About Higher Dimensional Transition Systems

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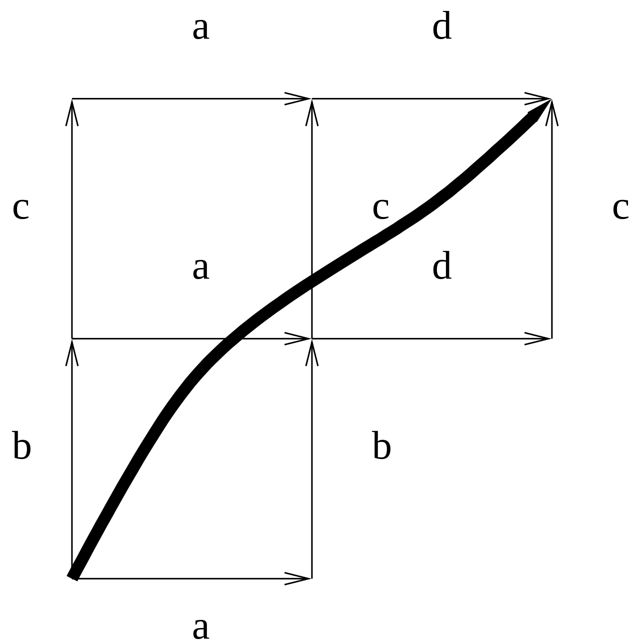
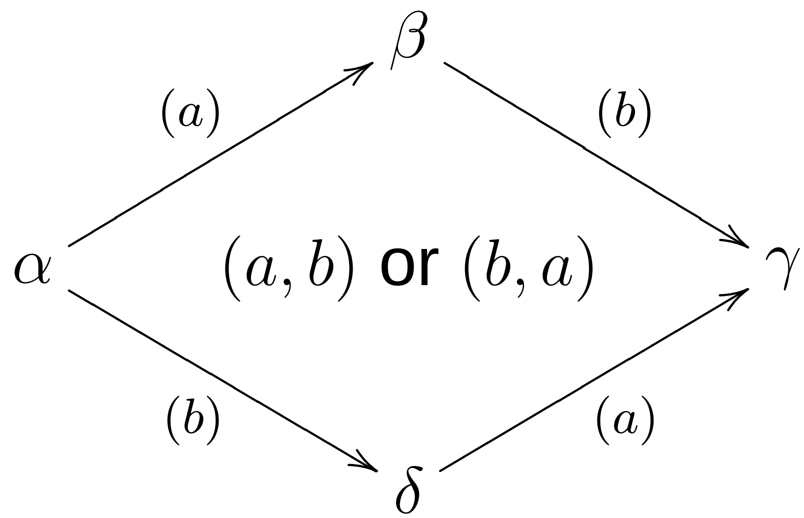
Organization of the talk

1. Weak higher dimensional transition system (HDTS)
2. Labelled symmetric precubical set
3. Realizing a labelled symmetric precubical set as a weak HDTS
4. Categorical equivalence
5. Homotopy theory of weak HDTS and bisimulation

Concurrent execution of two actions





- $\{\alpha, \beta, \gamma, \delta\}$ set of states and of 0-cubes
- $\{(\alpha, a, \beta), (\beta, b, \gamma), (\alpha, b, \delta), (\delta, a, \gamma), (\alpha, a, b, \gamma), (\alpha, b, a, \gamma)\}$ set of transitions

Weak HDTS

- Σ a nonempty set of labels
- A weak HDTS $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$ with set of states S, set of actions L, labelling map μ, set of n-transitions $T_n \subset S \times L^n \times S$ with $n \geqslant 1$
  - Multiset axiom $(\alpha, u_1, \dots, u_n, \beta) \in T$ implies $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta) \in T$ for every permutation σ
  - Coherence axiom if $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ belong to T, then $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2) \in T$
- Note: the Coherence axiom is automatically satisfied in a cube

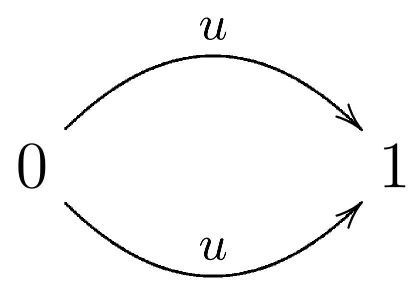
The category WHDTS

- $X = (S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n)$ and $X' = (S', \mu' : L' \to \Sigma, T' = \bigcup_{n \geqslant 1} T'_n)$
- A map of weak HDTS $f : X \to X'$ consists of
  - a set map $f_0 : S \to S'$
  - a map $\widetilde{f} \in (\mathbf{Set}{\downarrow}\Sigma)(\mu, \mu')$

  such that if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition, then $(f_0(\alpha), \widetilde{f}(u_1), \dots, \widetilde{f}(u_n), f_0(\beta))$ is a transition
- The forgetful functor $\omega : \mathbf{WHDTS} \to \mathbf{Set}^{\{s\} \cup \Sigma}$: $\omega(X) = (S, (\mu^{-1}(x))_{x \in \Sigma})$ is concrete topological
- $\mathbf{WHDTS}$ is locally finitely presentable

Two additional axioms

- First Cattani-Sassone Axiom CSA1 If (α, u, β) and (α, u', β) are two transitions such that $\mu(u) = \mu(u')$, then $u = u'$. Impossible situation:



  - CSA1 always satisfied for concrete examples
- Unique Intermediate State Axiom CSA2 for $n \geqslant 2$ and $1 \leqslant p < n$, for every transition $(\alpha, u_1, \ldots, u_n, \beta)$, there exists a unique state ν such that $(\alpha, u_1, \ldots, u_p, \nu)$ and $(\nu, u_{p+1}, \ldots, u_n, \beta)$ are transitions
  - CSA2 plays the role of the face maps in a precubical set

The category HDTS

- HDTS : weak HDTS satisfying CSA1 and CSA2
  - **HDTS** locally finitely presentable, but not topological
  - **HDTS** full reflective subcategory of **WHDTS**
- The pure n-transition $C_n[a_1, \ldots, a_n]^{ext}$
  - Set of states $S = \{I, F\}$
  - Set of actions $L = \{(a_1, 1), \ldots, (a_n, n)\}$
  - Labelling map $\mu(a_i, i) = a_i$
  - $T = \{(I, (a_{\sigma(1)}, \sigma(1)), \ldots, (a_{\sigma(n)}, \sigma(n)), F)\}$
- The n-cube $C_n[a_1, \ldots, a_n]$ is the reflection of $C_n[a_1, \ldots, a_n]^{ext}$: 2^n states since $\{(a_1, 1), \ldots, (a_n, n)\}$ contains n elements

Cattani-Sassone HDTS

- A Cattani-Sassone HDTS is a HDTS with the Coherence axiom replaced by CSA3 below
- Third Cattani-Sassone axiom CSA3 if the nine tuples $(\alpha, u_1, \ldots, u_n, \beta)$, $(\alpha, u_1, \ldots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \ldots, u_n, \beta)$, $(\nu_1, u_{p+1}, \ldots, u_{p+q}, \nu_2)$, $(\nu_2, u_{p+q+1}, \ldots, u_n, \beta)$, $(\alpha, u_1, \ldots, u_{p+q}, \nu_2')$, $(\nu_2', u_{p+q+1}, \ldots, u_n, \beta)$, $(\alpha, u_1, \ldots, u_p, \nu_1')$ and $(\nu_1', u_{p+1}, \ldots, u_{p+q}, \nu_2')$ are transitions, then $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$
- The HDTS are the Cattani-Sassone HDTS
- The Coherence axiom is the topological part of CSA3
- The remaining part of CSA3 is algebraic and already contained in CSA2

HDTS as a small-orthogonality class

- Every HDTS is orthogonal to the set of maps $C_n[a_1,\ldots,a_n]^{ext} \to C_n[a_1,\ldots,a_n]$ for $n \geqslant 0$ and $a_1,\ldots,a_n \in \Sigma$
- Every HDTS is orthogonal to the set of maps $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$
- Every weak HDTS orthogonal to the set of maps $C_n[a_1,\ldots,a_n]^{ext} \to C_n[a_1,\ldots,a_n]$ for $n \geqslant 0$ and $a_1,\ldots,a_n \in \Sigma$ satisfies CSA2
- Every weak HDTS orthogonal to the set of maps $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$ satisfies CSA1
- HDTS is a small-orthogonality class

Symmetric precubical set

- Symmetric precubical set
  - Family $(K_n)_{n\geqslant 0}$ of sets ($x \in K_n$ is called a n-cube), of face maps $\partial_i^\alpha : K_n \to K_{n-1}$ satisfying the cubical relations and of symmetry maps $s_i : K_n \to K_n$ with $1 \leqslant i \leqslant n-1$ satisfying the Moore relations and $\partial_j^\alpha s_i = s_{i-1}\partial_j^\alpha$ for $j < i$, $\partial_j^\alpha s_i = \partial_{i+1}^\alpha$ for $j = i$, $\partial_j^\alpha s_i = \partial_i^\alpha$ for $j = i+1$ and $\partial_j^\alpha s_i = s_i \partial_j^\alpha$ for $j > i+1$
- The symmetric precubical set of labels
  - $(!^S\Sigma)_0 = \{()\}$
  - $(!^S\Sigma)_n = \Sigma^n$ for $n \geqslant 1$
  - $\partial_i^\alpha(a_1, \ldots, a_n) = (a_1, \ldots, \widehat{a_i}, \ldots, a_n)$
  - $s_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, a_n)$

Labelled symmetric precubical set

- Labelled symmetric precubical set: map $K \to !^S\Sigma$
  - The category $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$ is locally finitely presentable
- $[n] = \{0,1\}^n$, the n-cube $\square_S[n]$ defined by: $f \in \square_S[n]_p$ set map from $[p]$ to $[n]$ which is a composite of
  - $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_r) = (\epsilon_1, \dots, [\alpha]_i, \dots, \epsilon_r)$
  - $\sigma_i(\epsilon_1, \dots, \epsilon_r) = (\epsilon_1, \dots, \epsilon_{i+1}, \epsilon_i, \dots, \epsilon_r)$
- The labelled n-cube $\square_S[a_1, \dots, a_n]$ with $a_1, \dots, a_n \in \Sigma$ is the map $\square_S[n] \to !^S\Sigma$ which takes $\mathrm{Id}_{[n]}$ to $(a_1, \dots, a_n)$
- The boundary $\partial\square_S[a_1, \dots, a_n]$ of $\square_S[a_1, \dots, a_n]$ is the labelled n-cube with the $n!$ n-cubes removed

From precubical set to WHDTS

- There exists an isomorphism of categories

$$\boxed{\mathbb{T} : \left\{\text{Cubes of } \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma\right\} \cong \{\text{Cubes of } \mathbf{HDTS}\}}$$

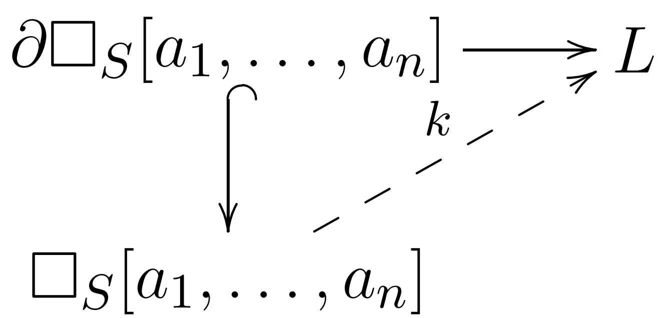
- $\mathbb{T}$ extended to a functor $\mathbb{T} : \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma \to \mathbf{WHDTS}$ by

$$\boxed{\mathbb{T}(K) := \varinjlim_{\square_S[a_1,\dots,a_n]\to K} C_n[a_1,\dots,a_n]}$$

  - $\mathbb{T} : \square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma \to \mathbf{WHDTS}$ is a left adjoint
  - $\mathbb{T}$ is not faithful, not full and not essentially surjective

About faithfulness

- For $n \geqslant 2$, the two maps
  $\square_S[a_1, \ldots, a_n] \rightrightarrows \square_S[a_1, \ldots, a_n] \sqcup_{\partial\square_S[a_1,\ldots,a_n]} \square_S[a_1, \ldots, a_n]$
  have same image by $\mathbb{T}$
  $\mathrm{Id}_{C_n[a_1,\ldots,a_n]} : C_n[a_1, \ldots, a_n] \rightarrow C_n[a_1, \ldots, a_n]$
- The map $(\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma)(K, L) \rightarrow \mathbf{WHDTS}(\mathbb{T}(K), \mathbb{T}(L))$ is one-to-one when there exists at most one lift k



for $n \geqslant 2$ and $a_1, \ldots, a_n \in \Sigma$ HDA paradigm

About the HDA paradigm

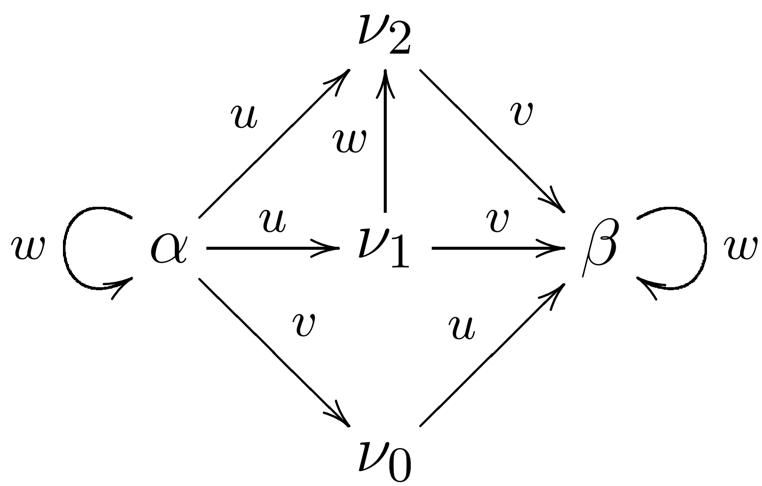
- A labelled symmetric precubical set L satisfies the HDA paradigm if and only if L is orthogonal to the maps

  $$\square_S[a_1, \ldots, a_n] \sqcup_{\partial \square_S[a_1,\ldots,a_n]} \square_S[a_1, \ldots, a_n] \to \square_S[a_1, \ldots, a_n]$$

  for $n \geqslant 2$ and $a_1, \ldots, a_n \in \Sigma$
- The full subcategory $\mathbf{HDA}^\Sigma$ of L satisfying the HDA paradigm is reflective locally finitely presentable
- The restriction functor $\mathbb{T} : \mathbf{HDA}^\Sigma \to \mathbf{WHDTS}$ is faithful
- It is not full, nor essentially surjective...

About fullness (I)

- Let $C_2[u,v]$ be the HDTS with set of states $\{\alpha, \beta, \nu_0, \nu_2\}$
- Let X be the image by $\mathbb{T}$ of



- The inclusion $C_2[u,v] \subset X$ does not come from a map of labelled symmetric precubical sets since there are no squares with the vertices $\alpha, \beta, \nu_0, \nu_2$

About fullness (II)

- Let K and L be two labelled symmetric precubical sets such that L satisfies the HDA paradigm and such that $\mathbb{T}(L)$ satisfies the Unique intermediate state axiom. Then the set map

$$(\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma)(K,L) \xrightarrow{f\mapsto\mathbb{T}(f)} \mathbf{WHDTS}(\mathbb{T}(K),\mathbb{T}(L))$$

is bijective

About essential surjectivity

- $\underline{x} = (\varnothing, \{x\} \subset \Sigma, \varnothing) \neq \mathbb{T}(\varnothing)$ with $x \in \Sigma$
- $\left\{ \begin{array}{c} \mathbb{T}(\square_1[x] \sqcup \square_1[x]) \\ \xrightarrow{x_1} \\ \xrightarrow{x_2} \end{array} \right. \ncong \left\{ \begin{array}{c} \varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x]) \\ \xrightarrow{x} \\ \xrightarrow{x} \end{array} \right.$
- We must consider the categorical localization $\mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$ where

$$\underline{\mathrm{Cub}}(X) = \varinjlim_{C_n[a_1,\dots,a_n]\to X} C_n[a_1, \dots, a_n]$$

- $\underline{\mathrm{Cub}}(\varnothing) \cong \underline{\mathrm{Cub}}(\underline{x})$
- $\underline{\mathrm{Cub}}(\mathbb{T}(\square_1[x] \sqcup \square_1[x])) \cong \underline{\mathrm{Cub}}\, \varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x])$

The category equivalence

- A category equivalence

$$\mathbf{HDA}^{\Sigma}_{hdts} := \mathbb{T}^{-1}(\mathbf{HDTS}) \cap \mathbf{HDA}^{\Sigma} \simeq \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$$

- $\mathbf{HDA}^{\Sigma}_{hdts}$ is reflective locally finitely presentable in $\mathbf{HDA}^{\Sigma}$, and in $\square_S^{op}\mathbf{Set}{\downarrow}!^S\Sigma$
- Note: a HDTS $\mathbb{T}(K)$ satisfies CSA1 if and only if K is orthogonal to the maps $\square_S[a] \sqcup_{\partial\square_S[a]} \square_S[a] \to \square_S[a]$ for $a \in \Sigma$
- Question: is $\mathbf{HDA}^{\Sigma}_{hdts}$ a small-orthogonality class ?

Homotopy theory of weak HDTS

- A cofibration of weak HDTS is a map inducing an injection on the set of actions
- There exists a left-determined model category structure on $\mathbf{WHDTS}$ with respect to the class of cofibrations
  - i.e. the smallest localizer with respect to the cofibrations is the class of weak equivalences of a model category structure
  - A left proper combinatorial model category
- Consider the Bousfield localization with respect to the inclusions
  - $C_n[a_1, \dots, a_n]^{ext} \subset C_n[a_1, \dots, a_n]$
  - $\underline{x} \subset C_1[x]$

Cubical transition system

- Wrong definition: colimits of cubes because $C_1[x] \sqcup C_1[x]$ and $\varinjlim(C_1[x] \leftarrow \underline{x} \rightarrow C_1[x])$ must be two non-isomorphic cubical transition systems
- A cubical transition system is a weak HDTS X injective w.r.t. the maps
  - $\underline{x} \subset C_1[x]$ for $x \in \Sigma$
  - $C_n[a_1, \dots, a_n]^{ext} \subset C_n[a_1, \dots, a_n]$ for $n \geqslant 0$ and $a_1, \dots, a_n \in \Sigma$
- The full subcategory $\mathbf{CTS}$ is locally presentable, coreflective, not concretely coreflective and not topological

Homotopy theory in CTS

- A cofibration of cubical transition systems is a map inducing an injection on the set of actions
- There exists a left-determined model category structure on $\mathbf{CTS}$ with respect to the class of cofibrations
  - i.e. the smallest localizer with respect to the cofibrations is the class of weak equivalences of a model category structure
  - A left proper combinatorial model category
- The adjunction $\mathbf{CTS} \leftrightarrows \mathbf{WHDTS}$ is a Quillen equivalence

The homotopy category of WHDTS

- The canonical map $X \to \mathbf{1}$ functorially factors as a composite

$$X \longrightarrow \mathrm{CSA}_1(X) \longrightarrow \mathbf{1}$$

  with $X \to \mathrm{CSA}_1(X)$ transfinite composition of pushouts of $C_1[x] \sqcup_{\partial C_1[x]} C_1[x] \to C_1[x]$ for $x \in \Sigma$ and $\mathrm{CSA}_1(X)$ safisfying CSA1

- The homotopy category of $\mathbf{WHDTS}$ is equivalent to $\mathbf{CTS}[\mathrm{CSA}_1^{-1}]$: in particular, two weakly equivalent cubical transition systems satisfying CSA1 (e.g. HDTS) are isomorphic

- $C_1[x] \sqcup C_1[x] \to \varinjlim(C_1[x] \leftarrow \underline{x} \to C_1[x])$ is not a weak equivalence !

Bousfield localizing w.r.t. $\underline{\mathrm{Cub}}$

- There exists a model structure on $\mathbf{WHDTS}$ such that the homotopy category is $\mathbf{CTS}[\underline{\mathrm{Cub}}^{-1}]$
- In this new model category, the map

$$C_1[x] \sqcup C_1[x] \to \varinjlim(C_1[x] \leftarrow \underline{x} \to C_1[x])$$

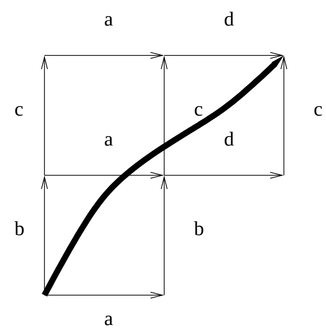
  is a weak equivalence
- This model structure is obtained by Bousfield localizing $\mathbf{WHDTS}$ w.r.t. the class of maps

$$\left\{g : X \to Y \text{ s.t. } \underline{\mathrm{Cub}}(g^{fib}) \text{ weak equivalence}\right\}$$

where $(-)^{fib}$ is the fibrant replacement functor

Path in a weak HDTS

- A path is a cubical transition system such that:
  - there is a unique initial state I
  - there is a unique final state F
  - there is a finite number of states, all reachable from I



- Class of paths denoted by $\mathcal{P}$; a path from a CTS X is a map $P \longrightarrow X$

Bisimulation in CTS

- Let X and X' be two CTS with set of states S and S'; X' simulates X if there exists a binary relation $\mathcal{R} \subset S \times S'$ such that for every $(\alpha, \alpha') \in \mathcal{R}$, for every path P, for every map $c : P \to X$ such that $c(I) = \alpha$, there exists a map $c' : P' \to X'$ such that $c'(I) = \alpha'$ and such that $(c(F), c'(F)) \in \mathcal{R}$.
- (Usual definition) X and X' are bisimilar if X' simulates X and X simulates X'.
- (Joyal-Nielsen-Winskel's definition) X and X' are bisimilar iff there exists a diagram of maps of cubical transition systems $X \leftarrow Z \to X'$ such that the maps $Z \to X$ and $Z \to X'$ both satisfies the RLP with respect to the inclusion $\{I\} \subset P$ for every path P

Homotopy and bisimulation

- Two weakly equivalent cubical transition systems are bisimilar
- Two weakly equivalent weak HDTS are not necessarily bisimilar: e.g. $C_n[a_1, \dots, a_n]^{ext}$ and $C_n[a_1, \dots, a_n]$
- Some last remarks:
  - Does the categorical equivalence

$$\mathbf{HDA}^{\Sigma}_{hdts} := \mathbb{T}^{-1}(\mathbf{HDTS}) \cap \mathbf{HDA}^{\Sigma} \simeq \mathbf{HDTS}[\underline{\mathrm{Cub}}^{-1}]$$

    come from a Quillen equivalence ?
  - Does the Bousfied localization w.r.t. the class of all bisimulations exist ?