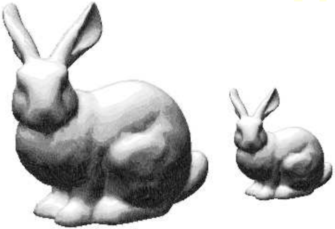

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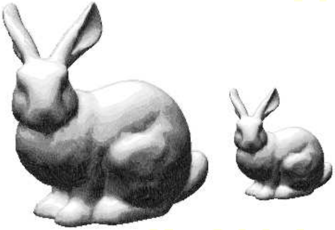

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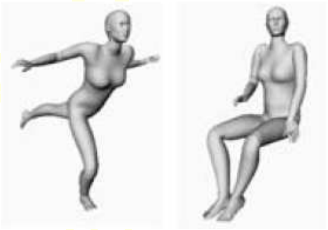
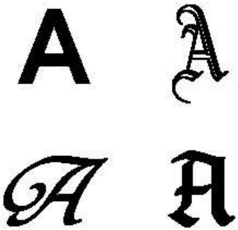


. . . but not only they do.

Shape = Topology?

Sometimes homeomorphisms are deceptive.

Shape = Psychology?



Size pairs

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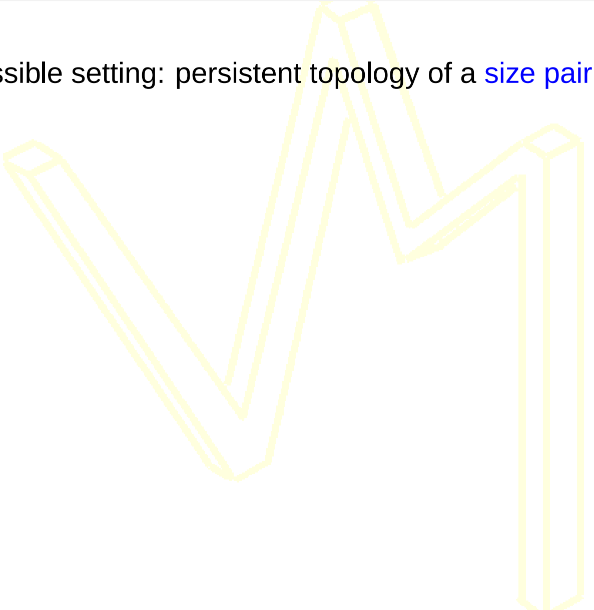

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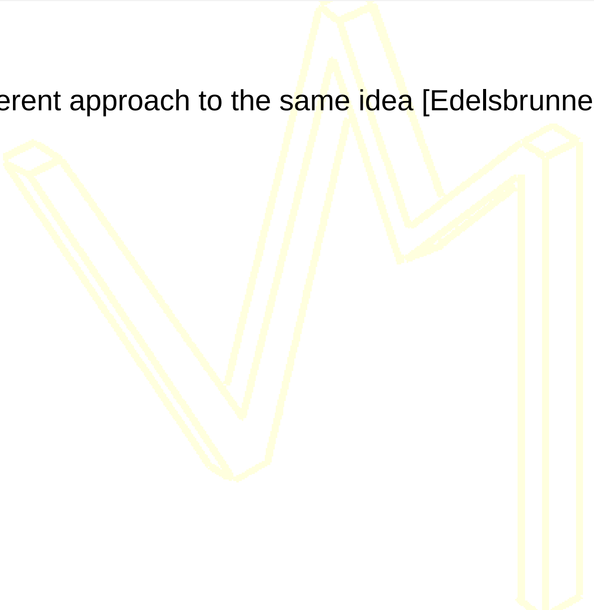
(Claudia Landi and Patrizio Frosini will further elaborate on that. In fact, what I am presenting is largely the product of team work at our Vision Mathematics group in Bologna.)

Persistence

- A different approach to the same idea [Edelsbrunner et al. 2000]:



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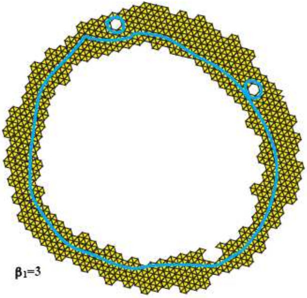

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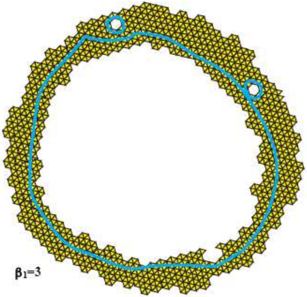

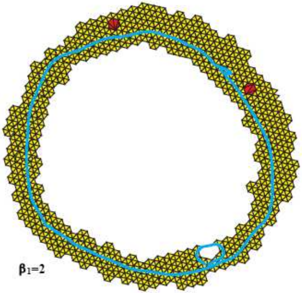

Natural pseudodistance

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- Such a measure is the natural pseudodistance d between pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$, where X and Y are homeomorphic compact spaces. d is defined as

$$d\left((X,\vec{\varphi}),(Y,\vec{\psi})\right) = \inf_{f} \max_{P\in X} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty$$

where f varies among all homeomorphisms from X to Y [Frosini et al. 1999].

Rank invariant [Carlsson et al. 2007]

- We define the following relation $\preceq$ ($\prec$) in $\mathbb{R}^n$: if $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$, we write $\vec{u} \preceq \vec{v}$ ($\vec{u} \prec \vec{v}$) if and only if $u_j \leq v_j$ ($u_j < v_j$) for $j = 1, \ldots, n$. Let also Δ^+ be the set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \vec{u} \prec \vec{v}\}$.

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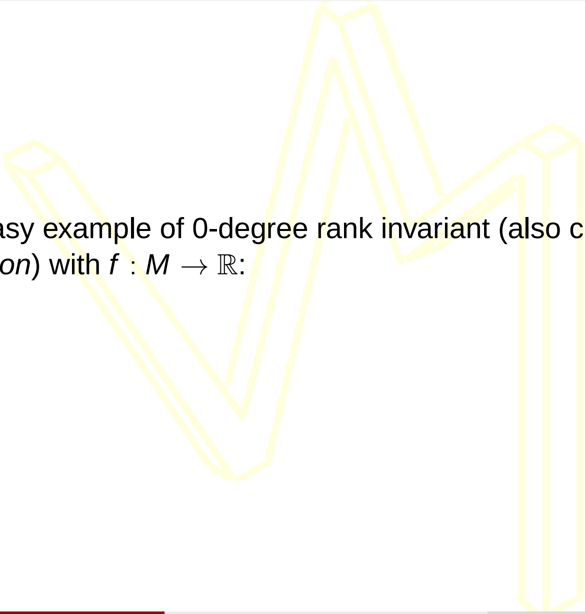
- We denote by $X\langle \vec{f} \preceq \vec{u} \rangle$ the lower level set $\{p \in X \mid \overrightarrow{f}(p) \preceq \overrightarrow{u}\}$.
- For each $i \in \mathbb{Z}$, the i-th rank invariant of $(X, \vec{f})$ is $\rho_{(X,\vec{f},i)} : \Delta^+ \to \mathbb{N}$ defined as $\rho_{(X,\vec{f},i)}(\vec{u}, \vec{v}) = \dim(Imf_{\vec{u}}^{\vec{v}}),\ \vec{u} \prec \vec{v}$ with

$$f_{\vec{u}}^{\vec{v}} : H_i(X\langle \vec{f} \preceq \vec{u} \rangle) \to H_i(X\langle \vec{f} \preceq \vec{v} \rangle),$$

where $f_{\vec{u}}^{\vec{v}}$ is the homomorphism induced by the inclusion map of lower level sets $X\langle \vec{f} \preceq \vec{u} \rangle \subseteq X\langle \vec{f} \preceq \vec{v} \rangle$

Size functions [Frosini 1991]



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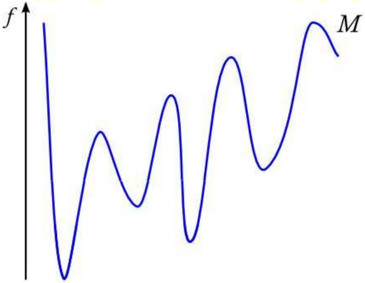

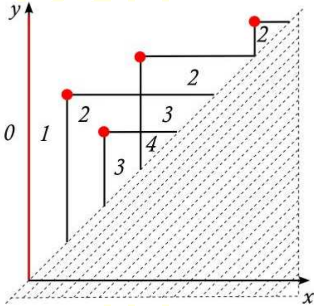

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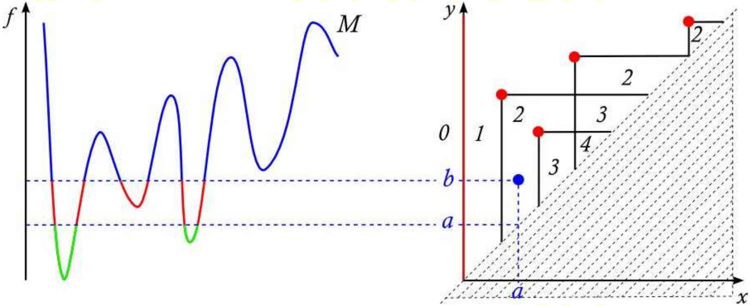

Precursors

Other precursors of the rank invariant:

- *Size homotopy groups* [Frosini et al. 1999]
- *Persistent Betti numbers* [Edelsbrunner et al. 2000]
- *Persistence barcodes* [Carlsson et al. 2005]
- *Size functor* [Cagliari et al. 2001]

Cornerpoints

When the codomain of the measuring function is one-dimensional, the rank invariant is built by superimposition of (possibly unbounded) triangles. So all of its information is condensed in their vertices (possibly at infinity) with multiplicity, i.e. in the so called *persistence diagram* and can be treated as a formal series of points [Frosini et al. 2001].

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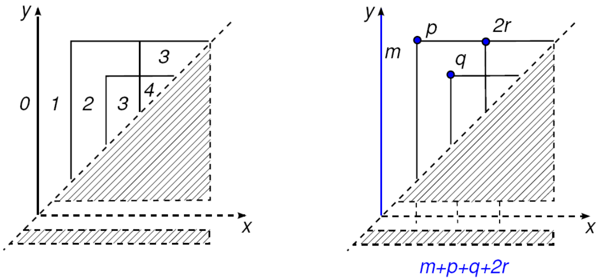

A lower bound from size functions

- Recall the natural pseudodistance between size pairs (X, φ), (Y, ψ) (with one-dimensional codomains):

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A lower bound from size functions

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$$d((X,\varphi),(Y,\psi)) = \inf_{f} \max_{P\in X} |\varphi(P) - \psi(f(P))|$$

- If there exist $\bar{x}, \tilde{x}, \bar{y}, \tilde{y} \in \mathbb{R}$ such that $\rho_{(X,\varphi,0)}(\bar{x},\bar{y}) > \rho_{(Y,\psi,0)}(\tilde{x},\tilde{y})$, then

$$\min\{\tilde{x} - \bar{x}, \bar{y} - \tilde{y}\} \leq d((X,\varphi),(Y,\psi))$$

Matching distance between rank invariants (one-dimensional case)

It is possible to define a matching distance $d_{match}(\rho_1, \rho_2)$ between the rank invariants ρ_1, ρ_2 of two size pairs, at the same degree, by minimizing the greatest distance between corresponding cornerpoints of the two persistence diagrams (having added a suitable set of points on the diagonal).

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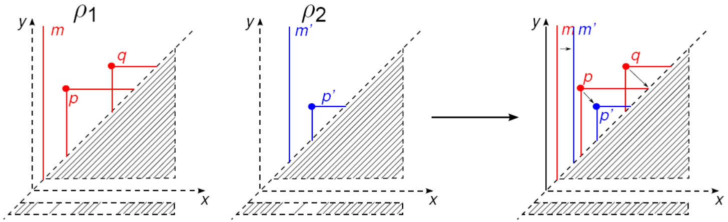

A lower bound from the matching distance

Still in the case of one-dimensional domain, the matching distance between size functions provides a lower bound for the natural pseudodistance:

$$d_{match}\left(\rho_{(X,\varphi,0)},\rho_{(Y,\psi,0)}\right) \leq d\left((X,\varphi),(Y,\psi)\right)$$

The need for a multidimensional codomain

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- Next pictures illustrate the case of $(\mathcal{C}, \vec{\varphi}')$, $(\mathcal{S}, \vec{\varphi}'')$, where $\mathcal{C}$ and $\mathcal{S}$ are a cube of edge length 2 and a sphere of diameter 2 respectively, and $\vec{\varphi}'(x, y, z) = \vec{\varphi}''(x, y, z) = (|x|, |y|)$.

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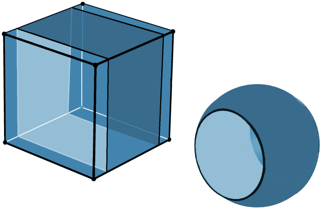
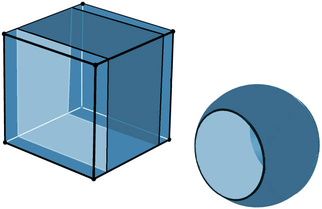
- Lower level sets of the single components are homotopically circles in both cases, whereas they differ if the whole functions are taken into account.



Lower level sets of the component $|y| \leq \frac{\sqrt{2}}{2}$. The ones of $|x|$ are just rotated versions.



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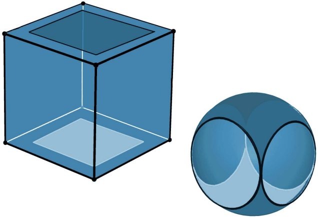

Lower level sets of $\vec{\varphi}' = \vec{\varphi}'' \preceq (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

The problem of discontinuities

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- In general, in the $2n$-dimensional Δ^+, the rôle of cornerpoints is played by a $(2n - 2)$-dimensional submanifold.
- No more possibility of representing the invariant by a formal series? No more matching distance?

Admissible pairs

- For size functions [Biasotti et al. 2008] then generally for rank invariants [Cagliari et al. 2009] we have proved that suitable foliations of Δ^+ exist, along which the computation can be reduced to the one-dimensional case, so back to cornerpoints.

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- For size functions [Biasotti et al. 2008] then generally for rank invariants [Cagliari et al. 2009] we have proved that suitable foliations of Δ^+ exist, along which the computation can be reduced to the one-dimensional case, so back to cornerpoints.
- For every vector $\vec{l} = (l_1, \ldots, l_n)$ in $\mathbb{R}^n$ such that $\sqrt{\sum_{j=1}^{n} l_j^2} = 1$, and $l_j > 0$ for $j = 1, \ldots, n$, and for every vector $\vec{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ such that $\sum_{j=1}^{n} b_j = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is admissible. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} \vec{u} &= s\vec{l} + \vec{b} \\ \vec{v} &= t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

Tame functions

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- Let $(X, \vec{\varphi})$ be a size pair. We shall say that $\vec{\varphi}$ is max-tame if, for every admissible pair $(\vec{l}, \vec{b})$, the function $g(P) = \max_{j=1,\dots,n} \left\{ \frac{\varphi_j(P) - b_j}{l_j} \right\}$ is tame.

Reduction

Theorem

Let $(\vec{l}, \vec{b})$ be an admissible pair and $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n) : X \to \mathbb{R}^n$ be a max-tame function. Then, for every $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$, and for

$$g(P) = \max_{j=1,\ldots,n} \left\{ \frac{\varphi_j(P) - b_j}{l_j} \right\}$$

the equality

$$\rho_{(X,\vec{\varphi},i)}(\vec{u}, \vec{v}) = \rho_{(X,g,i)}(s, t)$$

holds for all $i \in \mathbb{Z}$ and $s, t \in \mathbb{R}$ with $s < t$.

Matching distance (multidimensional case)

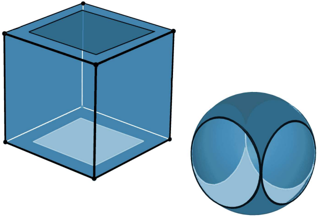
The previous theorem gives us the opportunity of defining a distance between n-dimensional rank invariants.

Let $(X, \vec{\varphi}')$, $(Y, \vec{\varphi}'')$ be two max-tame size pairs and $\bar{\rho}'_{X,i}$, $\bar{\rho}''_{Y,i}$ be the respective rank invariants. Then the i-th multidimensional matching distance is defined as

$$D(\bar{\rho}'_{X,i}, \bar{\rho}''_{Y,i}) = \sup_{(\vec{l},\vec{b})} \min_{j=1,\ldots,n} l_j \cdot d(\rho'_{X,i}, \rho''_{Y,i})$$

where $(\vec{l}, \vec{b})$ varies among all admissible pairs, and for each such pair $\rho'_{X,i}$, $\rho''_{Y,i}$ are the rank invariants of the corresponding one-dimensional functions.

Example



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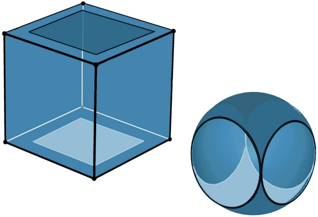
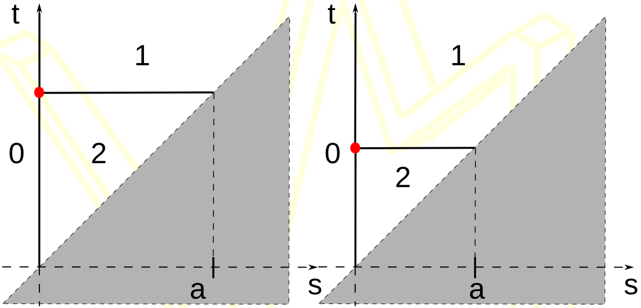


We choose $\vec{l} = (\cos\theta, \sin\theta)$ with $0 < \theta < \frac{\pi}{2}$, and $\vec{b} = (a, -a)$ with $a \in \mathbb{R}$. The corresponding half-plane is parameterized as

$$\begin{cases} u_1 = s\cos\theta + a \\ u_2 = s\sin\theta - a \\ v_1 = t\cos\theta + a \\ v_2 = t\sin\theta - a \end{cases}$$

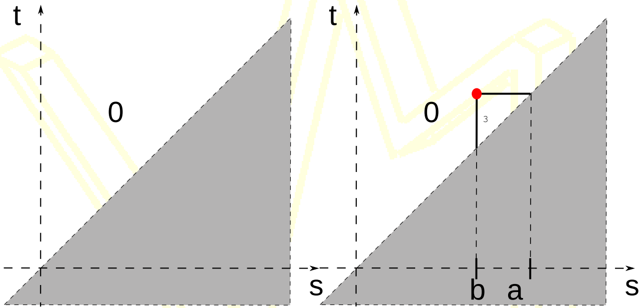
with $s, t \in \mathbb{R}, s < t$. In particular, we consider $\theta = \frac{\pi}{4}, a = 0$.

$i = 0$ Left: cube; right: sphere.



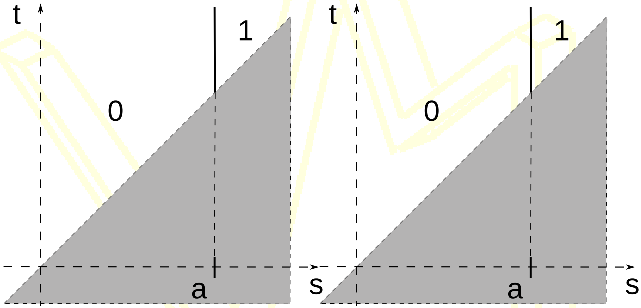
$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},l}, \overline{\rho}''_{\mathcal{S},l}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},l}, \rho''_{\mathcal{S},l}) = \frac{\sqrt{2}}{2}(\sqrt{2} - 1)$$

$i = 1$ Left: cube; right: sphere.



$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},\infty}, \overline{\rho}''_{\mathcal{S},\infty}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},\infty}\ \rho''_{\mathcal{S},\infty}) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}-1}{2}\right)$$

$i = 2$ Left: cube; right: sphere.



$$\Rightarrow D(\overline{\rho}'_{\mathcal{C},\in}, \overline{\rho}''_{\mathcal{S},\in}) \geq \frac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},\in}\ \rho''_{\mathcal{S},\in}) = 0$$

Approximations

We hardly have access to a full, formal description of an object of interest, in our case a compact, Riemannian submanifold X of $\mathbb{R}^m$.

We normally have a cloud of points belonging to it or even to a narrow neighborhood.

Still, it is possible to get inequalities relating the rank invariant of the considered object and the one of a suitable ball covering together. This is possible because of the following theorem.

Retracts

In the following, τ is the largest number such that every open normal bundle B about X of radius s is embedded in $\mathbb{R}^m$ for $s < \tau$.

Theorem (Niyogi et al., 2008)

Let $L = \{l_1, \ldots, l_k\}$ be a collection of points of X, and let $U = \bigcup_{j=1,\ldots,k} B(l_j, \delta)$ be the union of balls of $\mathbb{R}^m$ with center at the points of L and radius δ. Now, if L is such that for every point $p \in X$ there exists an $l_j \in L$ such that $\|p - l_j\| < \dfrac{\delta}{2}$, then, for every $\delta < \sqrt{\dfrac{3}{5}}\tau$, X is a deformation retract of U. So they have the same homology.

Inequalities

Luckily, the construction of the previous theorem can be carried over to lower level sets, so we get a double inequality.
Let $\vec{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. Then, for $\varepsilon \in \mathbb{R}^+$, the modulus of continuity $\Omega(\varepsilon)$ of $\vec{f}$ is:
$\Omega(\varepsilon) = \max_{j=1,\dots,n} \sup \left\{ \mathrm{abs}(f_j(\vec{p}) - f_j(\vec{p'})) \mid \vec{p}, \vec{p'} \in \mathbb{R}^m,\ \|\vec{p} - \vec{p'}\| \leq \varepsilon \right\}$

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With L, δ and U as before, we have:

Theorem (Cavazza et al. 2010)

If $(\vec{u}, \vec{v})$ is a point of Δ^+ and if $\vec{u} + \vec{\omega}(\delta) < \vec{v} - \vec{\omega}(\delta)$, where $\vec{\omega}(\delta) = (\Omega(\delta), \dots, \Omega(\delta)) \in \mathbb{R}^n$, then

$$\rho_{(U,\vec{f}_U,i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta)) \le \rho_{(X,\vec{f}_X,i)}(\vec{u}, \vec{v}) \le \rho_{(U,\vec{f}_U,i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta))$$

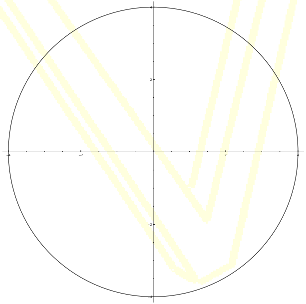
Blind strips

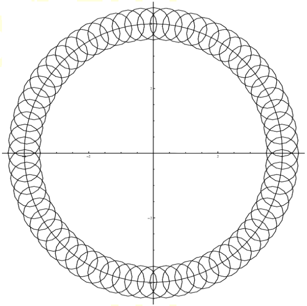
If the two extremes coincide, also the middle term equals them. An interesting consequence of this is that in a neighborhood of the discontinuity sets we may have strict inequalities, but out of it, the rank invariant of the unknown X coincides with that of the well-known U.

This is particularly interesting in the case of $n = 1$ (to which, as we have seen, it is still possible to bring back the computation). With $n = 1$, the regions of uncertainty, which we call *blind strips*, are just 2ω-wide strips around the discontinuity segments of the persistence diagram for U.

Example

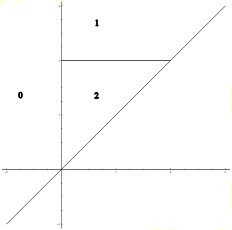
A circle X in the plane and a covering of it, made of balls centered at points of X. Measuring function is $|y|$.

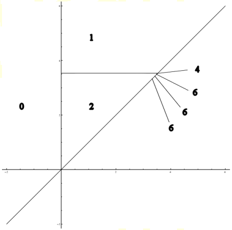




Example

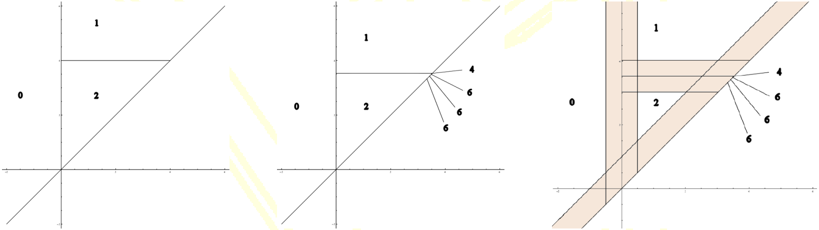
The 0-th degree rank invariant (size function) of X and U





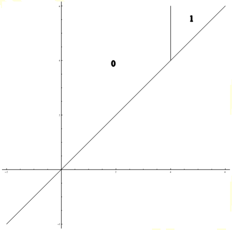
Example

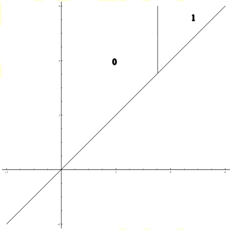
The 0-th degree rank invariant (size function) of X and U and the blind strips



Example

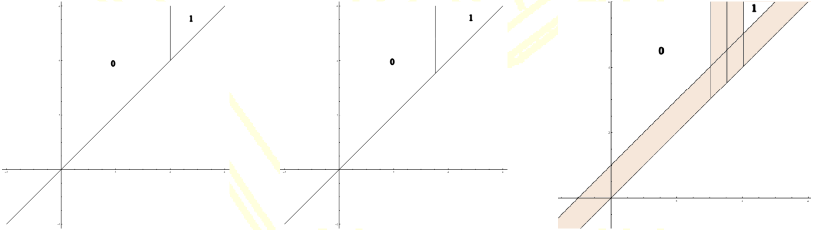
The 1-st degree rank invariant of X and U





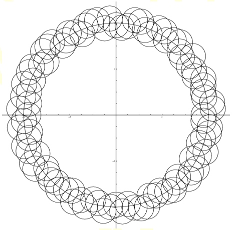
Example

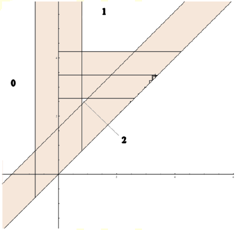
The 1-st degree rank invariant of X and U and the blind strips



Points near X

Once more based on a theorem by Nyiogi, Smale and Weinberger, it is possible to get a (more complicated) inequality for points picked up in a narrow neighborhood of X.





Finiteness

The finiteness of the sample suggests the possibility of a combinatorial representation.
This is indeed possible by means of a construction of Edelsbrunner.

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Out of a finite set B of balls, it is possible to build its *Voronoi diagram* $\mathcal{V}$ which gives rise to the *dual complex* $\mathcal{K}$. For $\mathcal{S} = |\mathcal{K}|$ and the union U of the balls of B it holds

Theorem (Edelsbrunner 1993)

$\mathcal{S}$ is a deformation retract of U.

Voronoi diagram

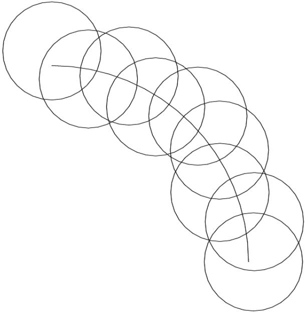


Figure: A quarter of circle of radius 4 covered by nine balls of radius 1.

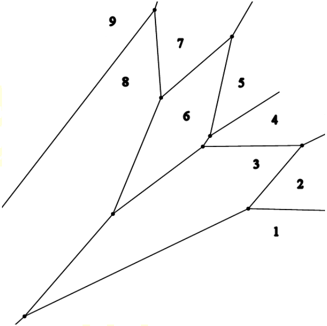


Figure: The Voronoi Diagram $\mathcal{V}$ of B.

Dual shape

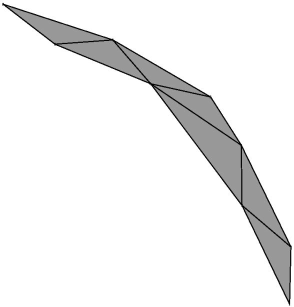


Figure: The dual complex $\mathcal{K}$.

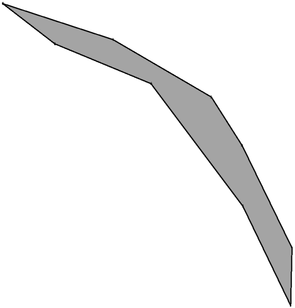
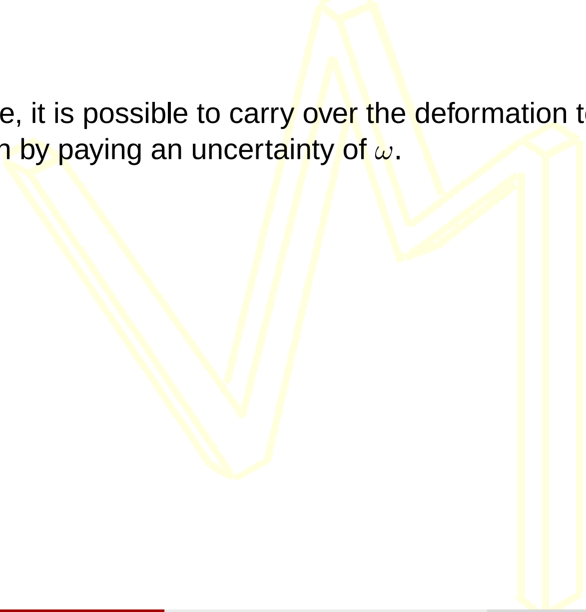


Figure: The dual shape $\mathcal{S}$.

Ball union and dual shape

Once more, it is possible to carry over the deformation to lower level sets, again by paying an uncertainty of ω.



Ball union and dual shape

Once more, it is possible to carry over the deformation to lower level sets, again by paying an uncertainty of ω.

Lemma

If $(\vec{u}, \vec{v})$ *is a point of* Δ^+ *and if* $\vec{u} + \vec{\omega}(\delta) < \vec{v} - \vec{\omega}(\delta)$*, where* $\vec{\omega}(\delta) = (\Omega(\delta), \ldots, \Omega(\delta)) \in \mathbb{R}^n$ *, then*

$$\rho_{(U, \vec{f}_U, i)}(\vec{u} - \vec{\omega}(\delta), \vec{v} + \vec{\omega}(\delta)) \leq \rho_{(\mathcal{S}, \vec{f}_\mathcal{S}, i)}(\vec{u}, \vec{v}) \leq \rho_{(U, \vec{f}_U, i)}(\vec{u} + \vec{\omega}(\delta), \vec{v} - \vec{\omega}(\delta)).$$

Submanifold and dual shape

By combining the two results concerning U, we get a double inequality relating the initial submanifold X and the dual shape $\mathcal{S}$ (so also the dual complex $\mathcal{K}$). The outcoming blind strips are now 4ω wide.

Theorem (Cavazza et al. 2010)

If $(\vec{u}, \vec{v})$ is a point of Δ^+ and if $\vec{u} + 2\vec{\omega}(\delta) < \vec{v} - 2\vec{\omega}(\delta)$, where $\vec{\omega}(\delta) = (\Omega(\delta), \ldots, \Omega(\delta)) \in \mathbb{R}^n$, then

$$\rho_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u}-2\vec{\omega}(\delta), \vec{v}+2\vec{\omega}(\delta)) \leq \rho_{(X,\vec{f}_X,i)}(\vec{u}, \vec{v}) \leq \rho_{(\mathcal{S},\vec{f}_{\mathcal{S}},i)}(\vec{u}+2\vec{\omega}(\delta), \vec{v}-2\vec{\omega}(\delta)).$$

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- The reduction of the computation of the rank invariant — in the case of a multidimensional codomain of the measuring function — to the one-dimensional case, by means of a suitable foliation.
- The possibility of estimating the rank invariant of a submanifold of $\mathbb{R}^m$ by computing it for a ball union covering it, or for a related polyhedron.

THANKS FOR THE ATTENTION !

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